数理解析研究所講究録
第 798 巻 1992 年 1-8

ON ε-CONVEX MULTIOBJECTIVE PROGRAMMING PROBLEMS

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1. Introduction

In this note, we consider ε-solutions of convex multiobjective programming problems and ε-saddle points of vector Lagrangian.

Pareto solutions of multiobjective programming problems and saddle points of vector Lagrangian were investigated by many authors. To guarantee the existence of the Pareto solutions and the saddle points, certain compactness assumption is necessary in general. However, Loridan established the existence results of ε-solutions of multiobjective programming problems without the compactness assumption in [4].

We assume none of the compactness condition in this note. Our results obtained extend the ones of Lai and Ho [2] and Loridan [3]. This note is organized as follows: In section 2, we formulate a multiobjective programming problem and recall the definition of ε-quasi Pareto solution of the problem due to [4]. In section 3, corresponding to the results of [2], we show the necessary and sufficient condition to obtain the ε-quasi Pareto solution. In section 4, we establish the existence result of ε-saddle points of vector Lagrangian. Also, by investigating the ε-saddle point, the existence result of some ε-solution of the multiobjective programming problem is shown to be verified.

2. Preliminaries

Let X be a Banach space with the dual space X^*.

In this note, we consider the following multiobjective programming problem:

(P) minimize $f(x)$

subject to $g(x) \leq 0$

where $f = (f_1, \ldots, f_n)$ and $g = (g_1, \ldots, g_m)$,

$f_k (1 \leq k \leq n)$, $g_i (1 \leq i \leq m) : X \to \mathbb{R}$,

$g(x) \leq 0$ means that $g_i(x) \leq 0$ for each i.

We denote the feasible set $\{x \in X | g(x) \leq 0\}$ by K.

Throughout this note, we assume the following:

Assumption (A). f_k is continuous convex and bounded from below for each $k = 1, \ldots, n$.

g_i is continuous convex for each $i = 1, \ldots, m$.

The feasible set K is nonempty.

Each element of $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{R}^n$ is positive.

With this assumption, Loridan [4] showed the existence theorem of ε-solutions of the multiobjective programming problem (P).

Theorem 2.1. [4, Proposition 4.2] Assume that (A) is satisfied. Let $u \in K$ be a $(\Sigma_{j=1}^{n} \varepsilon_j)$-solution of

$\inf\{\Sigma_{k=1}^{n} f_k(x) | x \in K\}$

i.e.

$$\Sigma_{k=1}^{n} f_k(u) \le \inf\{\Sigma_{k=1}^{n} f_k(x) \mid x \in K\} + \Sigma_{j=1}^{n} \varepsilon_j.$$

Then, there exists $x_\varepsilon \in K$ such that

(1) there is no $x \in K$ such that $f_k(x) \le f_k(x_\varepsilon) - \varepsilon_k$, $k = 1, \ldots, n$, with at least one strict inequality.

(2) $\Sigma_{k=1}^{n} f_k(x_\varepsilon) \le \Sigma_{k=1}^{n} f_k(u)$.

(3) $\|u - x_\varepsilon\| \le \sqrt{(\Sigma_{j=1}^{n} \varepsilon_j)}$

(4) there is no $x \in K$ such that $f_k(x) + (\varepsilon_k/\sqrt{\Sigma\varepsilon_j})\|x - x_\varepsilon\| \le f_k(x_\varepsilon)$, $k = 1, \ldots, n$, with at least one strict inequality.

Associated with the above theorem, Loridan defined an ε-approximate solution of (P) as follows:

Definition 2.1.[4] An element $x_\varepsilon \in K$ is said to be an ε-quasi Pareto solution of (P) if and only if x_ε satisfies (4) of Theorem 2.1.

