数理解析研究所講究録
第 801 巻 1992 年 144-154

On types of blowing-ups of ideals in Buchsbaum rings

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1. *Introduction*

Throughout this note $(A, \mathfrak{m}, k)$ denotes a Noetherian local ring of dimension $d > 0$ and $\mathfrak{q}$ is a parameter ideal of A. Let us denote

$R(\mathfrak{q}) := \bigoplus_{n \geqq 0} \mathfrak{q}^n$, the Rees algebra of $\mathfrak{q}$;

Proj $R(\mathfrak{q})$, the set of all homogeneous prime ideals of $R(\mathfrak{q})$ not containing $R(\mathfrak{q})_+ := \bigoplus_{n > 0} \mathfrak{q}^n$.

Moreover $H^i_{\mathfrak{m}}(*)$ stands for the i^{th} local cohomology functor with respect to $\mathfrak{m}$.

Recall that the local ring A is said to be a Buchsbaum ring if the difference $l_A(A/\mathfrak{q}) - e_{\mathfrak{q}}(A)$ is an invariant of A not depending on the choice of parameter ideal $\mathfrak{q}$ of A.

In 1982 S. Goto showed that $A/H^0_{\mathfrak{m}}(A)$ is a Buchsbaum ring (resp. a Gorenstein ring) if and only if Proj $R(\mathfrak{q})$ is a locally Cohen-Macaulay scheme (resp. a locally Gorenstein scheme) for every parameter ideal $\mathfrak{q}$ of A, and he also gave us similar characterizations of complete intersections and regular local rings; see (1.1), (1.2), (4.4) and (4.6) of [3]. However, there was given no statement for the local ring $A/H^0_{\mathfrak{m}}(A)$ to be a Cohen-Macaulay ring in terms of blowing-up. So our problem is stated as follows.

Problem. What is the condition concerning blowing-up for the local ring $A/H^0_{\mathfrak{m}}(A)$ to be a Cohen-Macaulay ring?

To describe our results let us recall more notations. We denote by $r(A)$ the type of A, i.e., $r(A) := l_A(\mathrm{Ext}_A^d(k, A))$, the d^{th} Bass number of A, where $d := \dim A$, cf. [1]. Moreover let K_A denote the canonical module of A if it exists, $\mu_A(*)$ the minimal number of generators of an A-module and $\hat{A}$ the completion of A with respect to the $\mathfrak{m}$-adic topology. Then our first result is:

Theorem (1.1). Let A be a Buchsbaum ring and $a_1, \dots, a_d$ a system of parameters for A. Let $\mathrm{R}(\underline{a}^n)$ denote the Rees algebra of the system of parameters $\underline{a}^n := a_1{}^n, \dots, a_d{}^n$ for A for an integer $n > 0$. Then the following equality

$$r(\mathrm{R}(\underline{a}^n)_{\mathfrak{P}}) = \mu_{\hat{A}}(K_{\hat{A}}) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} . l_A(\mathrm{H}_{\mathfrak{m}}^i(A))$$

holds for all $\mathfrak{P} \in \mathrm{Proj}\ \mathrm{R}(\underline{a}^n)$ with $\dim \mathrm{R}(\underline{a}^n)/\mathfrak{P} = 1$ and all $n \geqq 3$.

Combining this theorem and Goto's one in [3] we can state our second result, which is an answer to our problem described as above:

Theorem (1.2). The following two statements are equivalent.

(1) $A/\mathrm{H}_{\mathfrak{m}}^0(A)$ is a Cohen-Macaulay ring.

(2) For every parameter ideal $\mathfrak{q}$ of A, $\mathrm{Proj}\ \mathrm{R}(\mathfrak{q})$ is a locally Cohen-Macaulay scheme and the equality $r(\mathrm{R}(\mathfrak{q})_{\mathfrak{P}}) = \mu_{\hat{A}}(K_{\hat{A}})$ holds for all $\mathfrak{P} \in \mathrm{Proj}\ \mathrm{R}(\mathfrak{q})$ with $\dim \mathrm{R}(\mathfrak{q})/\mathfrak{P} = 1$.

