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THE MOD 2 COHOMOLOGY ALGEBRAS OF FINITE GROUPS WITH DIHEDRAL SYLOW 2–SUBGROUPS

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Abstract. The mod 2 cohomolgy algebras of finite groups with dihedral Sylow 2-subgroups are completely determined, by using the theory of relatively projective covers.

1. INTRODUCTION

Let G be a finite group with a dihedral Sylow 2-subgroup

$$D = \langle\, x, y \mid x^2 = 1, y^2 = 1, (xy)^{2^{n-1}} = 1 \,\rangle$$

and let

$$w = (xy)^{2^{n-2}}$$

be the central involution in D. Structures of such finite groups had been deeply investigated by D. Gorenstein, J. H. Walter and R. Brauer. But our starting point is the following simple fact.

Fact 1.1. *One of the following holds:*

(1) *both* $\langle x\rangle$ *and* $\langle y\rangle$ *are conjugate to* $\langle w\rangle$ *in* G*;*
(2) *one and only one of* $\langle x\rangle$ *and* $\langle y\rangle$ *is conjugate to* $\langle w\rangle$ *in* G*;*
(3) *neither of* $\langle x\rangle$ *and* $\langle y\rangle$ *are conjugate to* $\langle w\rangle$ *in* G*.*

We are concerned with the cohomology algebra

$$H^*(G, \boldsymbol{F}_2) = \bigoplus_{n=0}^{\infty} H^n(G, \boldsymbol{F}_2)$$

with coefficients in the field $\boldsymbol{F}_2$ of two elements.

Cohomology algebras of such groups have been determined individually. For instance

Example 1.2.

Type (1) $G = SL(3,2)$ [Benson-Carlson 4]

$$H^*(G, \boldsymbol{F}_2) \simeq \boldsymbol{F}_2[\varepsilon, \theta_1, \theta_2]/(\theta_1\theta_2)$$

where $\deg\varepsilon = 2$ *and* $\deg\theta_1 = \deg\theta_2 = 3$.

Type (2) $G = S_4$

$$H^*(G, \boldsymbol{F}_2) \simeq \boldsymbol{F}_2[\chi, \varepsilon, \theta]/(\chi\theta)$$

where $\deg\chi = 1$, $\deg\varepsilon = 2$, *and* $\deg\theta = 3$.

Type (3) $G = D_{2n}$ *dihedral group of order* $2n$.

$$H^*(G, \boldsymbol{F}_2) \simeq H^*(D, \boldsymbol{F}_2) \simeq \boldsymbol{F}_2[\xi, \eta, \alpha]/(\xi\eta)$$

where $\deg\xi = \deg\eta = 1$ *and* $\deg\alpha = 2$.

The purpose of this report is to show the preceding results hold not only for these groups but also for all groups of each type:

Main Theorem. *Let G be a finite group with a dihedral Sylow 2-subgroup D.*

(1) *If the group G is of type (1), then*

$$H^*(G,\boldsymbol{F}_2) \simeq \boldsymbol{F}_2[\varepsilon,\theta_1,\theta_2]/(\theta_1\theta_2)$$

where $\deg\varepsilon = 2$ *and* $\deg\theta_1 = \deg\theta_2 = 3$.

(2) *If the group G is of type (2), then*

$$H^*(G,\boldsymbol{F}_2) \simeq \boldsymbol{F}_2[\chi,\varepsilon,\theta]/(\chi\theta)$$

where $\deg\chi = 1$, $\deg\varepsilon = 2$, *and* $\deg\theta = 3$.

(3) *If the group G is of type (3), then*

$$H^*(G,\boldsymbol{F}_2) \simeq \boldsymbol{F}_2[\chi,\psi,\varepsilon]/(\chi\psi)$$

where $\deg\chi = \deg\psi = 1$ *and* $\deg\varepsilon = 2$.

One of our main tools is the theory of relatively injective hulls. The notion of relatively projective covers of modules was introduced by R. Knörr in [8]. Considering duals we can get the notion of relatively injective hulls. In his paper [1] T. Asai gave a dimension formula for the homogeneous submodules of the cohomology algebra $H^*(G,\boldsymbol{F}_2)$ by utilizing relatively injective hulls of trivial modules. Adding further consideration on generators and relations, we shall determine the structure of the cohomology algebras. It has been known that in the case (3) the group G has a normal 2-complement, which was proved by using the theory of fusions. We shall also give another proof to this fact by our method.

Recently the 2-localization of the classification spaces of finite groups with dihedral, generalized quaternion, or semidihedral Sylow 2-subgroups have been given by J. Martino and S. Priddy in [9]. As a consequence the mod 2 cohomology algebras of such finite groups are determined. On the other hand our methods are entirely algebraic and completely different from theirs.

2. Notation and preliminaries

In this section let G be an arbitrary finite group and let F be a field of characteristic p dividing the order of G. By an FG-module we shall always mean a finitely generated right FG-module.

