数理解析研究所講究録
第 800 巻 1992 年 84-93

CASCADE PAROCESS OF VORICES AND TURBULENCE STRUCTURE

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In this paper, modelling of turbulent boundary layer along a flat plate by means of an ensembel of "individual eddies" and transverse vortex filaments and eddy viscosity based on consideration of momentum and energy conservation are studied. In Fig.1, a brief sketch of turbulent boundary layer along a flat plate is shown, in which the Blasius laminar boundary layer starts at the leading edge, the transition region from laminar to turbulent flow and fully developed turbulent boundary layer are followed.

In the transition region, the burst phnomena are important and would produce many three-dimensional vortices. In this process, a part of a transverse vortex filament, which constitutes the Blasius flow, is lifted up, elongated and curled like a hairpin. The further deformed Ω-shaped vortex would be cut to be a vortex ring. This vortex ring is not definitely circular, but deformed, bent and wrinkled. The vortex is called "individual eddy"(Fig.2). The individual eddy is modelled by a circular ring vortex and has a translational velocity V, energy E and momentum P :

$$V=\frac{\Gamma}{4\pi a}(\log\frac{8a}{b}-\frac{1}{4}),\quad E=\frac{1}{2}\rho\Gamma^{2}a(\log\frac{8a}{b}-\frac{7}{4}),\quad P=\pi\rho\Gamma a^{2} \tag{1}$$

where Γ is the circulation, a the ring radius, b the radius of the vortex filament and ρ the density. The axis of the vortex ring is roughly perpendicular to the wall and translational velocity seems to be directed upwards from the consideration of the producing process of the vortex ring. These eddies travel with the mean velocity and go into the fully-developed turbulent region.

The fully-developed region would be described by an ensemble of individual eddies and transverse vortex filaments, the latter of which are of an approximation for the mean flow. Experiments show that the thickness of boundary layer increases roughly linearly in the flow direction in the fully-developed region. In this region, geometrical and dynamical similarities of the vortex motion and the mean flow could be assumed. Extension of a line of the edge of boundary layer may intersect the flat plate. The intersection point is called the origin of similarity, which is chosen as the origin of x-axis. The origin may be located in the Blasius flow region. The similarities really start about at $x=x_0$, which is called starting point of similarity (Fig.1). Thus, the rear part of the Blasius region, the transition region and the initial part of the fully-developed region would be between $x = 0$ and x_0 . In the fully-developed region, the molecular viscosity effect is almost negligible, and the characteristic outer parameters are only ρ and U, the velocity outside the boundary layer. Thus, we can not construct the characteristic lenght from the outer parameters. It is natural to assume a similarity that dynamical variables depend only on y/x , where y is the coordinate perpendicular to the plate.

The thickening of boundary layer in the fully-developed region is that the turbulent vortex region swells out to the free uniform flow region, entrains this region into itself and is called the "entrainment of the turbulent boundary layer". The vortex model gives a view that the entrainment phenomenon is due to the individual eddies in the edge of the boundary layer going upwards and into the uniform flow.

We choose the independent variables:

$$\xi = \frac{x}{x_0}, \qquad \eta = \frac{y}{\delta_0}, \qquad \zeta = \frac{y}{\delta(x)} \tag{2}$$

where $\delta(x) = \alpha x$ is the boundary layer thickness and δ_0 the boundary layer thickness at the starting point of similarity. The boundary layer in the fully-developed region is considered to be divided into three layers, $\zeta = 0 \sim \varepsilon$ ($\varepsilon \ll 1$, the viscous layer), $\zeta = \varepsilon \sim \zeta_i$ ($\zeta_i \simeq 0.5$, the log-layer) and $\zeta = \zeta_i \sim 1$, (the intermittency layer).