2. *Preliminalies*

Let $\mathfrak{q} = (a_1, \dots a_d)$ be a parameter ideal of A. Then we begin with the following.

Lemma (2.1). (1) $\mathfrak{m}.\mathrm{R}(\mathfrak{q}) \in \mathrm{Proj}\ \mathrm{R}(\mathfrak{q})$.

(2) Every $\mathfrak{P} \in \mathrm{Proj}\ \mathrm{R}(\mathfrak{q})$ such that $\dim \mathrm{R}(\mathfrak{q})/\mathfrak{P} = 1$ contains $\mathfrak{m}.\mathrm{R}(\mathfrak{q})$.

(3) $R(q)/\mathfrak{m}.R(q)$ is isomorphic to the polynomial ring over the residue field k of A with d-indeterminates.

We know that

$$\mathrm{Proj}\ R(q) \cong \coprod_{i=1}^{d} \mathrm{Spec}\ A[\frac{x}{a_i} \mid x \in q] .$$

Moreover we have

Lemma (2.2). Let $\mathfrak{P} \in \mathrm{Proj}\ R(q)$ and put $B := A[\frac{x}{a_d} \mid x \in q]$. Assume that the given $\mathfrak{P}$ corresponds to the prime ideal P of B . Then $R(q)_{\mathfrak{P}}$ is isomorphic to the local ring $B[T]_{P.B[T]}$ as A-algebras, where T is an indeterminate over B . Thus $r(R(q)_{\mathfrak{P}}) = r(B_P)$ holds.

From now on, we assume that A is a Buchsbaum ring and we use the following notations:

$$B := A[\frac{a_1}{a_d} , \dots , \frac{a_{d-1}}{a_d}] \text{ in } A[\frac{1}{a_d}] ;$$

$$Q := (\frac{a_1}{a_d} , \dots , \frac{a_{d-1}}{a_d}, a_d).B ;$$

$$M := \mathfrak{m}.B + Q .$$

Then we have

Lemma (2.3)(cf. [3], (3.4), (3.6)). (1) The local ring B_M is a Cohen-Macaulay ring of dimension d .

(2) $Q.B_M$ is a parameter ideal of B_M .

(3) $B_M/Q.B_M \cong A/[q_{d-1} : a_d] + (a_d)$ as A-algebras, where we put $q_{d-1} := (a_1, \dots , a_{d-1})$.

Lemma (2.4). $r(B_M) = l_A(\mathrm{Hom}_A(k, A/[q_{d-1} : a_d] + (a_d)))$.

For a system of parameters $a_1, \dots , a_d$ for A we define the ideal $\Sigma(a_1, \ldots, a_d)$ of A as follows:

$$\Sigma(a_1, \ldots, a_d) := \sum_{i=1}^{d} [(a_1, \ldots, \hat{a}_i, \ldots, a_d) : a_i] + (a_1, \ldots, a_d) .$$

When we set $\underline{a} = a_1, \ldots, a_d$ or $\mathfrak{q} = (a_1, \ldots, a_d)$, we also use the notation $\Sigma(\underline{a})$ or $\Sigma(\mathfrak{q})$.

Then as $\mathfrak{q} \subset [\mathfrak{q}_{d-1} : a_d] + (a_d) \subset \Sigma(\mathfrak{q})$ and $\mathfrak{m}.\Sigma(\mathfrak{q}) \subset \mathfrak{q}$, there is an exact sequence of k-vector spaces:

$$0 \longrightarrow \frac{\Sigma(\mathfrak{q})}{[\mathfrak{q}_{d-1} : a_d] + (a_d)} \longrightarrow \mathrm{Hom}_A(k, B_M/QB_M) \longrightarrow \mathrm{Hom}_A(k, A/\Sigma(\mathfrak{q})) .$$

Thus we get the following.