For $H \leq G$, M an FG-module and $\alpha \in H^n(G, M)$ we denote by α_H the restriction $\mathrm{Res}_H^G(\alpha)$ of α to H.

The nth cohomology group $H^n(G, F)$ is isomorphic with the vector space $\mathrm{Hom}_{FG}(\Omega^n(F), F)$. For an element $\alpha \in H^n(G, F)$ we denote by $\widehat{\alpha}$ the FG-homomorphism of $\Omega^n(F)$ to F which corresponds to α. Also we denote by L_α the kernel of $\widehat{\alpha} : \Omega^n(F) \longrightarrow F$.

Our aim is to determine generators of $H^*(G, F)$ and relations. In general if the group G has p-rank n, then there exist n homogeneous elements $\zeta_1, \ldots, \zeta_n$ for which the cohomology algebra $H^*(G, F)$ is finitely generated over the subalgebra $F[\zeta_1, \ldots, \zeta_n]$. This condition is equivalent to

$$\boxed{L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_n} \text{ is projective}}$$

When p-rank $= 2$, we can say about bases over $F[\zeta_1, \zeta_2]$:

Lemma 2.1. [Okuyama-Sasaki 11] *For $\zeta \in H^r(G, F)$ and $\eta \in H^s(G, F)$, if the tensor product $L_\zeta \otimes L_\eta$ is a projective module, then it holds that* *for* $n \geq r+s-1$

$$H^n(G,F) = H^{n-r}(G,F)\zeta + H^{n-s}(G,F)\eta.$$

Namely

$$\boxed{H^*(G,F) = [\ \bigoplus_{n=1}^{r+s-2} H^n(G,F)\ ]F[\zeta,\eta]}$$

Also useful is to determine the dimensions of homogeneous submodules. The following lemma will be applied to our situation.

Lemma 2.2. [Asai 1] *Let*

$$0 \longrightarrow M \xrightarrow{f} U \longrightarrow N \longrightarrow 0$$

be a short exact sequence of FG*-modules. Suppose that the modules* M *and* N *are projective free and that the module* U *is periodic so that* $U \otimes L_\gamma$ *is projective for an element* $\gamma \in H^r(G,F)$*. For* S *a simple* FG*-module, if*

$$[\ f_n^* : \mathrm{Ext}_{FG}^n(U,S) \longrightarrow \mathrm{Ext}_{FG}^n(M,S)\ ] = 0 \ \ \textit{for}\ \ 0 \leq n \leq r-1$$

then

$$f_n^* = 0 \ \ \textit{for all}\ \ n \geq 0.$$

If this happens, the long exact Ext-sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{FG}(N,S) \to \mathrm{Hom}_{FG}(U,S) \to \mathrm{Hom}_{FG}(M,S) \to \\ \mathrm{Ext}_{FG}^1(N,S) \to \cdots \to \mathrm{Ext}_{FG}^n(N,S) \to \mathrm{Ext}_{FG}^n(U,S) \xrightarrow{f_n^*} \\ \mathrm{Ext}_{FG}^n(M,S) \xrightarrow{\Delta} \mathrm{Ext}_{FG}^{n+1}(N,S) \to \mathrm{Ext}_{FG}^{n+1}(U,S) \xrightarrow{f_{n+1}^*} \\ \mathrm{Ext}_{FG}^{n+1}(M,S) \to \cdots \end{aligned}$$

breaks into short exact sequences

$$0 \longrightarrow \operatorname{Ext}^n_{FG}(M,S) \xrightarrow{\Delta} \operatorname{Ext}^{n+1}_{FG}(N,S) \longrightarrow \operatorname{Ext}^{n+1}_{FG}(U,S) \longrightarrow 0$$

$n = 0, 1, 2, \dots$.

Especially

Corollary 2.3. *Under the same assumption of Lemma 2.2, for all $n \geq 0$*

$$\boxed{\dim \operatorname{Ext}^{n+1}_{FG}(N,S) = \dim \operatorname{Ext}^n_{FG}(M,S) + \dim \operatorname{Ext}^{n+1}_{FG}(U,S)}$$

3. Cohomolgy algebra of dihedral 2-group

Henceforth we let

$$\boldsymbol{F} = \boldsymbol{F}_2.$$

Before discussing general cases we have to consider the cohomology algebra of the dihedral 2-group D. Let

$$\xi \text{ and } \eta \in H^1(D, \boldsymbol{F})$$

be the elements which satisfy

$$\begin{cases} \xi(x) = 1 \\ \xi(y) = 0 \end{cases} \qquad \begin{cases} \eta(x) = 0 \\ \eta(y) = 1 \end{cases}$$

regarding $H^1(D, \boldsymbol{F})$ as $\operatorname{Hom}(D, \boldsymbol{F})$.

