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A secretary problem with uncertain employment and restricted offering chances

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Abstract

A version of the secretary problem with no recall, in which an offer of acceptance is refused by the applicant with a fixed known probability $1-p$, $(0 \le p \le 1)$ and the offering chances, until the decision maker gets one applicant, are at most M times is treated. This problem is an extension of Smith (1975). The optimal strategy of this problem is obtained in Section 2. In Section 3, as example,the optimal offering strategies and the maximum probabilities of selecting the best applicant are given for the problems with $M = 1, 2, 3$ respectively.

DYNAMIC PROGRAMMING;OPTIMAL STOPPING;SECRETARY PROBLEM;UNCERTAIN EMPLOYMENT

1. Introduction

We consider a variation of the sequential observation and selection problem, often referred to as the secretary problem and studied extensively by Gilbert and Mosteller (1966). The basic framework of the classical secretary problem can be described as follows. N applicants appear one by one in random order with all $N!$ orderings being equally likely. We are able, at any time, to rank the applicants that have so far appeared according to some order of preference. As each applicant appears, we must decide whether or not to make an offer to that applicant with the objective of maximizing the probability of choosing the best applicant. It is assumed that each applicant accepts an offer of employment with certainty and that an applicant to whom an offer is not made cannot be recalled later.

There are many interesting modifications of this problem, for an excellent review of the published work to date, see Freeman(1983) or Ferguson(1989). Smith(1975) is the first to consider the problem with uncertain employment where each applicant has the right to decline an offer of employment with a known fixed probability, $1-p(=q, 0 \le p \le 1)$, independent of his/her rank and the arrangement of the other applicants. In Smith's problem, we can make as many offers as we wish. The problem we consider here puts restriction on the number of offers and allows us to make offer at most M times, where $M(\le N)$ is the predetermined number. We call our problem m-problem if we are allowed to give m more offers in the future. As easily seen, to solve the M-problem completely, we must also solve the $(M-1)$-,$(M-2)$-,$\cdots$,1−problems. The event that we can employ the overall best is called *success* and our objective is to find a strategy of maximizing the probability of success. Another modification of Smith's problem was considered in Tamaki(1991). We derive the optimal strategy of the problem in Section 2 and investigate in detail,the $1, 2, 3-$problems in Section 3.

2. The optimal strategy of the problem

Let X_j denote the relative rank of the jth applicant among the first j applicants (rank 1 being relative best). Then,since the applicants appear in random order, it is easy to see that

(i) the X_j are independent random variables, and

(ii) $P(X_j = i) = 1/j$ for $i = 1, 2, \cdots, j$, for $j = 1, 2, \cdots, N$.

The n-th applicant is sometimes called a *candidate* if he/she is relative best, that is, $X_n = 1$.

Define the state of the process as (n, m), $1 \le n \le N, 0 \le m \le M$, when we confront the m-problem and observe that the nth applicant is a candidate. In state (n, m), we must decide either to give an offer or not to the current candidate. Let $w_n^{(m)}$ be the probability of success starting from state (n, m). Also

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let $u_n^{(m)}(v_n^{(m)})$ be the corresponding probability when we make an offer (when we decline to make an offer) to the current candidate in state (n,m) and proceed optimally thereafter. Then from the principle of optimality

$$w_n^{(m)} = max\{u_n^{(m)}, v_n^{(m)}\}, \quad m \geq 1, 1 \leq n \leq N, \tag{1}$$

where

$$u_n^{(m)} = p\frac{n}{N} + qv_n^{(m-1)}, \tag{2}$$

$$v_n^{(m)} = \frac{1}{n+1}w_{n+1}^{(m)} + (1 - \frac{1}{n+1})v_{n+1}^{(m)}. \tag{3}$$

The boundary conditions are $v_n^{(0)} \equiv 0, v_N^{(m)} \equiv 0, w_N^{(m)} = u_N^{(m)} \equiv p$ for $m \geq 1$. Eq.(2) follows since the availability of the applicant can be ascertained by giving an offer and the m-problem enters into the $(m-1)$-problem once an offer is declined. Eq.(1),(2) and (3) can be solved recursively to yield the optimal strategy and the probability of success $v^{(m)} \equiv w_1^{(m)}$. The following lemma gives the monotonicity property of $v_n^{(m)}$.