Lemma (2.5). (1) $r(B_M) \geqq \sum_{i=1}^{d-1} \binom{d-1}{i-1} . l_A(\mathrm{H}^i_{\mathfrak{m}}(A))$.

(2) $r(B_M) \leqq l_A(\mathrm{Hom}_A(k, A/\Sigma(\mathfrak{q})) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} . l_A(\mathrm{H}^i_{\mathfrak{m}}(A))$.

To estimate the value $l_A(\mathrm{Hom}_A(k, A/\Sigma(\mathfrak{q})))$, let us recall a well-known fact that the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(A)$ is isomorphic to the direct limit $\varinjlim_n \mathrm{H}^i(\underline{a}^n; A)$ defined by the Koszul (co-)homology modules $\mathrm{H}^i(\underline{a}^n; A)$ of A with respect to the system $\underline{a}^n := a_1{}^n, \ldots, a_d{}^n$ $(n > 0)$, for each i. In particular for the case $i = d$ we have the following direct system:

$$A/(\underline{a}) \longrightarrow A/(\underline{a}^2) \longrightarrow \ldots \longrightarrow \varinjlim_n A/(\underline{a}^n) \cong \mathrm{H}^d_{\mathfrak{m}}(A) ,$$

where the map $A/(\underline{a}^n) \longrightarrow A/(\underline{a}^{n+l})$ is multiplication by the element $a_1{}^l \ldots a_d{}^l$. As we have the equality

$$\Sigma(a_1{}^{l_1+n_1}, \ldots, a_d{}^{l_d+n_d}) : a_1{}^{l_1} \ldots a_d{}^{l_d} = \Sigma(a_1{}^{n_1}, \ldots, a_d{}^{n_d})$$

for any integers $l_1, \ldots, l_d \geqq 0$ and $n_1, \ldots, n_d > 0$, this direct system induces the new one as follows:

$$A/\Sigma(\underline{a}) \longrightarrow A/\Sigma(\underline{a}^2) \longrightarrow \ldots \longrightarrow \varinjlim_n A/\Sigma(\underline{a}^n) \cong \mathrm{H}^d_{\mathfrak{m}}(A) ,$$

where all the induced maps $A/\Sigma(\underline{a}^n) \longrightarrow A/\Sigma(\underline{a}^{n+l})$ are injective, cf.

[7], p. 456. On the other hand, we know

$$\mu_{\hat{A}}(K_{\hat{A}}) = l_A(\mathrm{Hom}_A(k, \mathrm{H}^d_{\mathfrak{m}}(A))) \quad .$$

Therefore we get

Proposition (2.6). $r(B_M) \leqq \mu_{\hat{A}}(K_{\hat{A}}) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} \cdot l_A(\mathrm{H}^i_{\mathfrak{m}}(A))$.

Example (2.7). The equality of the formula in Proposition (2.6) is not true in general.

Let A be a Buchsbaum ring with minimal multiplicity, namely the equality

$$e(A) = 1 + \sum_{i=1}^{d-1} \binom{d-1}{i-1} \cdot l_A(\mathrm{H}^i_{\mathfrak{m}}(A))$$

holds, see [5]. Suppose that $d \geqq 2$ and $\mathrm{H}^i_{\mathfrak{m}}(A) \neq (0)$ for some i $(1 \leqq i < d)$. Then we have

$$\mathrm{Hom}_A(k, B_M/QB_M) \cong \mathrm{Hom}_A(k, A/[\mathfrak{q}_{d-1} : a_d] + (a_d)) = \mathfrak{m}/[\mathfrak{q}_{d-1} : a_d] + (a_d) \ ,$$

namely the map $\mathrm{Hom}_A(k, B_M/QB_M) \longrightarrow \mathrm{Hom}_A(k, A/\Sigma(\mathfrak{q}))$ is a zero map. This implies

$$r(B_M) = \sum_{i=1}^{d-1} \binom{d-1}{i-1} \cdot l_A(\mathrm{H}^i_{\mathfrak{m}}(A)) < \mu_{\hat{A}}(K_{\hat{A}}) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} \cdot l_A(\mathrm{H}^i_{\mathfrak{m}}(A)) \quad .$$

3. *When does the equality occur in Proposition (2.6)?*

Concerning the formula in Proposition (2.6) described as above, we have a natural question as follows:

Question. When does the equality occur in Proposition (2.6)?