By direct calculation we have

Proposition 3.1. $\Omega^{-1}_{\{\langle x\rangle,\langle y\rangle\}}(\boldsymbol{F}_D) = \Omega(\boldsymbol{F}_D)$:

$$\boxed{0 \longrightarrow \boldsymbol{F} \longrightarrow \boldsymbol{F}^D_{\langle x\rangle} \oplus \boldsymbol{F}^D_{\langle y\rangle} \underset{g}{\longrightarrow} \Omega(\boldsymbol{F}) \longrightarrow 0}$$

where

$$g : \begin{cases} (1\otimes 1, 0) & \mapsto x-1 \\ (0, 1\otimes 1) & \mapsto y-1. \end{cases}$$

Let

$$\alpha \in H^2(D, \boldsymbol{F})$$

be the element corresponding to the extension

$$0 \longrightarrow \boldsymbol{F} \underset{f}{\longrightarrow} \boldsymbol{F}^D_{\langle x\rangle} \oplus \boldsymbol{F}^D_{\langle y\rangle} \underset{g}{\longrightarrow} \Omega(\boldsymbol{F}) \longrightarrow 0$$

and let

$$\begin{aligned} z &= xy \\ \zeta &= \xi + \eta. \end{aligned}$$

Proposition 3.2. (1) *The restriction $\alpha_{\langle z\rangle}$ does not vanish. In particular α is not a zero-diviser in $H^*(D, \boldsymbol{F})$.*
(2) *The tensor product $L_\alpha \otimes L_\zeta$ is projective. Hence for $n \geq 2$*

$$\boxed{H^n(D,\boldsymbol{F}) = H^{n-2}(D,\boldsymbol{F})\alpha + H^{n-1}(D,\boldsymbol{F})\zeta}$$

Proof. (1) The restriction of the extension above to the subgroup $\langle z\rangle$ does not split.
(2) This follows from the fact that the restriction of the tensor product to each four subgroup of D is projective. Lemma 2.1 gives the second assertion. □

Let

$$U = \boldsymbol{F}^D_{\langle x\rangle} \oplus \boldsymbol{F}^D_{\langle y\rangle}.$$

Then

$$\begin{aligned} U \otimes L_\zeta &= \left(\boldsymbol{F}^D_{\langle x\rangle} \oplus \boldsymbol{F}^D_{\langle y\rangle}\right) \otimes L_\zeta \\ &= L_\zeta|^D_{\langle x\rangle} \oplus L_\zeta|^D_{\langle y\rangle} \quad : \text{ projective,} \end{aligned}$$

because both $\zeta_{\langle x\rangle}$ and $\zeta_{\langle y\rangle}$ are nonzero elements. Since

$$\mathrm{im} f \subset \mathrm{soc}(U) \subset \mathrm{rad}(U)$$

we see that

$$[\ f_0^* : \mathrm{Hom}_{\boldsymbol{F}D}(U, \boldsymbol{F}) \longrightarrow \mathrm{Hom}_{\boldsymbol{F}D}(\boldsymbol{F}, \boldsymbol{F})\ ] = 0.$$

Hence Corollary 2.3 gives a dimension formula

$$\begin{aligned} \dim \mathrm{Ext}^{n+1}_{\boldsymbol{F}D}(\Omega(\boldsymbol{F}), \boldsymbol{F}) &= \dim \mathrm{Ext}^n_{\boldsymbol{F}D}(\boldsymbol{F}, \boldsymbol{F}) + \dim \mathrm{Ext}^{n+1}_{\boldsymbol{F}D}(U, \boldsymbol{F}) \\ &= \dim \mathrm{Ext}^n_{\boldsymbol{F}D}(\boldsymbol{F}, \boldsymbol{F}) + 2. \end{aligned}$$

Namely

$$\dim H^{n+2}(D, \boldsymbol{F}) = \dim H^n(D, \boldsymbol{F}) + 2.$$

This together with the facts that $\dim H^0(D, \boldsymbol{F}) = 1$ and $\dim H^1(D, \boldsymbol{F}) = 2$ yields the following:

Proposition 3.3.

$$\boxed{\dim H^n(D, \boldsymbol{F}) = n + 1}$$

Summarizing we have obtained that

$$H^*(D, \boldsymbol{F}) = H^1(D, \boldsymbol{F})\boldsymbol{F}[\zeta, \alpha]$$

and

$$\dim H^n(D, \boldsymbol{F}) = n + 1.$$

We must determine the defining relations. Useful is:

Lemma 3.4. *For* $\omega \in H^n(D, \boldsymbol{F}),\ n \geq 2$

$$\boxed{\omega_{\langle x\rangle} = 0 \text{ and } \omega_{\langle y\rangle} = 0 \Longrightarrow \alpha|\omega}$$