Lemma 1. **(i)** $v_n^{(m)}$ *is non-increasing in n.*
(ii) $v_n^{(m)}$ *is non-decreasing in m.*

Proof. (i) is evident from (3). (ii) shall be shown by backward induction on n. The assertion holds for $n = N$ and all m because $v_N^{(m+1)} - v_N^{(m)} = 0$ by definition. Assume now that the assertion holds for $n = k(\leq N)$ and for all m. We have from (3)

$$\begin{aligned} v_{k-1}^{(m+1)} - v_{k-1}^{(m)} &= \frac{1}{k}[\max\{p\frac{k}{N} + qv_k^{(m)}, v_k^{(m+1)}\} \\ &\quad - \max\{p\frac{k}{N} + qv_k^{(m-1)}, v_k^{(m)}\}] \\ &\quad +(1 - \frac{1}{k})(v_k^{(m+1)} - v_k^{(m)}). \end{aligned}$$

Applying the fact that $\max(a_1, b_1) - \max(a_2, b_2) \geq \min(a_1 - a_2, b_1 - b_2)$ to the right-hand side, we obtain

$$\begin{aligned} v_{k-1}^{(m+1)} - v_{k-1}^{(m)} &\geq \frac{1}{k}\min\{q(v_k^{(m)} - v_k^{(m-1)}), v_k^{(m+1)} - v_k^{(m)}\} \\ &\quad +(1 - \frac{1}{k})(v_k^{(m+1)} - v_k^{(m)}). \end{aligned}$$

The right-hand side in the above inequality is nonnegative from the induction hypothesis and the proof is complete.

Repeated use of (3) yields

$$v_n^{(m)} = \sum_{j=n+1}^{N} \frac{n}{j(j-1)}w_j^{(m)}. \tag{4}$$

Throughout this paper , the vacuous sum is assumed to be zero. Now let

$$\tilde{v}_n^{(m)} = \sum_{j=n+1}^{N} \frac{n}{j(j-1)}u_j^{(m)}. \tag{5}$$

Then $\tilde{v}_n^{(m)}$ represents the probability of success attainable by giving an offer to the first candidate that appears after leaving state (n,m) and preceding optimally thereafter. Let B_m be the one-stage look-ahead (OLA) stopping region for the m-problem, that is, B_m is the set of state (n,m) for which giving an offer immediately is at least as good as waiting for the first candidate to appear to whom an offer is given. Thus

$$B_m = \{(n,m) : u_n^{(m)} \geq \tilde{v}_n^{(m)}\}.$$

From (2) and (5) we have

$$u_n^{(m)} - \tilde{v}_n^{(m)} = p\frac{n}{N}(1 - \psi_n) + q\{v_n^{(m-1)} - \sum_{j=n+1}^{N} \frac{n}{j(j-1)}v_j^{(m-1)}\},$$

where $\psi_n \equiv \sum_{j=n+1}^{N} \frac{1}{j-1}$. Define $A_n^{(m)}$ as $(u_n^{(m)} - \tilde{v}_n^{(m)})/n$, that is,

$$A_n^{(m)} = \frac{p}{N}(1-\psi_n) + \frac{q}{n}\{v_n^{(m-1)} - \sum_{j=n+1}^{N} \frac{n}{j(j-1)} v_j^{(m-1)}\}, \quad m \geq 1, 1 \leq n \leq N. \tag{6}$$

Applying (4) to $v_n^{(m-1)}$ of the above equation, we have

$$A_n^{(m)} = \frac{p}{N}(1-\psi_n) + q \sum_{j=n+1}^{N} \frac{1}{j(j-1)}\{w_j^{(m-1)} - v_j^{(m-1)}\}. \tag{7}$$

Then B_m can be written as

$$B_m = \{(n,m) : A_n^{(m)} \geq 0\}.$$

It is well known that if B_m is closed, i.e., $B_m = \{(n,m) : n \geq s_m^*\}$ for some specified value s_m^*, then the optimal strategy in state (n,m) is to give an offer as soon as state enters B_m (see,e.g., Ross or Chow et al). The following theorem is the main result of this paper.