In this section we shall try to give an answer to this question. Throughout this section we still assume that A is a Buchsbaum ring and $\mathfrak{q} = (a_1, \ldots, a_d)$ is a parameter ideal of A . Let $n \geqq 2$ be an integer and we put

$b_1 := a_1{}^n, \ \ldots \ , b_d := a_d{}^n$, and $c := a_d{}^{n-1}$;

$$B' := A[\frac{b_1}{b_d}, \dots, \frac{b_{d-1}}{b_d}] \text{ in } A[\frac{1}{b_d}] \ ;$$

$$Q' := (\frac{b_1}{b_d}, \dots, \frac{b_{d-1}}{b_d}, c)B' \ ;$$

$$M' := \mathfrak{m}B' + Q' \ .$$

Then we have the following as same as (2.3).

Lemma (3.1). (1) The local ring $B'_{M'}$ is a Cohen-Macaulay ring of dimension d and $Q'B'_{M'}$ is a parameter ideal of $B'_{M'}$.
(2) $B'_{M'}/Q'B'_{M'} \cong A/[(b_1, \dots, b_{d-1}) : c] + (c)$ as A-algebras.

Proof. (1) As $b_1, \dots, b_d$ forms a system of parameters for A too, this comes from (2.3) at once.

(2) We may assume that depth $A > 0$. Hence the parameter a_d (and hence c too) is a non-zero-divisor on A. Let x be an element of A such that

$$x = \frac{b_1}{b_d}.\frac{x_1}{b_d{}^l} + \dots + \frac{b_{d-1}}{b_d}.\frac{x_{d-1}}{b_d{}^l} + c.\frac{x_d}{b_d{}^l} ,$$

for some elements $x_1, \dots, x_d \in (\underline{b})^l$ and some integer $l \geqq n$. As $(\underline{b})^l = (b_d{}^l) + (b_1, \dots, b_{d-1}).(\underline{b})^{l-1}$ we have $x_d = b_d{}^l u + v$ for some elements $u \in A$ and $v \in (b_1, \dots, b_{d-1}).(\underline{b})^{l-1}$. As $b_d{}^l = b_d c$ by our definition of the element c, we have

$$b_d{}^{l+1}(x - cu) = b_1 x_1 + \dots + b_{d-1}x_{d-1} + b_d cv \ .$$

As $[(b_1, \dots, b_{d-1}) : b_d{}^{l+1}] = [(b_1, \dots, b_{d-1}) : c]$ we get that

$$x \in [(b_1, \dots, b_{d-1}) : c] + (c) \ .$$

Now recall the direct system

$$A/\Sigma(\underline{a}) \longrightarrow A/\Sigma(\underline{a}^2) \longrightarrow \dots \longrightarrow A/\Sigma(\underline{a}^n) \longrightarrow \dots \longrightarrow \mathrm{H}^d_{\mathfrak{m}}(A) \ ,$$

and that all maps of this system are injective. Then consider the following commutative diagram:

$$\begin{array}{ccc}
A/\sum(\underline{a}^{n-1}) & \xrightarrow{a_1 \cdots a_d} & A/\sum(\underline{a}^{n}) \\
{\scriptstyle a_1 \cdots a_{d-1}} \downarrow & & \uparrow {\scriptstyle a_d} \\
A/[(\underline{b'}) : c] + (c) & \longrightarrow & A/\sum(\underline{b'}, c)
\end{array}$$

where we put $\underline{b'} := b_1, \ldots, b_{d-1}$. Notice that the two vertical maps on both sides of this diagram are injective too. By (2) of (3.1) this diagram induces the next one.