Proof. Since $\widehat{\omega}_{\langle x\rangle}$ and $\widehat{\omega}_{\langle y\rangle}$ are projective maps and

$$0 \longrightarrow \boldsymbol{F} \longrightarrow \boldsymbol{F}^D_{\langle x\rangle} \oplus \boldsymbol{F}^D_{\langle y\rangle} \longrightarrow \Omega(\boldsymbol{F}) \longrightarrow 0$$

is a $\{\langle x\rangle, \langle y\rangle\}$-injective hull, the homomorphism $\widehat{\omega}$ can be extended to a homomorphism ϕ of P_n, the injective hull of $\Omega^n(\boldsymbol{F})$, to $\boldsymbol{F}^D_{\langle x\rangle} \oplus \boldsymbol{F}^D_{\langle y\rangle}$. Let $\widehat{\tau}$ be the homomorphism of $\Omega^{n-1}(\boldsymbol{F})$ to $\Omega(\boldsymbol{F})$ which is induced from ϕ. Then, letting τ denote the element in $H^{n-2}(D, \boldsymbol{F})$ represented by $\widehat{\tau}$, we see that $\omega = \alpha\tau$.

$$\begin{tikzcd}
\Omega^n(\boldsymbol{F}) \arrow[r] \arrow[d, "\widehat{\omega}"'] & P_n \arrow[r] \arrow[d, "\phi"'] & \Omega^{n-1}(\boldsymbol{F}) \arrow[d, "\widehat{\tau}"'] \\
\boldsymbol{F} \arrow[r] & \boldsymbol{F}^D_{\langle x\rangle} \oplus \boldsymbol{F}^D_{\langle y\rangle} \arrow[r] & \Omega(\boldsymbol{F})
\end{tikzcd}$$

□

Lemma 3.5.

$$\boxed{\xi\eta = 0}$$

Proof. This follows from the facts that $(\xi\eta)_{\langle x\rangle} = 0$ and $(\xi\eta)_{\langle y\rangle} = 0$. □

Considering the dimensions of the homogeneous submodules of the subalgebra $\boldsymbol{F}[\xi, \eta, \alpha]$, we have

Theorem 3.6.

$$\boxed{H^*(D, \boldsymbol{F}) \simeq \boldsymbol{F}[\xi, \eta, \alpha]/(\xi\eta)}$$

where $\deg \xi = \deg \eta = 1$ *and* $\deg \alpha = 2$.

4. General cases

First we shall treat the case (3).

Proposition 4.1. *If a finite group G with a dihedral Sylow 2-subgroup D is of type (3), then G has a normal 2-complement. In particular the cohomology algebra $H^*(G, \boldsymbol{F})$ is isomorphic with that of the Sylow 2-subgroup D.*

Proof. A $\langle z\rangle$-injective hull of the trivial $\boldsymbol{F}G$-module $\boldsymbol{F}$ is of the form $0 \to \boldsymbol{F} \to Sc\langle z\rangle \to M \to 0$, where $Sc\langle z\rangle$ is the Scott module with vertex $\langle z\rangle$ and M is an indecomposable $\boldsymbol{F}G$-module with vertex D. We claim that the right-hand module M is isomorphic with $\boldsymbol{F}$. Since the group G is of type (3), we see that $\{\langle z\rangle\} \cap_G D = \{\langle z\rangle\}$. Hence, restricting the extension above to D, we see that the restriction M_D is the direct sum of the trivial module $\boldsymbol{F}_D$ and a $\langle z\rangle$-injective module. Namely the module M has a trivial source. We also note that the head of the module M has the trivial module $\boldsymbol{F}$ as a direct summand. Thus we have that the module M is isomorphic with $\boldsymbol{F}$, as desired. Namely there exists an extension $0 \to \boldsymbol{F} \to Sc\langle z\rangle \to \boldsymbol{F} \to 0$. Such an extension splits over the subgroup $O^2(G)$, because it corresponds to an element in $H^1(G, \boldsymbol{F}) \simeq \mathrm{Hom}(G, \boldsymbol{F})$ and the restriction of $\mathrm{Hom}(G, \boldsymbol{F})$ to $O^2(G)$ is the zero-module. Therefore the subgroup $O^2(G)$ acts trivially on $Sc\langle z\rangle$ so that a Sylow 2-subgroup of $O^2(G)$ is contained in a vertex of $Sc\langle z\rangle$. Consequently the subgroup $O^2(G)$ has a normal 2-complement, which means that $O^2(G)$ is itself a normal 2-complement of the group G. $\square$

Now we proceed to the cases (1) and (2). Similarly to the case of dihedral 2-groups our methods are:

(1) to find homogeneous elements ε and σ for which

the tensor product $L_\varepsilon \otimes L_\sigma$ is projective;

(2) to get the dimension formula

$$\dim H^n(G, \boldsymbol{F}) = ?\ ;$$

(3) to determine the defining relations.