Theorem 1. *Let s_m^* be specified as $s_m^* = min\{n : A_n^{(m)} \geq 0\}$. Then B_m is closed and gives an optimal offering region for the m-problem. Moreover s_m^* is non-increasing in m.*

Proof. It suffices to show that for $k \geq 1$, **(H-1)** $A_n^{(k)}$ is non-decreasing in n and **(H-2)** $A_n^{(k+1)} \geq A_n^{(k)}$ for $n = 1, \cdots, N$. We show these by induction on k. The assertion for $k = 1$ is immediate since we have from (7)

$$A_n^{(1)} = \frac{p}{N}(1-\psi_n),$$

and

$$A_n^{(2)} - A_n^{(1)} = \frac{q}{n}\sum_{j=n+1}^{N} \frac{n}{j(j-1)}\{w_j^{(1)} - v_j^{(1)}\} \geq 0.$$

Assume both (H-1) and (H-2) hold for $k = m-1$, that is, assume $A_n^{(m-1)}$ is non-decreasing in n, $A_n^{(m)} \geq A_n^{(m-1)}$ and define $s_{m-1}^* = \min\{n : A_n^{(m-1)} \geq 0\}$. Then, from the induction hypothesis and (2),

$$w_j^{(m-1)} - v_j^{(m-1)} = \begin{cases} 0 & ,j \leq s_{m-1}^* - 1 \\ u_j^{(m-1)} - v_j^{(m-1)} & ,j \geq s_{m-1}^*. \end{cases}$$

$$= \begin{cases} 0 & ,j \leq s_{m-1}^* - 1 \\ p\frac{j}{N} - \{v_j^{(m-1)} - qv_j^{(m-2)}\} & ,j \geq s_{m-1}^*. \end{cases} \tag{8}$$

Substituting (8) into (7), we obtain

$$A_n^{(m)} = \begin{cases} \frac{p}{N}(1-\psi_n) + q\sum_{j=s_{m-1}^*}^{N} \frac{1}{j(j-1)}\{w_j^{(m-1)} - v_j^{(m-1)}\} & ,n+1 \leq s_{m-1}^* \\ \frac{p}{N}(1-p\psi_n) - q\sum_{j=n+1}^{N} \frac{1}{j(j-1)}\{v_j^{(m-1)} - qv_j^{(m-2)}\} & ,n+1 \geq s_{m-1}^* - 1. \end{cases} \tag{9}$$

When $n+1 \leq s_{m-1}^*$, $A_n^{(m)}$ is clearly non-decreasing in n. When $n+1 \geq s_{m-1}^* - 1$, $A_n^{(m)}$ is also non-decreasing in n from Lemma 1 (ii). Thus (H-1) for $k = m$ is established.

From (7) we have

$$A_n^{(m+1)} - A_n^{(m)} = q\sum_{j=n+1}^{N} \frac{1}{j(j-1)}(\{w_j^{(m)} - v_j^{(m)}\} - \{w_j^{(m-1)} - v_j^{(m-1)}\}). \tag{10}$$

As $A_n^{(m)}$ is non-decreasing in n, we can define s_m^* as $s_m^* = min\{n : A_n^{(m)} \geq 0\}$ such that

$$w_j^{(m)} - v_j^{(m)} = \begin{cases} 0 & ,j \leq s_m^* - 1 \\ u_j^{(m)} - v_j^{(m)} & ,j \geq s_m^*. \end{cases}$$

Considering that $v_j^{(m)} = \tilde{v}_j^{(m)}$ for $j \geq s_m^*$, we have for $j \geq s_m^*$

$$\begin{aligned} u_j^{(m)} - v_j^{(m)} &= u_j^{(m)} - \sum_{i=j+1}^{N} \frac{j}{i(i-1)} u_i^{(m)} \\ &= \frac{jp}{N}(1-\psi_j) + q(v_j^{(m-1)} - \sum_{i=j+1}^{N} \frac{j}{i(i-1)} v_i^{(m-1)}) \\ &= jA_j^{(m)}. \end{aligned}$$