Remark. Since there exists a $(\Sigma_{j=1}^{n}\varepsilon_j)$-solution of $\inf\{\Sigma_{k=1}^{n} f_k(x) \mid x \in K\}$, the existence of the ε-quasi Pareto solution of (P) is also verified under the assumption (A). However, the (exact) Pareto solution of (P) is not necessarily attained under the same assumption (A). The condition to obtain the Pareto solution of (P) was derived in [5].

3. ε-Optimality Condition

In this section, associated with the result of Lai and Ho [2], we show the necessary and sufficient condition to obtain an ε-quasi

Pareto solution of (P).

To derive our results, we introduce the regular condition associated with the constraint qualification of Slater type [1].

Assumption (A1). For each j, there exists a point x such that

$g(x) < 0$ and $F_k(x) < F_k(x_\varepsilon)$ for any $k \neq j$

where

x_ε is an ε-quasi Pareto solution of (P),

$F_k(x) = f_k(x) + \bar{\varepsilon}_k \|x - x_\varepsilon\|$ with $\bar{\varepsilon}_k = \varepsilon_k / \sqrt{(\Sigma_{j=1}^{n} \varepsilon_j)}$.

Under the above assumption, we show the necessary and sufficient condition to obtain an ε-quasi Pareto solution of (P).

Theorem 3.1. Assume that (A1) is satisfied. Let $x_\varepsilon \in K$. Then, x_ε is an ε-quasi Pareto solution of (P) if and only if there exist $a_\varepsilon \in A$ and $b_\varepsilon \in B$ such that

$0 \in \partial f(x_\varepsilon) a_\varepsilon + \partial g(x_\varepsilon) b_\varepsilon + \bar{\varepsilon} a_\varepsilon B^*$,

$g(x_\varepsilon) b_\varepsilon = 0$.

where

A and B denote the sets of matrices defined by

$A = \{a = (a_{jk})_{n\times n} \mid a_{jk} \geq 0$ and $a_{kk} = 1,\ j, k = 1, \ldots, n\}$,

$B = \{b = (b_{ik})_{m\times n} \mid b_{ik} \geq 0,\ i = 1, \ldots, m,\ k = 1, \ldots, n\}$.

$\partial f(x) = (\partial f_1(x), \ldots, \partial f_n(x)) = \{(x_1^*, \ldots, x_n^*) \mid x_k^* \in \partial f_k(x)\}$,

$\partial g(x) = (\partial g_1(x), \ldots, \partial g_m(x)) = \{(x_1^*, \ldots, x_m^*) \mid x_i^* \in \partial g_i(x)\}$,

with $\partial f_k(x)$ $(\partial g_i(x))$ is the subdifferential of f_k(resp. g_i) at x.

$\bar{\varepsilon} = (\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_n)$ with $\bar{\varepsilon}_j = \varepsilon_j / (\sqrt{\Sigma_{k=1}^{n} \varepsilon_k})$.

B^* is the unit ball in X^*.

4. ε-Quasi Saddle Point of Vector Lagrangian

In this section, we establish the existence result of ε-saddle points of vector Lagrangian by using the scalarized Lagrangian. Also, the existence result of some ε-solutions of (P) is presented.

Definition 4.1.[2] The vector Lagrangian $L : X \times U \to \mathbb{R}^n$ is defined by

$$L(x, \lambda) = f(x) + g(x)\lambda$$

where

$$\lambda \in U = \{\lambda = (\lambda_{ik})_{m\times n} \mid \lambda_{ik} \geq 0 \text{ for any } i = 1, \ldots, m,\ k = 1, \ldots, n\}$$