Let

$$\mathcal{H} = \{\ \langle x\rangle, \langle y\rangle, \langle z\rangle\ \}.$$

Useful is the $\mathcal{H}$-injective hull of the trivial module. We begin with

Proposition 4.2. $\Omega_{\mathcal{H}}^{-1}(\boldsymbol{F}_D) = \Omega^2(\boldsymbol{F}_D)$*:*

$$\boxed{0 \longrightarrow \boldsymbol{F} \longrightarrow \boldsymbol{F}_{\langle x\rangle}^D \oplus \boldsymbol{F}_{\langle y\rangle}^D \oplus \boldsymbol{F}_{\langle z\rangle}^D \longrightarrow \Omega^2(\boldsymbol{F}) \longrightarrow 0}$$

Proof. This can be verified by direct computation. $\square$

Let

$$T = \boldsymbol{F}_{\langle x\rangle}^D \oplus \boldsymbol{F}_{\langle y\rangle}^D \oplus \boldsymbol{F}_{\langle z\rangle}^D.$$

For a group G of type (2) we assume that

$$x \underset{G}{\sim} w \quad \text{but} \quad y \underset{G}{\not\sim} w.$$

We let

$$\mathcal{H}' = \begin{cases} \{\ \langle z\rangle\ \} & \text{case (1)} \\ \{\ \langle y\rangle, \langle z\rangle\ \} & \text{case (2).} \end{cases}$$

Then by [Asai 1, Lemma 2.1] an $\mathcal{H}$-injective hull S of the $\boldsymbol{F}G$-module $\boldsymbol{F}_G$ is given by

$$S = \bigoplus_{H \in \mathcal{H}'} ScH$$

where ScH is the Scott module with vertex $H \in \mathcal{H}'$.

Proposition 4.3. $\Omega_{\mathcal{H}}^{-1}(\boldsymbol{F}_G) = \Omega^2(\boldsymbol{F}_G)$:

$$\boxed{0 \longrightarrow \boldsymbol{F} \longrightarrow S \longrightarrow \Omega^2(\boldsymbol{F}) \longrightarrow 0}$$

Proof. Let $0 \to \boldsymbol{F} \to S \to M \to 0$ be an $\mathcal{H}$-injective hull. Because $\mathcal{H} \cap_G D = \mathcal{H}$, the restriction of the extension above to the Sylow 2-subgroup D contains the extension $0 \to \boldsymbol{F}_D \to T \to \Omega^2(\boldsymbol{F}_D) \to 0$ as a direct summand. Hence $\Omega^2(\boldsymbol{F}_D)$ is a direct summand of M_D. Now we note that the centralizer $C_G(w)$ has a normal 2-complement. Using the Green correspondence with respect to $(G, D, C_G(w))$, we have that $M = \Omega^2(\boldsymbol{F})$. □

Let

$$\sigma \in H^3(G, \boldsymbol{F})$$

be the element corresponding to the extension

$$0 \longrightarrow \boldsymbol{F} \xrightarrow[f]{} S \xrightarrow[g]{} \Omega^2(\boldsymbol{F}) \longrightarrow 0.$$

We note that the restriction of the extension above to D is

$$0 \longrightarrow \boldsymbol{F}_D \xrightarrow[f]{} T \oplus X \xrightarrow[g]{} \Omega^2(\boldsymbol{F}_D) \oplus X \longrightarrow 0$$

where X is projective.

Theorem 4.4. *There exists a homogeneous element $\varepsilon \in H^2(G, \boldsymbol{F})$ such that*

$$\varepsilon_H \neq 0 \text{ for each } H \in \mathcal{H}'$$

so that

the tensor product $S \otimes L_\varepsilon$ is projective.

It also holds that

the tensor product $L_\varepsilon \otimes L_\sigma$ *is projective.*

Hence one has for $n \geq 4$

$$H^n(G, \boldsymbol{F}) = H^{n-2}(G, \boldsymbol{F})\varepsilon + H^{n-3}(G, \boldsymbol{F})\sigma.$$

Namely

$$\boxed{H^*(G, \boldsymbol{F}) = \Big[\ \bigoplus_{n=1}^{3} H^n(G, \boldsymbol{F})\ \Big]\ \boldsymbol{F}[\varepsilon, \sigma]}$$

Proof. We can choose a homomorphism $\lambda : S \to \boldsymbol{F}$ such that

$$\lambda_H \text{ is not projective for all } H \in \mathcal{H}'.$$

Since

$$[\ f^* : \mathrm{Hom}_{\boldsymbol{F}G}(S, \boldsymbol{F}) \longrightarrow \mathrm{Hom}_{\boldsymbol{F}G}(\boldsymbol{F}, \boldsymbol{F})\ ] = 0$$

we have that

$$g^* : \mathrm{Hom}_{\boldsymbol{F}G}(\Omega^2(\boldsymbol{F}), \boldsymbol{F}) \simeq \mathrm{Hom}_{\boldsymbol{F}G}(S, \boldsymbol{F}).$$