$$\begin{array}{ccccccc}
\mathrm{Hom}_A(k, A/\sum(\underline{a}^{n-1})) & \longrightarrow & \mathrm{Hom}_A(k, A/\sum(\underline{a}^{n})) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}_A(k, \mathrm{H}^d_{\mathfrak{m}}(A)) \\
{\scriptstyle a_1 \cdots a_{d-1}} \downarrow & & \uparrow {\scriptstyle a_d} & & & & \\
\mathrm{Hom}_A(k, B'/Q'B'_{M'}) & \longrightarrow & \mathrm{Hom}_A(k, A/\sum(\underline{b'}, c)) & & & &
\end{array}$$

Choosing the integer n large enough we can make all maps of the top row of this diagram bijective. Then the vertical map on the right of this diagram must be bijective too, i.e.,

$$\mathrm{Hom}_A(k, A/\sum(\underline{b'}, c)) \xrightarrow[\cong]{a_d} \mathrm{Hom}_A(k, A/\sum(\underline{a}^{n})) ,$$

and hence the map on the bottom of this diagram is surjective, namely the following sequence of k-vector spaces must be exact.

$$0 \to \frac{\sum(\underline{b'}, c)}{[(\underline{b'}) : c] + (c)} \to \mathrm{Hom}_A(k, B'/Q'B'_{M'}) \to \mathrm{Hom}_A(k, A/\sum(\underline{b'}, c)) \to 0$$

Therefore we get the following result, which is an answer to our question described at the beginning of this section.

Proposition (3.2). Under the same notations described as above, one has the equality

$$r(B'_{M'}) = \mu_{\hat{A}}(K_{\hat{A}}) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} \cdot l_A(\mathrm{H}^i_{\mathfrak{m}}(A))$$

if the integer n is large enough. (More precisely, according to Proposition (3.8) of [7], this equality holds for all $n \geqq 3$.)

4. *Reduction step.*

In this section, we shall discuss how we reduce our arguments to the case that the residue field k of A is algebraically closed.

Let q be a parameter ideal of A, R(q) the Rees algebra of q and $\mathfrak{P}$ a homogeneous prime ideal of R(q) with $\dim R(q)/\mathfrak{P} = 1$. Then by (2.1) the ideal $\mathfrak{P}$ contains the homogeneous prime ideal $\mathfrak{m}.R(q)$. We recall the following result given by S. Goto [3].

Lemma (4.1)(cf. (3.5) of [3]). Let A be a Buchsbaum ring and let $\bar{k}$ denote the algebraic closure of the residue field k of A. Then there exists a Buchsbaum local A-algebra $\bar{A}$ with maximal ideal $\bar{\mathfrak{m}}$ such that (1) $\bar{A}$ is A-flat, (2) $\bar{\mathfrak{m}} = \mathfrak{m}.\bar{A}$, and (3) $\bar{k} \cong \bar{A}/\bar{\mathfrak{m}}$ as k-algebras.

Then it is clear that $q.\bar{A}$ is still a parameter ideal of $\bar{A}$ and the ring $R(q) \otimes_A \bar{A}$ coincides with the Rees algebra of the parameter ideal $q.\bar{A}$ of $\bar{A}$. We put $R := R(q)$ and $\bar{R} := R \otimes_A \bar{A}$. As $\dim \bar{R}/\mathfrak{P}.\bar{R} = 1$ there is a homogeneous prime ideal, say $\bar{\mathfrak{P}}$, of $\bar{R}$ sucht that $\bar{\mathfrak{P}} \supset \mathfrak{P}.\bar{R}$ and $\dim \bar{R}/\bar{\mathfrak{P}} = 1$. Then we claim the following.

Lemma (4.2). The fibre $\bar{R}_{\bar{\mathfrak{P}}} \otimes_{R_{\mathfrak{P}}} (R_{\mathfrak{P}}/\mathfrak{P}.R_{\mathfrak{P}})$ is a Gorenstein ring of dimension 0 .

