On types of blowing-ups of ideals in Buchsbaum rings

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1. Introduction

Throughout this note \((A, m, k)\) denotes a Noetherian local ring of dimension \(d > 0\) and \(q\) is a parameter ideal of \(A\). Let us denote

\[
R(q) := \bigoplus_{n \geq 0} q^n, \text{ the Rees algebra of } q;
\]

\[
\text{Proj } R(q), \text{ the set of all homogeneous prime ideals of } R(q) \text{ not containing } R(q)_+ := \bigoplus_{n > 0} q^n.
\]

Moreover \(H^i_m(*)\) stands for the \(i^{th}\) local cohomology functor with respect to \(m\).

Recall that the local ring \(A\) is said to be a Buchsbaum ring if the difference \(l_A'(A/q) - e_q(A)\) is an invariant of \(A\) not depending on the choice of parameter ideal \(q\) of \(A\).

In 1982 S. Goto showed that \(A/H^0_m(A)\) is a Buchsbaum ring (resp. a Gorenstein ring) if and only if \(\text{Proj } R(q)\) is a locally Cohen-Macaulay scheme (resp. a locally Gorenstein scheme) for every parameter ideal \(q\) of \(A\), and he also gave us similar characterizations of complete intersections and regular local rings; see (1.1), (1.2), (4.4) and (4.6) of [3]. However, there was given no statement for the local ring \(A/H^0_m(A)\) to be a Cohen-Macaulay ring in terms of blowing-up. So our problem is stated as follows.

Problem. What is the condition concerning blowing-up for the local ring \(A/H^0_m(A)\) to be a Cohen-Macaulay ring?
To describe our results let us recall more notations. We denote by 
\( r(A) \) the type of \( A \), i.e., 
\[ r(A) := l_A^d(\operatorname{Ext}_A^d(k, A)) \]
the \( d \)th Bass number of \( A \), where \( d := \dim A \), cf. [1]. Moreover let \( K_A \) denote the canonical module of \( A \) if it exists, \( \mu_A(\ast) \) the minimal number of generators of an \( A \)-module and \( \hat{A} \) the completion of \( A \) with respect to the \( m \)-adic topology. Then our first result is:

**Theorem (1.1).** Let \( A \) be a Buchsbaum ring and \( a_1, \ldots, a_d \) a system of parameters for \( A \). Let \( R(\mathfrak{a}_n^\infty) \) denote the Rees algebra of the system of parameters \( \mathfrak{a}_n^\infty := a_1^n, \ldots, a_d^n \) for \( A \) for an integer \( n > 0 \).
Then the following equality
\[ r(R(\mathfrak{a}_n^\infty)_\mathfrak{p}) = \mu_{\hat{A}}(K_{\mathfrak{p}}) + \sum_{i=1}^{d-1} \binom{d-1}{i-1} l_A^i(H_{\mathfrak{m}}^i(A)) \]
holds for all \( \mathfrak{p} \in \operatorname{Proj} R(\mathfrak{a}_n^\infty) \) with \( \dim R(\mathfrak{a}_n^\infty)/\mathfrak{p} = 1 \) and all \( n \geq 3 \).

Combining this theorem and Goto's one in [3] we can state our second result, which is an answer to our problem described as above:

**Theorem (1.2).** The following two statements are equivalent.

1. \( A/H_\mathfrak{m}^0(A) \) is a Cohen-Macaulay ring.
2. For every parameter ideal \( q \) of \( A \), \( \operatorname{Proj} R(q) \) is a locally Cohen-Macaulay scheme and the equality \( r(R(q)_\mathfrak{p}) = \mu_{\hat{A}}(K_{\mathfrak{p}}) \) holds for all \( \mathfrak{p} \in \operatorname{Proj} R(q) \) with \( \dim R(q)/\mathfrak{p} = 1 \).

2. Preliminaries

Let \( q = (a_1, \ldots, a_d) \) be a parameter ideal of \( A \). Then we begin with the following.

**Lemma (2.1).** (1) \( \mathfrak{m}.R(q) \in \operatorname{Proj} R(q) \).
(2) Every \( \mathfrak{p} \in \operatorname{Proj} R(q) \) such that \( \dim R(q)/\mathfrak{p} = 1 \) contains \( \mathfrak{m}.R(q) \).
(3) $R(q)/\mathfrak{m}.R(q)$ is isomorphic to the polynomial ring over the residue field $k$ of $A$ with $d$-indeterminates.