Hence

$$w_j^{(m)} - v_j^{(m)} = \begin{cases} 0 & ,j \leq s_m^* - 1 \\ jA_j^{(m)} & ,j \geq s_m^*, \end{cases} \tag{11}$$

and similarly we have

$$w_j^{(m-1)} - v_j^{(m-1)} = \begin{cases} 0 & ,j \leq s_{m-1}^* - 1 \\ jA_j^{(m-1)} & ,j \geq s_{m-1}^*, \end{cases} \tag{12}$$

Since $s_m^* \leq s_{m-1}^*$, (11) and (12) yield

$$\begin{aligned} \{w_j^{(m)} - v_j^{(m)}\} - \{w_j^{(m-1)} - v_j^{(m-1)}\} &= \begin{cases} 0 & ,j \leq s_m^* - 1 \\ jA_j^{(m)} & ,s_m^* \leq j \leq s_{m-1}^* - 1 \\ j(A_j^{(m)} - A_j^{(m-1)}) & ,j \geq s_{m-1}^*, \end{cases} \\ &\geq 0, \end{aligned} \tag{13}$$

where the last inequality follows from the definition of s_m^* and the induction hypothesis. Applying (13) to (10) immediately yields

$$A_n^{(m+1)} - A_n^{(m)} \geq 0, \quad 1 \leq n \leq N,$$

which proves (H-2) for $k = m$.

From Theorem 1, the optimal strategy of the $M-$problem can be summarized as follows: We pass over the first $s_M^* - 1$ applicants and give an offer to the first candidate that appears thereafter. If the $M - m$ offers are all declined, the next offer is only given the candidate that appears on or after s_m^*, $m = 1, 2, \cdots, M-1$.

Tables 1,2,3 and 4 give the values of s_1^*, s_2^*, s_3^* and the maximum probabilities of success for various values of p and N, the values for $p = 1.0$ being taken from Table 2 in Gilbert and Mosteller (1966). Moreover Tables 5 and 6 give the values of $s_4^*, s_5^*, s_6^*, s_7^*$ and the probabilities of success.

Let $\tilde{s}_m^*$ be the unique root between 0 and 1 of the equation $\lim_{N,n\to\infty} A_{\frac{n}{N}}^{(m)} = 0$ for $m \geq 1$ and writing $v_n^{(m)}$ as $v^{(m)}(\frac{n}{N})$ when $N, n \to \infty$, with $\frac{n}{N} \to x$, then substituting (4) into (9) we have from $s_m^* \leq s_{m-1}^*$

$$\lim_{N,n\to\infty} A_{\frac{n}{N}}^{(m)} = p(1 + logx) + q(\frac{v^{(m-1)}(\tilde{s}_{m-1}^*)}{\tilde{s}_{m-1}^*} - \int_{\tilde{s}_{m-1}^*}^{1} \frac{v^{(m-1)}(y)}{y^2} dy).$$

Thus $\tilde{s}_m^*$ and $v^{(m)}(x)$ are easily found to be the forms in the following corollary.

Corollary 1. *For $m \geq 1$,*

$$\tilde{s}_m^* = exp\{-(1 + \frac{q}{p}\{\frac{v^{(m-1)}(\tilde{s}_{m-1}^*)}{\tilde{s}_{m-1}^*} - \int_{\tilde{s}_{m-1}^*}^{1} \frac{v^{(m-1)}(y)}{y^2} dy\})\},$$

$$v^{(m)}(x) = \begin{cases} v^{(m)}(\tilde{s}_m^*) & ,0 < x \leq \tilde{s}_m^* \\ -pxlogx + q\int_x^1 \frac{x}{y^2} v^{(m-1)}(y)dy & ,\tilde{s}_m^* \leq x < 1. \end{cases}$$

and the limiting value of the maximum probability of success is given by $v^{(m)}(0+) = v^{(m)}(\tilde{s}_m^*)$.