The mean flow velocity is discussed here, based on the approximate mean flow equations. We adopt the following turbulent boundary layer equations:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial y}\nu_T\frac{\partial \bar{u}}{\partial y}, \tag{3}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \tag{4}$$

where ν_T is the eddy viscosity, which is a function of the position and the pressure term is dropped like the Blasius equation. A following similarity solution would be assumed for this equation:

$$\bar{u} = Uf(\zeta), \qquad\qquad \bar{v} = Ug(\zeta). \tag{5}$$

From (4), we have

$$-\frac{\dot{\delta}}{\delta}\zeta\frac{\mathrm{d}Uf}{\mathrm{d}\zeta} + \frac{1}{\delta}\frac{\mathrm{d}Ug}{\mathrm{d}\zeta} = \frac{U}{\delta}[-\dot{\delta}\zeta\frac{\mathrm{d}f}{\mathrm{d}\zeta} + \frac{\mathrm{d}g}{\mathrm{d}\zeta}] = 0$$

or

$$\frac{\mathrm{d}g}{\mathrm{d}\zeta} = \dot{\delta}\zeta\frac{\mathrm{d}f}{\mathrm{d}\zeta} \tag{6}$$

where $\dot{\delta} = \mathrm{d}\delta/\mathrm{d}x = \text{const}$. Thus, we have

$$g = \dot{\delta}\int_0^\zeta \zeta\frac{\mathrm{d}f}{\mathrm{d}\zeta}\mathrm{d}\zeta = \dot{\delta}[\zeta f(\zeta) - \int_0^\zeta f(\zeta)\mathrm{d}\zeta] \tag{7}$$

Further,

$$\frac{1}{\zeta}\frac{\mathrm{d}}{\mathrm{d}\zeta}\nu_T\frac{\mathrm{d}\bar{u}}{\mathrm{d}y} = -U^2\frac{\dot{\delta}}{\delta}f'\int_0^\zeta f\mathrm{d}\zeta$$

is derived from (3). Integrating this, we have

$$\nu_T\frac{\mathrm{d}\bar{u}}{\mathrm{d}y} = -U^2\dot{\delta}\int_0^\zeta f'(\sigma)\int_0^\sigma f(\tau)\mathrm{d}\tau\mathrm{d}\sigma + \frac{\tau_*}{\rho}$$

or

$$U\nu_T\frac{1}{\delta}\frac{\mathrm{d}f}{\mathrm{d}\zeta} = -U^2\dot{\delta}\int_0^\zeta f'(\sigma)\int_0^\sigma f(\tau)\mathrm{d}\tau\mathrm{d}\sigma + v_*^2$$

where $v_* = \sqrt{\tau_*/\rho}$ is the friction vecocity and τ_* the tangential stress at $y = 0$. The similarity solution of f requires that $\tau_* = \mathrm{const.}$ The similarity requirement gives also

$$\nu_T = U\delta(x)\hat{\nu}_T(\zeta), \tag{8}$$

where $\hat{\nu}_T$ is a nondimensional function of ζ.Then, we have

$$\hat{\nu}_T\frac{\mathrm{d}f}{\mathrm{d}\zeta} = -\dot{\delta}[f\int_0^\zeta f\mathrm{d}\zeta - \int_0^\zeta f^2\mathrm{d}\zeta] + (\frac{v_*}{U})^2 \tag{9}$$

The boundary condition for f:

$$\hat{\nu}_T\frac{\mathrm{d}f}{\mathrm{d}\zeta} \longrightarrow 0 \qquad as \qquad \zeta \longrightarrow \infty$$

gives

$$(\frac{v_*}{U^2})^2 = \dot{\delta}\lim_{\zeta\longrightarrow\infty}[f\int_0^\zeta f\mathrm{d}\zeta - \int_0^\zeta f^2\mathrm{d}\zeta,] \tag{10}$$

But, further analysis based on (9) would be difficult. The mean velocity profile in the intermittency and log layers is roughly expressed by

$$\bar{u} = -\frac{U - u_0}{\log\varepsilon}\log\zeta + U. \tag{11}$$

where $u_0 = \bar{u}(y = \varepsilon\delta(x))$. We discuss the turbulence development in the intermittency layer and log-layer. For brevity, we adopt a discretized model (Fig.3), i.e. the intermittency and log layers are divided into many cells, which are separated by

$$\begin{aligned}&\xi = \xi_0 = 1,\ \xi_1 = 2,\ \cdots,\ \xi_l = 2^l,\ \cdots,\\ &\eta = \eta_0 = 2^{-M}, \eta_1 = 2^{-M+1},\ \cdots,\ \eta_m = 2^{-M+m},\ \cdots.\end{aligned} \tag{12}$$