We know that

$$\text{Proj } R(q) \cong \bigcup_{i=1}^{d} \text{Spec } \mathbb{A}[\frac{x}{a_i} | x \in q].$$

Moreover we have

Lemma (2.2). Let $\mathfrak{B} \in \text{Proj } R(q)$ and put $B := \mathbb{A}[\frac{x}{a_d} | x \in q].$

Assume that the given $\mathfrak{B}$ corresponds to the prime ideal $\mathfrak{p}$ of $B$.

Then $R(q)_{\mathfrak{B}}$ is isomorphic to the local ring $B[t]_{\mathfrak{p}}.B[t]$ as $A$-algebras, where $t$ is an indeterminate over $B$. Thus $r(R(q)_{\mathfrak{B}}) = r(B_{\mathfrak{p}})$ holds.

From now on, we assume that $A$ is a Buchsbaum ring and we use the following notations:

$$B := \mathbb{A}[\frac{a_1}{a_d}, \ldots, \frac{a_{d-1}}{a_d}] \text{ in } \mathbb{A}[\frac{1}{a_d}];$$

$$Q := \langle \frac{a_1}{a_d}, \ldots, \frac{a_{d-1}}{a_d}, a_d \rangle . B;$$

$$M := \mathfrak{m}.B + Q.$$

Then we have

Lemma (2.3) (cf. [3], (3.4), (3.6)). (1) The local ring $B_M$ is a Cohen-Macaulay ring of dimension $d$.

(2) $Q.B_M$ is a parameter ideal of $B_M$.

(3) $B_M/Q.B_M \cong A/[a_{d-1} : a_d] + (a_d)$ as $A$-algebras, where we put $q_{d-1} := \langle a_1, \ldots, a_{d-1} \rangle$.

Lemma (2.4). $r(B_M) = t_A(\text{Hom}_A(k, A/[q_{d-1} : a_d] + (a_d))).$

For a system of parameters $a_1, \ldots, a_d$ for $A$ we define the
ideal $\Sigma(a_1, \ldots, a_d)$ of $A$ as follows:

$$\Sigma(a_1, \ldots, a_d) := \sum_{i=1}^{d} [(a_1, \ldots, \hat{a}_i, \ldots, a_d) : a_i] + (a_1, \ldots, a_d).$$

When we set $a = a_1, \ldots, a_d$ or $q = (a_1, \ldots, a_d)$, we also use the notation $\Sigma(a)$ or $\Sigma(q)$.

Then as $q \subset [q_{d-1} : a_d] + (a_d) \subset \Sigma(q)$ and $m \cdot \Sigma(q) \subset q$, there is an exact sequence of $k$-vector spaces:

$$0 \to \frac{\Sigma(q)}{[q_{d-1} : a_d] + (a_d)} \to \text{Hom}_A(k, B_H/QB_H) \to \text{Hom}_A(k, A/\Sigma(q)).$$

Thus we get the following.

Lemma (2.5). (1) $r(B_H) \geq \sum_{i=1}^{d-1} \frac{(d-1)}{i} \cdot l_A(H^i_m(A))$.

(2) $r(B_H) \leq l_A(\text{Hom}_A(k, A/\Sigma(q))) + \sum_{i=1}^{d-1} \frac{(d-1)}{i} \cdot l_A(H^i_m(A))$.

To estimate the value $l_A(\text{Hom}_A(k, A/\Sigma(q)))$, let us recall a well-known fact that the local cohomology modules $H^i_m(A)$ is isomorphic to the direct limit $\lim_{n \to \infty} H^i_m(a^n; A)$ defined by the Koszul (co-)homology modules $H^i_m(a^n; A)$ of $A$ with respect to the system $a^n := a^n_1, \ldots, a^n_d$ ($n > 0$), for each $i$. In particular for the case $i = d$ we have the following direct system:

$$A/(a^n) \to A/(a^{n+1}) \to \cdots \to \lim_{n \to \infty} A/(a^n) \cong H^d_m(A),$$

where the map $A/(a^n) \to A/(a^{n+1})$ is multiplication by the element $a^n_1 \cdots a^n_d$. As we have the equality

$$\sum(a_1^{l_1 n_1}, \ldots, a_d^{l_d n_d}) : a_1^{l_1} \cdots a_d^{l_d} = \sum(a_1^{n_1}, \ldots, a_d^{n_d})$$