Let $\widehat{\varepsilon}$ be the element in $\mathrm{Hom}_{\boldsymbol{F}G}(\Omega^2(\boldsymbol{F}), \boldsymbol{F})$ such that $g^*(\widehat{\varepsilon}) = \lambda$. Then it holds that

$$\varepsilon_H \neq 0 \text{ for each } H \in \mathcal{H}'.$$

Because each subgroup $H \in \mathcal{H}'$ is cyclic, the restriction $L_\varepsilon|_H$ is projective. Hence we have that

$$\begin{aligned} S \otimes L_\varepsilon \ &\Big|\ \Big(\bigoplus_{H \in \mathcal{H}'} \boldsymbol{F}_H^G\Big) \otimes L_\varepsilon \\ &= \bigoplus_{H \in \mathcal{H}'} L_\varepsilon|_H^G \ : \text{projective.} \end{aligned}$$

Since $\mathrm{soc}(S) \subset \mathrm{rad}(S)$, there exists an essential epimorphism $\rho : P_2 \to S$ such that $\partial_2 = g\rho$. We obtain a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L_\sigma \arrow[r] \arrow[d] & \Omega(S) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^3(\boldsymbol{F}) \arrow[r] \arrow[d, "\widetilde{\sigma}"'] & P_2 \arrow[r, "\partial_2"'] \arrow[d, "\rho"'] & \Omega^2(\boldsymbol{F}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \boldsymbol{F} \arrow[r, "f"'] \arrow[d] & S \arrow[r, "g"'] \arrow[d] & \Omega^2(\boldsymbol{F}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The second assertion holds from $L_\sigma \simeq \Omega(S)$. $\square$

Next let us determine the dimensions of the homogeneous submodules. We have observed in the proof of Theorem 4.4 that

$$[\ f_0^* : \mathrm{Hom}_{\boldsymbol{F}G}(S, \boldsymbol{F}) \longrightarrow \mathrm{Hom}_{\boldsymbol{F}G}(\boldsymbol{F}, \boldsymbol{F})\ ] = 0.$$

If

$$[\ f_1^* : \mathrm{Ext}^1_{\boldsymbol{F}G}(S, \boldsymbol{F}) \longrightarrow \mathrm{Ext}^1_{\boldsymbol{F}G}(\boldsymbol{F}, \boldsymbol{F})\ ] = 0$$

then we will obtain by Corollary 2.3 that

$$\dim \mathrm{Ext}^{n+1}_{\boldsymbol{F}G}(\Omega^2(\boldsymbol{F}), \boldsymbol{F}) = \dim \mathrm{Ext}^{n}_{\boldsymbol{F}G}(\boldsymbol{F}, \boldsymbol{F}) + \dim \mathrm{Ext}^{n+1}_{\boldsymbol{F}G}(S, \boldsymbol{F}).$$

Now it is sufficient to verify that

$$[\ f^*|_D : \mathrm{Ext}^1_{\boldsymbol{F}D}(S|_D, \boldsymbol{F}) \longrightarrow \mathrm{Ext}^1_{\boldsymbol{F}D}(\boldsymbol{F}, \boldsymbol{F})\ ] = 0.$$

This is equivalent to

$$[\ f_1^* : \mathrm{Ext}^1_{\boldsymbol{F}D}(T, \boldsymbol{F}) \longrightarrow \mathrm{Ext}^1_{\boldsymbol{F}D}(\boldsymbol{F}, \boldsymbol{F})\ ] = 0$$

where f_1^* is induced from the extension

$$0 \longrightarrow \boldsymbol{F} \underset{f}{\longrightarrow} \boldsymbol{F}^D_{\langle x\rangle} \oplus \boldsymbol{F}^D_{\langle y\rangle} \oplus \boldsymbol{F}^D_{\langle z\rangle} \underset{g}{\longrightarrow} \Omega^2(\boldsymbol{F}) \longrightarrow 0.$$

In the Ext-exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\boldsymbol{F}D}(\Omega^2(\boldsymbol{F}), \boldsymbol{F}) &\overset{g_0^*}{\to} \mathrm{Hom}_{\boldsymbol{F}D}(T, \boldsymbol{F}) \overset{f_0^*}{\to} \mathrm{Hom}_{\boldsymbol{F}D}(\boldsymbol{F}, \boldsymbol{F}) \overset{\Delta}{\to} \\ &\mathrm{Ext}^1_{\boldsymbol{F}D}(\Omega^2(\boldsymbol{F}), \boldsymbol{F}) \overset{g_1^*}{\to} \mathrm{Ext}^1_{\boldsymbol{F}D}(T, \boldsymbol{F}) \overset{f_1^*}{\to} \mathrm{Ext}^1_{\boldsymbol{F}D}(\boldsymbol{F}, \boldsymbol{F}) \to \cdots \end{aligned}$$