The intermittency and log layers are bounded by $\xi \leq 1$, $2^{-M} \leq \eta/\xi(=\zeta) \leq 1$, where $\zeta = 1$ corresponds to the edge of the boundary layer and $\zeta = 2^{-M}$ is the approximate lacation of the buffer layer (between the viscous and log layer). The mean flow stream lines would be almost parallel to the wall, i.e. $\eta =$ const. and the mean flow is approximated by many equal vortex sheets, which are located at

$\eta = \eta_m$ for $m = 0, 1, \cdots$. The vortex sheets at $\eta = \eta_m$ for $m = 0, \cdots, M$ start at $\xi = 1$, but those at $\eta = \eta_m$ for $m = M+1, \cdots$ start at $\xi = \xi_{M+1}, \cdots$.According to (11), discretized mean flow velocity (Fig.4) is given as

$$\left.\begin{array}{l} \bar{u} = u^M = U, \quad \zeta > \zeta^M = 1, \\ \bar{u} = u^n = U + (-M+n)\Delta\bar{u}, \quad \zeta^n < \zeta < \zeta^{n+1}, \\ \qquad n = 1, 2, \cdots, M-1, \\ \bar{u} = u^0 = U - M\Delta\bar{u}, \quad \zeta = \zeta^0 \end{array}\right\} \tag{13}$$

$$\Delta\bar{u} = \frac{U - u_0}{M}, \quad \zeta^n = 2^{-M+n}, \quad \zeta^M = 1, \quad \zeta^0 = 2^{-M} = \varepsilon \tag{14}$$

The vortex sheet could be modelled further to be a train of many vortex filaments (a vortex street) of the interval d with the same circulation Γ:

$$\Gamma = 2\ \Delta\bar{u}\ d \tag{15}$$

Each vortex composing the vortex streets is directed to the negative z-axis (Fig.3).

The individual eddies, which are produced in the transion region, have the advection velocity and the self-propulsive, translational velocity, parallel and perpendicular to the wall, respectively. The similarity arugument implies that the individual eddies would travel roughly along ζ = const. Thus, we would have trains of individual eddies along ζ =consts. Let us consider a train of individual eddies on $\zeta = \zeta^n = 2^{-M+n}$ for $n = 1, 2. \cdots, M-1$, which passes through $(\xi, \eta) = (\xi_l, \eta_{n+l})$, for $l = 0, 1, \cdots$ (Fig.5). The train of individual eddies goes diagonally through each similarity cell of $C^n = \{C_l^n\}_{l=0,1,\cdots}$, whose dialonal corners are on (ξ_l, η_{n+l}) and $(\xi_{l+1}, \eta_{n+l+1})$ for $l = 0, 1, \cdots$. This is called a series of similarity cells (Fig.6).

The advection velocity of individual eddies travelling on ζ_n would be preserved. Thus, the translational velocity should be also preserved, in order that these eddies do not leave ζ_n. The geomtrical similarity gives that the size of eddies is proportional to ξ.The most simple model for the trains of eddies satisfying those similarity requirements, is the one, in which the unification of two similar eddies and then the merging of two similar unified eddies or *vice versa* would happen for every travelling from (ξ_l, η_{n+l}) to $(\xi_{l+1}, \eta_{n+l+1})$ (Fig.5). The unification of two similar eddies would give a one twice in size and with the same ciculatioin as before. After the merging of two similar eddies, the size would remain as before and the circulation would be twice.Then, we would have an eddy twice both in size and circulation from four similar eddies, after the unification and merging. The above consideration gives

$$\left.\begin{array}{c} \Gamma_l^n = 2^l\Gamma^n, \quad a_l^n = 2^l a^n, \quad b_l^n = 2^l b^n, \\ a^n = a_0^n, \quad \Gamma^n = \Gamma_0^n, \quad b^n = b_0^n, \end{array}\right\} \tag{16}$$

$$V_l^n = V^n, \quad E_l^n = 2^{3l}E^n, \quad P_l^n = 2^{3l}P^n, N_l^n = 2^{-2l}N^n, \tag{17}$$

from (1). Here, a_l^n is the value of a at (ξ_l, η_{n+l}) and so on and N_l^n is the number of individual eddies passing that point per unit time. In our model, there are the vortex sheets or trains of vortex filaments at $\eta = \eta_m$ for $m = 0, 1, \cdots$, which are parallel to the wall, but the trains of eddies on $\zeta = \zeta_n$ are radial from the origion of similarity and should pass through the vortex street.