According to the theorem of [2], these lemmas imply the following.

Proposition (4.3). Under the same notations described as above, one has the equality $r(R_{\mathfrak{P}}) = r(\bar{R}_{\bar{\mathfrak{P}}})$.

5. *Proof of Theorem (1.1).*

We use the same notations as in Section 3. Let $\mathfrak{P}$ be a homogeneous prime ideal of the Rees algebra $R(\underline{b})$, where we put $\underline{b} := b_1, \ldots , b_d$, such that $\dim R(\underline{b})/\mathfrak{P} = 1$. Recall that

$$\mathrm{Proj}\ R(\underline{b}) = \bigsqcup_{i=1}^{d} \mathrm{Spec}\ A[\ \frac{x}{b_i}\ |\ x \in (\underline{b})]\ .$$

It has no loss of generality to assume that the given homogeneous prime ideal $\mathfrak{P}$ corresponds with the prime ideal N of Spec B' , where $B' := A[\ \frac{x}{b_d}\ |\ x \in (\underline{b})]$. Then we have the following from (2.3) and (3.1) at once.

Lemma (5.1). (1) N is a maximal ideal of B' and $N \cap A = \mathfrak{m}$.
(2) B'_N is a Cohen-Macaulay local ring of dimension d .
(3) $B'/\mathfrak{m}.B'$ is isomorphic to the polynomial ring over the residue field k of A with $d - 1$ indetermonates, as A-algebras.
(4) $r(R(\underline{b})_{\mathfrak{P}}) = r(B'_N)$.

Now we are ready to prove Theorem (1.1).

By the reduction described in the preceding section, we may assume that the residue field k of A is algebraically closed. Then applying Hilbert's Nullstellensatz to the extension field B'/P of k , we can find elements $c_1, \ldots, c_{d-1}$ in A not contained in maximal ideal $\mathfrak{m}$ of A such that

$$N = (\ \frac{b_1}{b_d} - c_1, \ldots, \frac{b_{d-1}}{b_d} - c_{d-1}\ ) + \mathfrak{m}.B'\ .$$

Put $b_1' := b_1 - c_1 b_d$, ... , $b'_{d-1} := b_{d-1} - c_{d-1} b_d$. Then we have

$$(b_1, \ldots, b_{d-1}, b_d) = (b_1', \ldots, b'_{d-1}, b_d)\ ,$$

$$B' = A[\frac{b'_1}{b_d}, \ldots, \frac{b'_{d-1}}{b_d}]\ ,$$

$$N = (\ \frac{b'_1}{b_d}, \ldots, \frac{b'_{d-1}}{b_d}\ ) + \mathfrak{m}.B'\ .$$

We set $Q'' := (\ \frac{b'_1}{b_d}, \ldots, \frac{b'_{d-1}}{b_d}, c\ ).B'$. Thus we have

Lemma (5.2). (1) $B'_N/Q''B'_N \cong A/[(b'_1, \ldots, b'_{d-1}) : c] + (c)$ as A-algebras.

(2) $[(b'_1, \ldots, b'_{d-1}) : c] + (c) = [(b_1, \ldots, b_{d-1}) : c] + (c)$.

By Lemma (5.2) we get that

$$\begin{aligned} r(\mathrm{R}(\underline{b})_{\mathfrak{P}}) &= r(B'_N) = l_A(\ \mathrm{Hom}_A(k,\ B'_N/Q''B'_N)\ ) \\ &= l_A(\ \mathrm{Hom}_A(k,\ A/[(b'_1, \ldots, b'_{d-1}) : c] + (c))\ ) \\ &= l_A(\ \mathrm{Hom}_A(k,\ A/[(b_1, \ldots, b_{d-1}) : c] + (c))\ ) \\ &= l_A(\ \mathrm{Hom}_A(k,\ B'_{M'}/Q'B'_{M'})\ ) = r(B'_{M'}) \\ &= \mu_{\hat{A}}(K_{\hat{A}}) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} . l_A(\mathrm{H}^i_{\mathfrak{m}}(A)) \quad . \end{aligned}$$

6. *Proof of Theorem (1.2).*

We shall here prove Theorem (1.2). By Theorem (1.1) and Goto's blowing-up characterization of Buchsbaum rings, cf. (1.1) of [3], we get the implication (2) $\Longrightarrow$ (1) at once.