The asymptotic examples for $m = 1, 2, 3$ are as follows:

$$\tilde{s}_1^* = e^{-1}, \tilde{s}_2^* = e^{-(1+\frac{q}{2})}, \tilde{s}_3^* = e^{-(1+\frac{1}{2}q+\frac{1}{3}q^2+\frac{1}{8}q^3)}.$$

$$v^{(1)}(0+) = p\tilde{s}_1^*, v^{(2)}(0+) = p\tilde{s}_2^* + pq\tilde{s}_1^*, v^{(3)}(0+) = p\tilde{s}_3^* + pq\tilde{s}_2^* + pq^2\tilde{s}_1^*.$$

$$v^{(1)}(x) = \begin{cases} p\tilde{s}_1^* & ,0 < x \leq \tilde{s}_1^* \\ -pxlogx & ,\tilde{s}_1^* \leq x < 1. \end{cases}$$

$$v^{(2)}(x) = \begin{cases} p\tilde{s}_2^* + pq\tilde{s}_1^* & ,0 < x \leq \tilde{s}_2^* \\ -pxlogx - \frac{1}{2}pqx + pq\tilde{s}_1^* & ,\tilde{s}_2^* \leq x \leq \tilde{s}_1^* \\ -pxlogx + \frac{1}{2}pqxlog^2x & ,\tilde{s}_1^* \leq x < 1. \end{cases}$$

$$v^{(3)}(x) = \begin{cases} p\tilde{s}_3^* + pq\tilde{s}_2^* + pq^2\tilde{s}_3^* & ,0 < x \leq \tilde{s}_3^* \\ -pxlogx - (\frac{1}{2} + \frac{1}{3}q + \frac{1}{8}q^2)pqx + pq\tilde{s}_2^* + pq^2\tilde{s}_1^* & ,\tilde{s}_3^* \leq x \leq \tilde{s}_2^* \\ -pxlogx + \frac{1}{2}pqxlogx(q + logx) - \frac{1}{3}pq^2x + pq^2\tilde{s}_1^* & ,\tilde{s}_2^* \leq x \leq \tilde{s}_1^* \\ -pxlogx + \frac{1}{2}pqxlog^2x - \frac{1}{6}pq^2xlog^3x & ,\tilde{s}_1^* \leq x < 1. \end{cases}$$

From the above asymptotic values we conjecture $v^{(m)}(0) = p\tilde{s}_m^* + pq\tilde{s}_{m-1}^* + pq^2\tilde{s}_{m-2}^* + \cdots + pq^{m-1}\tilde{s}_1^*$ and $\tilde{s}_m^* \to p^{\frac{1}{q}} = exp\{-(\frac{-log(1-q)}{2})\} = exp\{-(1 + \frac{q}{2} + \frac{q^2}{3} + \cdots)\}$,as $m \to \infty$.

3. Concluding remarks

1. The proof of Theorem 1 is made by induction but it is complicated. If it is shown that $v_n^{(m)} - v_n^{(m-1)}$ is nonincreasing in n, we can more easily show, from the optimality equation (3), that the cutoff points s_m^* are unique for each m and B_m is closed.

2. It would be reasonable to allow the probability of refusing, $1 - p$, to be a decreasing function of the absolute rank of the applicant and to depend on the number of offer, and also reasonable to allow the additional offering cost. Uncertainty of employment and restricted offering chances could be extended to the class of problems considered by Gusein-Zade (1966).

3. we can consider the problem in which there is not uncertainty of employment and we can select at most m applicants, i,e,. multiple choice is allowed to be made. If we define the state of process as (n, m), when we observe that the nth applicant is a candidate and we can select more m applicants in future. Let $w_n^{(m)}, u_n^{(m)}$ and $v\ n^{(m)}$ be the same definition of $m-$problem, although the definitions of the state are different. Then the dynamic programming equation of this multiple choice problem can be derived by putting *conveniently* $p = q = 1$ in the dynamic programming equations (1)-(3) of the $m-$problem. Thus we can apply the same analysis of the $m-$problem to the multiple choice problem, noting that the relation $p + q = 1$ can not be apply. Hence we can easily derive the optimal strategy of the multiple choice problem.