Here, we consider the time rates of the x-component of momentum and energy in the physical area D_l^n , the size of which is $(2^l x_0) \times (2^{-M+n+l}\delta_s)$, corresponding to C_l^n:

$$\dot{P}_x \equiv \frac{\partial}{\partial t}\int\int_{D_l^n} \rho u \, \mathrm{d}^2\boldsymbol{x} = -\int_{\partial D_l^n} \rho u v_n \, \mathrm{d}s + \int_{\partial D_l^n} p_{nx}\mathrm{d}s, \tag{18}$$

$$\begin{aligned}\dot{E} &\equiv \frac{\partial}{\partial t}\int\int_{D_l^n} \frac{1}{2}\rho \boldsymbol{v}^2 \mathrm{d}^2\boldsymbol{x} \\ &= -\frac{1}{2}\int_{\partial D_l^n} \rho \boldsymbol{v}^2 v_n \mathrm{d}s - \int\int_{D_l^n} \{\text{dissipation function}\}\mathrm{d}^2\boldsymbol{x}.\end{aligned} \tag{19}$$

for the case of no body force, no heat source and flow. $p_{n\alpha}$ is the normal stress component on ∂D_l^n , the boundary of D_l^n and $\boldsymbol{n}$ the outward normal on ∂D_l^n . Our assumption of trains of individual eddies only on ζ^n, for $n = 1, \cdots, M-1$ ascertains that the individual eddies except on ζ^n do not pass through each similarity cell belonging C^n.

Here, (18) could berecognized both the turbulence and the mean flow equations. We consider

$$\dot{P}_x = -\int_{\partial D_l^n} \rho u v_n \, \mathrm{d}s, \tag{20}$$

the turbulence equation, i.e. the increase of x-momentum is due to the inflow of momentum through the boundary ∂D_l^n, which corresponds to that of the individual eddies. The individual eddy has the y-momentum, the third of (16), entering C_l^n from C_{l-1}^n , and also passing through a vortex street at $\eta = \eta_m$. Just after entering C_l^n, the individual eddy is slower than the surrounding fluid in the x-direction sense. It would be accelerated and have the same velocity with the surroundings in some time, after full mixing. This momentum increase of individual eddy may be $P_l^n \; \Delta\bar{u}/V^n$. Thus, we have the decrease of momentum as the reaction of the momentum increase of individual eddies:

$$\dot{P}_x = -\int_{\partial D_l^n} \rho u v_n \, \mathrm{d}s = -4 \; 2^l \pi^2 \rho \frac{(a^n)^2 N^n \Delta\bar{u}}{A_1^n} \tag{21}$$

where $A_1 = \log 8a/b - 1/4$. The momentum increase should be ballanced by the righthand side of (18) as the mean flow, in which the eddy viscosity relation would be used.

$$\int_{\partial D_l^n} p_{nx} ds = -\frac{1}{2} 2^{M-n} \frac{x_0}{\delta_s} \rho \nu_T \Delta \bar{u}, \tag{22}$$

where ν_T is the eddy viscosity, which is function of place, but takes an averrage value in D_l^n here. From (21) and (22), we have

$$\nu_T = 8\pi^2 \frac{\delta_s (a^n)^3 N^n}{x_0 A_1^n} \xi_l \eta^n \sim 8\pi^2 \frac{\delta_s}{x_0} (\frac{a^3 N}{A_1})(\eta) \xi \eta \tag{23}$$

The similar consideration for energy equation (19) gives

$$\nu_T = \frac{1}{(\Delta \bar{u})^2} \frac{\delta_s}{x_0} A_2^n (\Gamma^n)^2 a^n N^n \xi_l \eta^n \sim \frac{1}{(\Delta \bar{u})^2} \frac{\delta_s}{x_0} (A_2 \Gamma^2 a N)(\eta) \xi \eta \tag{24}$$

where $A_2 = \log(8a/b) - 7/4$. In (23) and (24), the last terms are in the expressions as continuous variables, translated from those of the discretized model.