(1) $\Longrightarrow$ (2) Let $\mathfrak{q} = (a_1, \ldots, a_d)$ be a parameter ideal of A . It is enough to show that $r(\mathrm{R}(\mathfrak{q})_{\mathfrak{P}}) = \mu_{\hat{A}}(K_{\hat{A}})$ holds for $\mathfrak{P} \in \mathrm{Proj}\ \mathrm{R}(\mathfrak{q})$ with $\dim \mathrm{R}(\mathfrak{q})/\mathfrak{P} = 1$.

Passing the redusction described in Section 4, we may assume that the residue field k of A is algebraically closed. Recall that

$$\mathrm{Proj}\ \mathrm{R}(\mathfrak{q}) = \bigsqcup_{i=1}^{d} \mathrm{Spec}\ A[\ \frac{x}{a_i}\ |\ x \in \mathfrak{q}\ ] \quad .$$

We may further assume that the given ideal $\mathfrak{P}$ corresponds with a maximal ideal N of B , where $B := A[\ \frac{x}{a_d}\ |\ x \in \mathfrak{q}\ ]$. As k is algebraically closed, we can find elements in A , say $e_1, \ldots, e_d$, such that

$$N = (\ \frac{a_1}{a_d} - e_1, \ldots, \frac{a_{d-1}}{a_d} - e_{d-1}\ ) + \mathfrak{m}.B \quad ,$$

by Hilbert's Nullstellensatz. Put $a_i' := a_i - e_i a_d$ for $1 \leq i < d$.

Then $(a'_1, \ldots, a'_{d-1}, a_d) = \mathfrak{q}$, namely $a'_1, \ldots, a'_{d-1}, a_d$ is a system of parameters for A too. Moreover we have the followings:

$$B = A[\frac{a'_1}{a_d}, \ldots, \frac{a'_{d-1}}{a_d}] ,$$

$$N = (\frac{a'_1}{a_d}, \ldots, \frac{a'_{d-1}}{a_d}) + \mathfrak{m}.B .$$

Therefore we get that

$$r(B_N) = l_A(\mathrm{Hom}_A(k, A/(a'_1, \ldots, a'_{d-1}, a_d))) = \mu_{\hat{A}}(K_{\hat{A}}) .$$

This finishes the proof of Theorem (1.2).

References

[1] H. Bass. On the ubiquity of Gorenstein rings. Math. Zeitschr. 82(1963), 8-28.

[2] H. -B. Foxby and A. Thorup. Minimal injective resolutions under flat base change. Proc. Amer. Math. Soc. 67(1977), 27-31.

[3] S. Goto. Blowing-up of Buchsbaum rings. Commutative Algbera: Durham 1981. London Math. Soc. Lecture Note Series 72 (Cambridge Univ. Press 1982), 140-162.

[4] S. Goto. Buchsbaum rings of maximal embedding dimension. J. Algebra 76(1982), 383-399.

[5] S. Goto. On the associated graded rings of parameter ideals in Buchsbaum rings. J. Algebra 85(1983), 490-534.

[6] J. Stückrad and W. Vogel. Buchsbaum Rings and Applications. (Springer-Verlag, 1986).

[7] K. Yamagishi. Idealizations of maximal Buchsbaum modules over a Buchsbaum ring. Math. Proc. Cambridge Philos. Soc. 104(1988), 451-478.

[8] K. Yamagishi. Bass number characterization of surjective Buchsbaum modules. Math. Proc. Cambridge Philos. Soc. 110(1991), 261-279.