Definition 4.2. A point $(x_\varepsilon, \lambda_\varepsilon) \in X \times U$ is said to be an ε-quasi saddle point of L if the following conditions are satisfied:

there is no $\lambda \in U$ and $x \in X$ such that

$$L(x_\varepsilon, \lambda) - \bar{\varepsilon}[\lambda - \lambda_\varepsilon] \geq L(x_\varepsilon, \lambda_\varepsilon),\ L(x_\varepsilon, \lambda) - \bar{\varepsilon}[\lambda - \lambda_\varepsilon] \neq L(x_\varepsilon, \lambda_\varepsilon)$$

$$L(x_\varepsilon, \lambda_\varepsilon) \geq L(x, \lambda_\varepsilon) + \bar{\varepsilon}\|x - x_\varepsilon\|,\ L(x_\varepsilon, \lambda_\varepsilon) \neq L(x, \lambda_\varepsilon) + \bar{\varepsilon}\|x - x_\varepsilon\|$$

where

$[\lambda] = ([\lambda]_{jk})_{n\times n}$ is a $n\times n$ matrix

with $[\lambda]_{kk} = \|\lambda_k\|$, $[\lambda]_{jk} = 0$ for $j \neq k$, $\lambda_k = (\lambda_{1k}, \ldots, \lambda_{mk})^T$.

Remark. If $k = 1$, the above definition reduces to [3, Definition 6.1].

We show the relation between the ε-quasi saddle point and the primal problem (P).

Proposition 4.1. Assume that (A1) is satisfied. Let x_ε be an ε-quasi Pareto solution of (P). Then, there exists $\lambda_\varepsilon \in U$ such that

$(x_\varepsilon, \lambda_\varepsilon)$ is an ε-quasi saddle point of vector Lagrangian L.

Proposition 4.2. Let $(x_\varepsilon, \lambda_\varepsilon)$ be an ε-quasi saddle point of L. Then, the following are satisfied:

there is no $x \in \{x \mid g(x) \leq g(x_\varepsilon)\}$ such that

$f_k(x) + \bar{\varepsilon}_k\|x - x_\varepsilon\| \leq f_k(x_\varepsilon)$ for $k = 1, \cdots, n$

with at least one strict inequality,

$g_i(x_\varepsilon) \leq \tilde{\varepsilon}$ for any i where $\tilde{\varepsilon} = \max\{\bar{\varepsilon}_k \mid k = 1, \ldots, n\}$,

$\lambda_{\varepsilon_{ik}} \neq 0$ for each k implies $-\tilde{\varepsilon} \leq g_i(x_\varepsilon)$.

Remark. If k = 1, the above result reduces to [3, Theorem 6.2].

Now, we introduce a scalarized Lagrangian associated with the vector Lagrangian.

Definition 4.3. [2] For fixed $\mu \in \text{int}\, \mathbb{R}^n_+$ (i.e. $\mu_k > 0$ for any $k = 1, \ldots, n$), the scalar Lagrangian L_μ is defined by $L_\mu(x, \lambda) = \langle \mu, L(x, \lambda) \rangle$.

We show the relation between L_μ and the vector Lagrangian L.

Proposition 4.3. Let $(x_\varepsilon, \lambda_\varepsilon)$ be an ε-quasi saddle point of L_μ in the sence of Loridan [3] i.e.

$$L_\mu(x_\varepsilon, \lambda) - \langle \mu, \bar{\varepsilon}[\lambda - \lambda_\varepsilon] \rangle$$
$$\leq L_\mu(x_\varepsilon, \lambda_\varepsilon)$$

$$\leq L_\mu(x, \lambda_\varepsilon) + \langle \mu, \bar{\varepsilon}\|x - x_\varepsilon\|\rangle \text{ for any } (x, \lambda) \in X \times U$$

Then, $(x_\varepsilon, \lambda_\varepsilon)$ is also an ε-quasi saddle point of L.

We establish the existence result of an ε-quasi saddle point of scalar Lagrangian.

Proposition 4.4. For some $\mu \in \text{int } \mathbb{R}^n_+$, there exists an ε-quasi saddle point of L_μ.

Remark. From Propositions 4.3 and 4.4, it is verified that there exists an ε-quasi saddle point of vector Lagrangian without the compactness assumption. Also, from Proposition 4.2, the existence of certain ε-solution of (P) is verified.

5. References.

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