We considered the vortex model for the turbulent boundary layer along a flat plate. In this model, the turbulence in the boudary layer is recognized as an ensemble of the individual eddies, which correspond to the turbulence and the vortex streets, which are modelling of the mean flow. Similarity consideration and the equations of momentum and energy give the expressions for the eddy viscosity, which is related to the size, the circulation and production rate of the individual eddies. Problems to determine the characteristics of individual eddies to refine the eddy viscosity analysis and to obtaine solutions of the mean flow equations are left in future.

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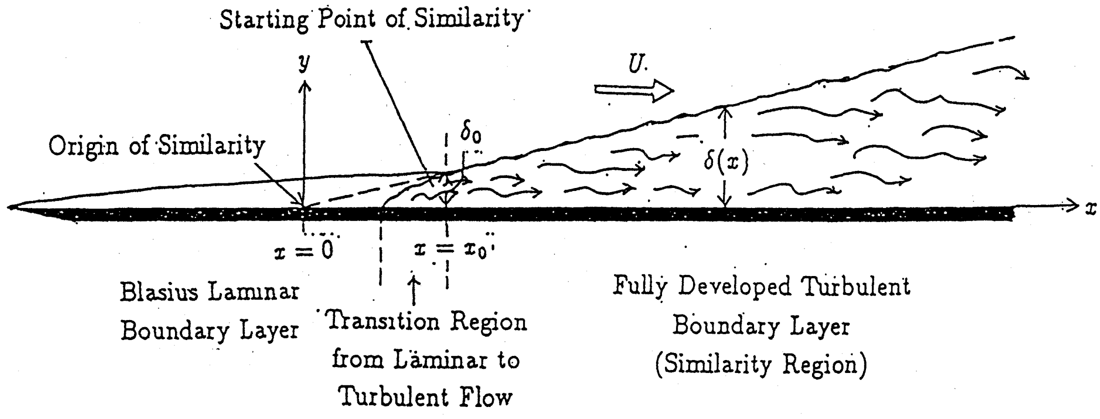

Fig.1. Turbulent Boundary Layer along a Flat Plate

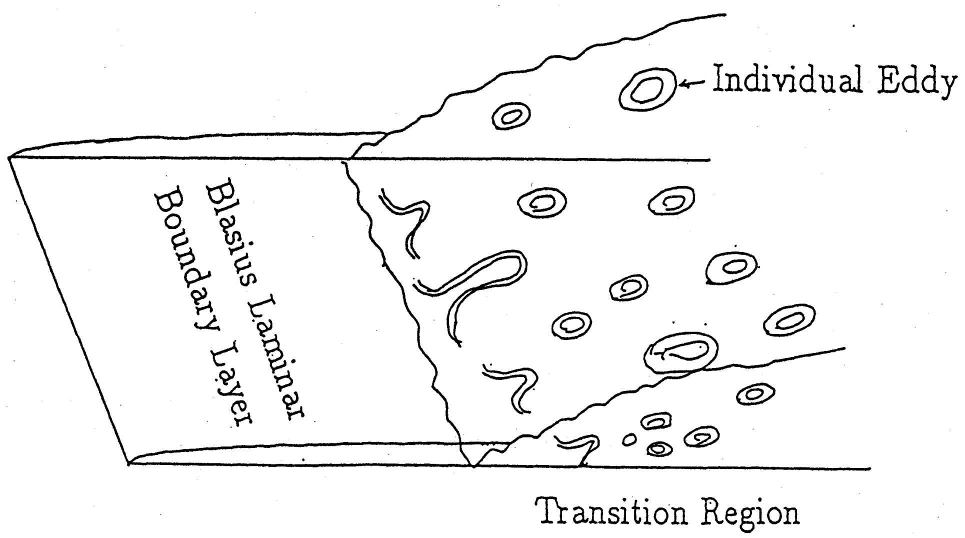

Fig.2. Appearance of Individual Eddies.