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Table 1: $p = 0.3$

N	s_1^*	s_2^*	s_3^*	$v^{(1)}$	$v^{(2)}$	$v^{(3)}$
2	1	1		0.1500000	0.2550000	
3	2	1	1	0.1500000	0.2050000	0.2295000
4	2	2	1	0.1375000	0.1900000	0.2080000
5	3	2	2	0.1300000	0.1862500	0.1985000
10	4	2	2	0.1196071	0.1686506	0.1860433
25	10	6	5	0.1142749	0.1604136	0.1774775
50	19	13	10	0.1122825	0.1576602	0.1745811
100	38	26	21	0.1113128	0.1563364	0.1731780
1000	369	259	211	0.1104587	0.1551576	0.1719363
∞	0.36788N	0.25924N	0.21093N	0.1103638	0.1550268	0.1717940

Table 2: $p = 0.5$

N	s_1^*	s_2^*	s_3^*	$v^{(1)}$	$v^{(2)}$	$v^{(3)}$
2	1	1		0.2500000	0.3750000	
3	2	1	1	0.2500000	0.2916667	0.3125000
4	2	2	1	0.2291667	0.2916667	0.2968750
5	3	2	1	0.2166667	0.2812500	0.2916667
10	4	3	3	0.1993452	0.2547098	0.2678205
25	10	7	7	0.1904582	0.2432968	0.2553796
50	19	14	13	0.1871375	0.2392014	0.2512869
100	38	28	26	0.1855214	0.2371828	0.2493212
1000	369	286	259	0.1840978	0.2354182	0.2475618
∞	0.36788N	0.28650N	0.25951N	0.1839397	0.2352223	0.2473664

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Table 3: $p = 0.9$

N	s_1^*	s_2^*	s_3^*	$v^{(1)}$	$v^{(2)}$	$v^{(3)}$
2	1	1		0.4500000	0.4950000	
3	2	1	1	0.4500000	0.4650000	0.4650000
4	2	2	1	0.4125000	0.4350000	0.4353750
5	3	3	3	0.3900000	0.4035000	0.4036500
10	4	4	4	0.3588214	0.3787527	0.3793534
25	10	10	10	0.3428247	0.3597193	0.3602341
50	19	18	10	0.3368475	0.3539776	0.3545874
100	38	35	35	0.3339385	0.3510086	0.3516011
1000	369	350	349	0.3313761	0.3489551	0.3489711
∞	0.36788N	0.34994N	0.34873N	0.3310915	0.3480531	0.3486620

Table 4: $p = 1.0$

N	s_1^*	s_2^*	s_3^*	$v^{(1)}$	$v^{(2)}$	$v^{(3)}$
2	1	1		0.5000000	0.5000000	
3	2	2	2	0.5000000	0.5000000	0.5000000
4	2	2	2	0.4583333	0.4583333	0.4583333
5	3	3	3	0.4333333	0.4333333	0.4333333
10	4	4	4	0.3986905	0.3986905	0.3986905
25	10	10	10	0.3809164	0.3809164	0.3809164
50	19	19	19	0.3742750	0.3742750	0.3742750
100	38	38	38	0.3710428	0.3710428	0.3710428
1000	369	369	369	0.3681956	0.3681956	0.3681956
∞	$e^{-1}N$	$e^{-1}N$	$e^{-1}N$	e^{-1}	e^{-1}	e^{-1}

Table 5: $p = 0.3$

N	s_4^*	s_5^*	s_6^*	s_7^*	$v^{(4)}$	$v^{(5)}$	$v^{(6)}$	$v^{(7)}$
5	2	2			0.1993575	0.1995502		
10	2	2	2	2	0.1896865	0.1909424	0.1910737	0.1910822
25	5	5	5	5	0.1825669	0.1838990	0.1841119	0.1841387
50	10	10	10	10	0.1799344	0.1812443	0.1814746	0.1815079
100	20	19	19	19	0.1785684	0.1799735	0.1802456	0.1802887
1000	191	183	180	180	0.1773773	0.1788235	0.1791325	0.1791869

Table 6: $p = 0.9$

N	s_4^*	s_5^*	s_6^*	s_7^*	$v^{(4)}$	$v^{(5)}$	$v^{(6)}$	$v^{(7)}$
5	3	3			0.4036500	0.4036500		
10	4	4	4	4	0.3793640	0.3793641	0.3793641	0.3793641
25	10	10	10	10	0.3602450	0.3602451	0.3602451	0.3602451
50	18	18	18	18	0.3546025	0.3546027	0.3546028	0.3546028
100	36	36	36	36	0.3516160	0.3516163	0.3516163	0.3516163
1000	349	349	349	349	0.3489711	0.3489715	0.3489715	0.3489715