it can easily be seen that $f_0^* = 0$, $\dim \mathrm{Ext}^1_{\boldsymbol{F}D}(\Omega^2(\boldsymbol{F}), \boldsymbol{F}) = 4$, and $\dim \mathrm{Ext}^1_{\boldsymbol{F}D}(T, \boldsymbol{F}) = 3$. Hence we obtain that

$$\begin{aligned} \mathrm{Ext}^1_{\boldsymbol{F}D}(T, \boldsymbol{F}) &\geq \ \ker f_1^* \\ &\simeq \mathrm{Ext}^1_{\boldsymbol{F}D}(\Omega^2(\boldsymbol{F}), \boldsymbol{F})/\mathrm{im}\ \Delta; \text{of dimension } 3 \end{aligned}$$

so that

$$\mathrm{Ext}^1_{\boldsymbol{F}D}(T, \boldsymbol{F}) = \ker f_1^*.$$

Namely it holds that

$$f_1^* = 0$$

as desired. Therefore, as we have mentioned

Lemma 4.5.

$$\dim H^{n+3}(G, \boldsymbol{F}) = \dim H^n(G, \boldsymbol{F}) + \dim \mathrm{Ext}^{n+1}_{\boldsymbol{F}G}(S, \boldsymbol{F}).$$

We must determine $\dim \mathrm{Ext}^n_{\boldsymbol{F}G}(S, \boldsymbol{F})$ and $\dim H^n(G, \boldsymbol{F})$, $0 \leq n \leq 2$.

Lemma 4.6. *For each $H \in \mathcal{H}$ one has*

$$\dim \operatorname{Ext}^n_{\boldsymbol{F}G}(ScH, \boldsymbol{F}) = 1 \textit{ for } n \geq 0$$

so that for $n \geq 0$

$$\dim \operatorname{Ext}^n_{\boldsymbol{F}G}(S, \boldsymbol{F}) = i \quad \textit{in case (i).}$$

Proof. The modules $Sc\langle x\rangle, Sc\langle y\rangle$, and $Sc\langle z\rangle$ are periodic of periods one, one, and two, respectively. Let $N = N_G(\langle z\rangle)$ and let L be the Green correspondent of $Sc\langle z\rangle$ with respect to $(G, \langle z\rangle, N)$. Then we have that $\operatorname{Ext}^1_{\boldsymbol{F}G}(Sc\langle z\rangle, \boldsymbol{F}) \simeq \operatorname{Ext}^1_{\boldsymbol{F}N}(L, \boldsymbol{F})$. Since the normalizer N has a normal 2-complement, the Green correspondent L is isomorphic with $\boldsymbol{F}^D_{\langle z\rangle}$. Hence it holds that $\operatorname{Ext}^1_{\boldsymbol{F}N}(L, \boldsymbol{F}) \simeq \operatorname{Ext}^1_{\boldsymbol{F}D}(\boldsymbol{F}^D_{\langle z\rangle}, \boldsymbol{F})$. □

Thus we have

Theorem 4.7. [Asai 1]

$$\boxed{\dim H^{n+3}(G, \boldsymbol{F}) = \dim H^n(G, \boldsymbol{F}) + i \quad \textit{in case (i)}}$$

Lemma 4.8.

$$\boxed{\begin{array}{c} \dim H^1(G, \boldsymbol{F}) = i - 1 \quad \textit{in case (i)} \\ \dim H^2(G, \boldsymbol{F}) = i \quad \textit{in case (i)} \end{array}}$$

Proof. Recall that $H^2(G, \boldsymbol{F}) \simeq \operatorname{Hom}_{\boldsymbol{F}G}(S, \boldsymbol{F})$, which implies the second assertion. The homomorphism $g_* : \operatorname{Hom}_{\boldsymbol{F}G}(\boldsymbol{F}, S) \to \operatorname{Hom}_{\boldsymbol{F}G}(\boldsymbol{F}, \Omega^2(\boldsymbol{F}))$ is epimorphic, because its restriction to D is epimorphic. The kernel of g_* is one dimensional. We note that the vector spaces $\operatorname{Hom}_{\boldsymbol{F}G}(\boldsymbol{F}, \Omega^2(\boldsymbol{F}))$

and $\mathrm{Hom}_{\boldsymbol{F}G}(\Omega(\boldsymbol{F}),\boldsymbol{F})$ have the same dimension. Thus we obtain that $\dim H^1(G,\boldsymbol{F}) = \dim \mathrm{Hom}_{\boldsymbol{F}G}(\boldsymbol{F},S) - 1$. □

By Theorem 4.7 and Lemma 4.8 the dimensions of homogeneous submodules are completely determined.

Now we are in a position to proceed to the final stage. We shall determine generators of $H^*(G,\boldsymbol{F})$ and relations in connection with those of $H^*(D,\boldsymbol{F})$.