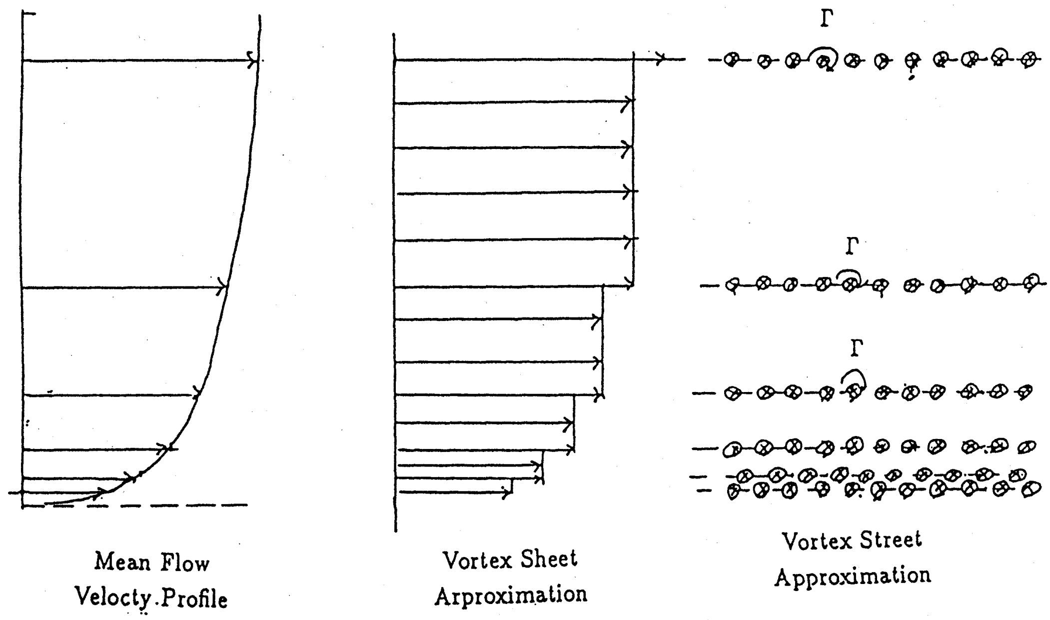

Fig.3. Modelling of Mean Flow by Many Vortex Streets

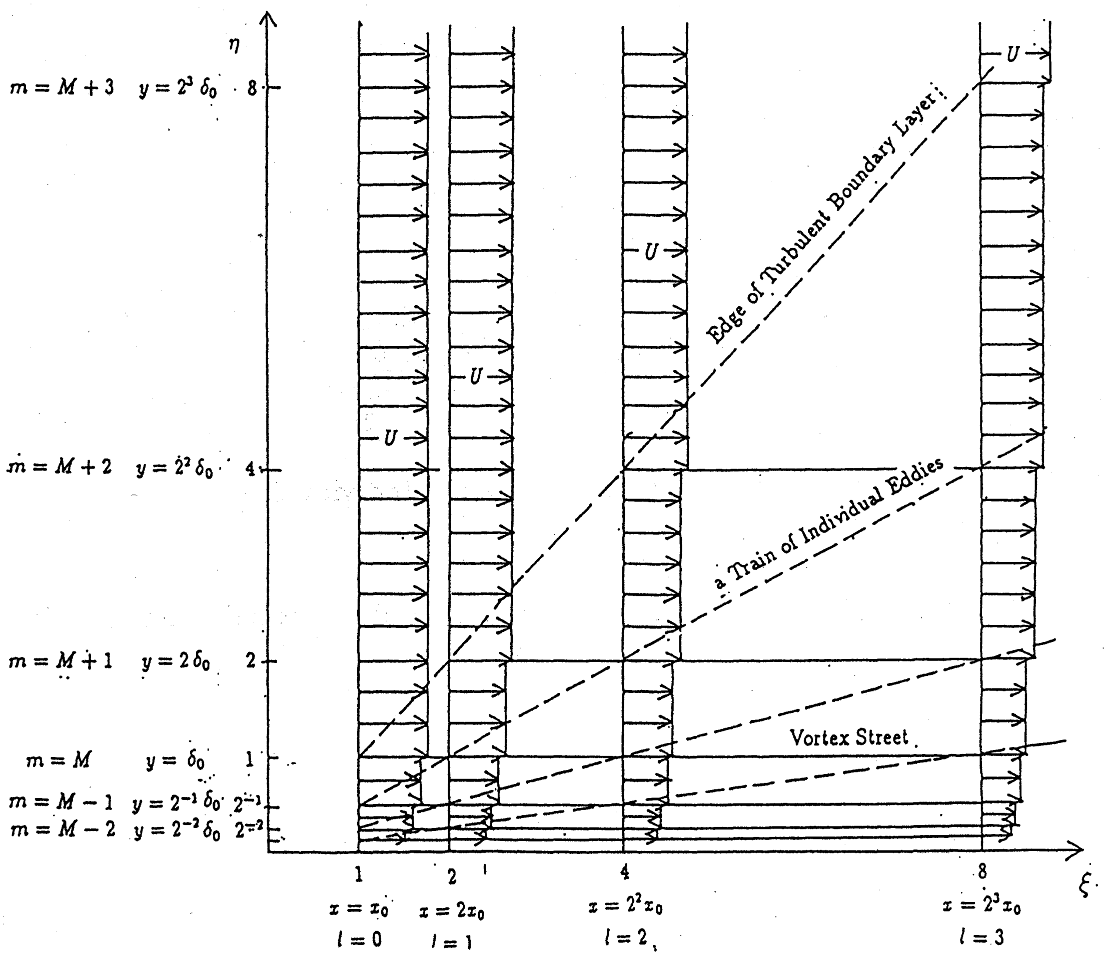