Lemma 4.9. *One has*

$$\begin{aligned}\varepsilon_D &= \alpha + \zeta^2\\ \sigma_D &= \alpha\zeta.\end{aligned}$$

Proof. First, recall that $\varepsilon_H \neq 0$ for each $H \in \mathcal{H}'$. Then one has that $(\varepsilon_D + \alpha + \zeta^2)_{\langle x\rangle} = 0$ and $(\varepsilon_D + \alpha + \zeta^2)_{\langle y\rangle} = 0$ so that, by Lemma 3.4, $\varepsilon_D + \alpha + \zeta^2 = \alpha$ or 0. Since $\varepsilon_{\langle z\rangle} \neq 0$, we see that $\varepsilon_D + \alpha + \zeta^2 = 0$. Second, recall that $\sigma_H = 0$ for each $H \in \mathcal{H}$ and that $\alpha_{\langle z\rangle} \neq 0$. Then we have by Lemma 3.4 that $\sigma_D = \alpha\zeta$. □

Case (1): x and y are conjugate to w.

Notice by Lemma 4.8 that $H^1(G,\boldsymbol{F}) = 0$ and $H^2(G,\boldsymbol{F}) = \langle\varepsilon\rangle$. Since $\dim H^3(G,\boldsymbol{F}) = 2$, we take another basis element θ:

$$H^3(G,\boldsymbol{F}) = \langle\sigma,\theta\rangle.$$

Because $|z| \geq 4$, we see that

$$\begin{aligned}\theta^2_{\langle z\rangle} &= (\theta_{\langle z\rangle})^2\\ &= 0.\end{aligned}$$

Then by our assumption we have that

$$\theta_{\langle x\rangle} = 0 \text{ and } \theta_{\langle y\rangle} = 0.$$

Hence by Lemma 3.4 there exists an element $\omega \in H^1(D, \boldsymbol{F})$ such that

$$\theta_D = \alpha\omega.$$

Because θ_D is linearly independent to $\sigma_D = \alpha\zeta$, it follows that

$$\theta_D = \alpha\xi \text{ or } \alpha\eta.$$

In either case we obtain that

$$\theta^2 = \sigma\theta.$$

By putting $\theta_1 = \theta$ and $\theta_2 = \sigma + \theta$, this is rewritten as

$$\theta_1\theta_2 = 0.$$

Summing up we see that

$$H^*(G, \boldsymbol{F}) = \boldsymbol{F}[\varepsilon, \theta_1, \theta_2] \quad \text{with} \quad \theta_1\theta_2 = 0.$$

Finally considering the dimensions of homogeneous submodules, we have

$$\boxed{\begin{array}{c} H^*(G, \boldsymbol{F}) \simeq \boldsymbol{F}[\varepsilon, \theta_1, \theta_2]/(\theta_1\theta_2) \\ \text{where } \deg \varepsilon = 2 \text{ and } \deg \theta_1 = \deg \theta_2 = 3 \end{array}}$$

Case (2): $x \underset{G}{\sim} w$ but $y \underset{G}{\not\sim} w$.

Recall that $\dim H^1(G, \boldsymbol{F}) = 1$. We put

$$H^1(G, \boldsymbol{F}) = \langle \chi \rangle.$$

We observe that

$$\chi_D = \eta.$$

Because the subgroup $\langle w \rangle$ is contatined in the commutator subgroup of the group G, the restriction $\chi_{\langle w \rangle}$ vanishes. By our assumption the restriction $\chi_{\langle x \rangle}$ also vanishes, which implies the assertoin above.

Since $\chi_D^2 = \eta^2$ and $\varepsilon_D = \alpha + \zeta^2$ are linearly independent and the second cohomology group $H^2(G, \boldsymbol{F})$ has dimensioin 2, we have that

$$H^2(G, \boldsymbol{F}) = \langle \chi^2, \varepsilon \rangle.$$

Similarly we see that

$$H^3(G, \boldsymbol{F}) = \langle \chi^3, \chi\varepsilon, \sigma \rangle.$$

The following can be verified by applying the restriction to D:

$$\chi^4 + \chi^2\varepsilon + \chi\sigma = 0.$$

By putting $\chi^3 + \chi\varepsilon + \sigma = \theta$, this can be rewritten as

$$\chi\theta = 0.$$

Thus we know that

$$H^*(G, \boldsymbol{F}) = \boldsymbol{F}[\chi, \varepsilon, \theta] \text{ with } \chi\theta = 0.$$

Finally, again considering the dimensions of homogeneous submodules, we see that the above relation is enough:

$$\boxed{\begin{array}{c} H^*(G, \boldsymbol{F}) \simeq \boldsymbol{F}[\chi, \varepsilon, \theta]/(\chi\theta) \\ \text{where } \deg \chi = 1, \ \deg \varepsilon = 2, \text{ and } \deg \theta = 3 \end{array}}$$

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