Fig.4. Discritized Mean Flow Velocity

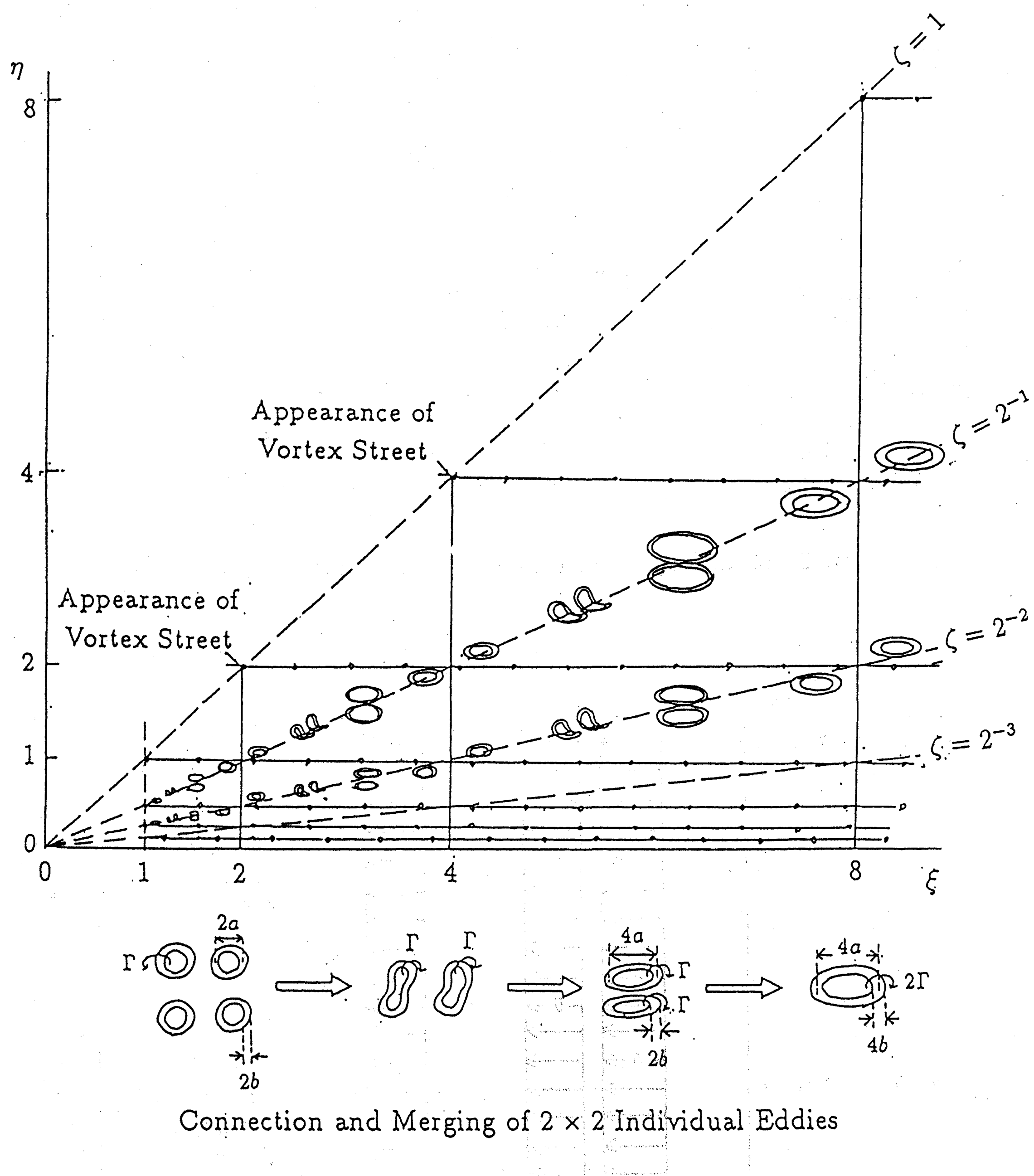

Fig.5. Trains of Individual Eddies and Connection and Merging of Individual Eddies

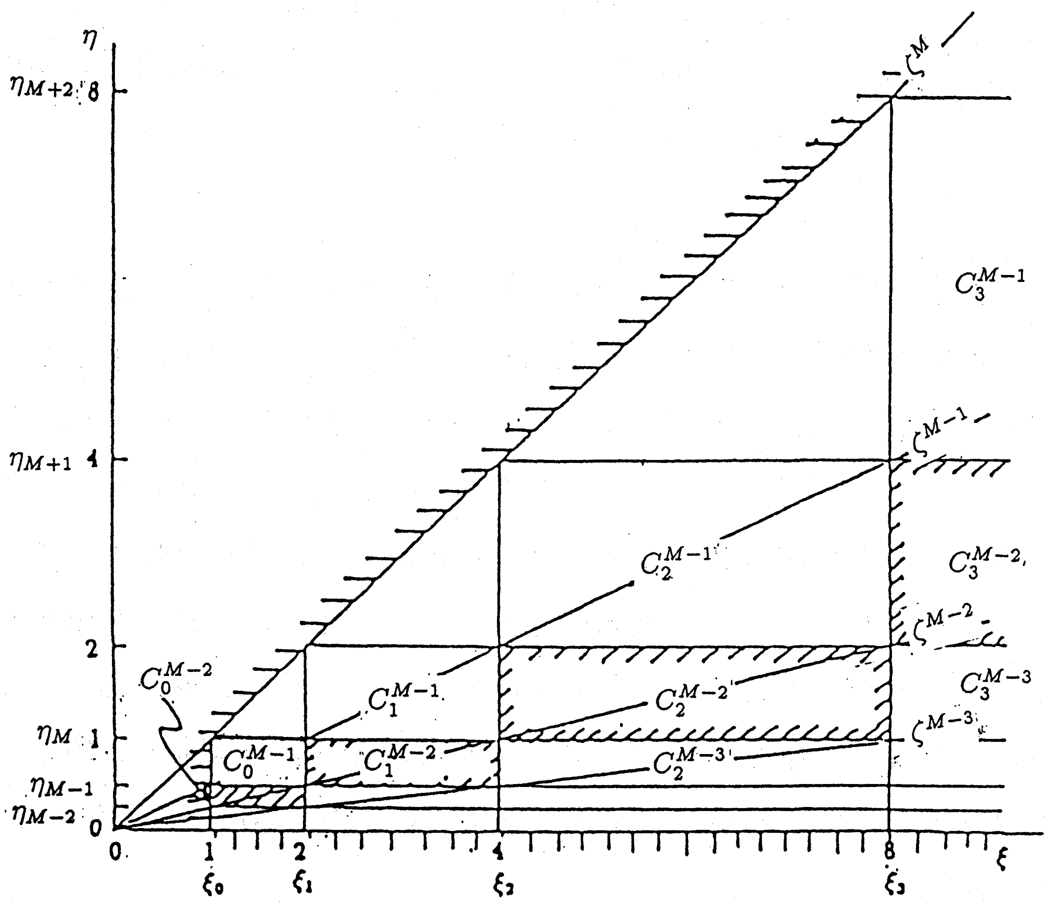

Fig.6. Normalized Coordinates and Series of Similarity Cells