for any integers $l_1, \ldots, l_d \geq 0$ and $n_1, \ldots, n_d > 0$, this direct system induces the new one as follows:

$$A/\Sigma(a^n) \to A/\Sigma(a^{n+1}) \to \cdots \to \lim_{n \to \infty} A/\Sigma(a^n) \cong H^d_m(A),$$

where all the induced maps $A/\Sigma(a^n) \to A/\Sigma(a^{n+1})$ are injective, cf.
[7], p. 456. On the other hand, we know

$$
\mu_A(K_A) = l_A(\text{Hom}_A(k, H^d_m(A))) .
$$

Therefore we get

Proposition (2.6). \( r(B^*_H) \leq \mu_A(K_A) + \sum_{i=1}^{d-1} \frac{(d-1)}{i-1} l_A(H^i_m(A)) . \)

Example (2.7). The equality of the formula in Proposition (2.6) is not true in general.

Let \( A \) be a Buchsbaum ring with minimal multiplicity, namely the equality

$$
\rho(A) = 1 + \sum_{i=1}^{d-1} \frac{(d-1)}{i-1} l_A(H^i_m(A))
$$

holds, see [5]. Suppose that \( d \geq 2 \) and \( H^i_m(A) \neq (0) \) for some \( i \) (\( 1 \leq i < d \)). Then we have

\[ \text{Hom}_A(k, B^*_H/QB^*_H) \cong \text{Hom}_A(k, A/[q_{d-1} : a_d] + (a_d)) = m/[q_{d-1} : a_d] + (a_d), \]

namely the map \( \text{Hom}_A(k, B^*_H/QB^*_H) \to \text{Hom}_A(k, A/\Sigma(q)) \) is a zero map. This implies

$$
r(B^*_H) = \sum_{i=1}^{d-1} \frac{(d-1)}{i-1} l_A(H^i_m(A)) < \mu_A(K_A) + \sum_{i=1}^{d-1} \frac{(d-1)}{i-1} l_A(H^i_m(A)) .
$$

3. When does the equality occur in Proposition (2.6)?

Concerning the formula in Proposition (2.6) described as above, we have a natural question as follows:

Question. When does the equality occur in Proposition (2.6)?

In this section we shall try to give an answer to this question.

Throughout this section we still assume that \( A \) is a Buchsbaum ring and \( q = (a_1, \ldots, a_d) \) is a parameter ideal of \( A \). Let \( n \geq 2 \) be an integer and we put

\[ b_1 := a_1^n, \ldots, b_d := a_d^n, \text{ and } c := a_d^{n-1} ; \]
\[ B' := A[\frac{b_1}{b_d}, \ldots, \frac{b_{d-1}}{b_d}] \text{ in } A[\frac{1}{b_d}] \; ; \]
\[ Q' := (\frac{b_1}{b_d}, \ldots, \frac{b_{d-1}}{b_d}, c)B' \; ; \]
\[ M' := mB' + Q' \; . \]

Then we have the following as same as \((2.3)\).

**Lemma (3.1).** (1) The local ring \( B'_M \) is a Cohen-Macaulay ring of dimension \( d \) and \( Q'B'_M \) is a parameter ideal of \( B'_M \).

(2) \( B'_M/Q'B'_M \cong A/[(b_1, \ldots, b_{d-1}) : c] + (c) \) as \( A \)-algebras.

**Proof.** (1) As \( b_1, \ldots, b_d \) forms a system of parameters for \( A \) too, this comes from \((2.3)\) at once.

(2) We may assume that \text{depth } A > 0 . Hence the parameter \( a_d \)
(and hence \( c \) too) is a non-zero-divisor on \( A \). Let \( x \) be an element of \( A \) such that

\[ x = \frac{b_1}{b_d}, \frac{x_1}{b_d}, \ldots, \frac{b_{d-1}}{b_d}, \frac{x_{d-1}}{b_d}, c, \frac{x_d}{b_d} \; , \]

for some elements \( x_1, \ldots, x_d \in (b_d)^l \) and some integer \( l \geq n \). As

\( (b_d)^l = (b_d)^l + (b_1, \ldots, b_{d-1}) \cdot (b_d)^{l-1} \) we have \( x_d = b_{d-1}^l u + v \) for some

\( u \in A \) and \( v \in (b_1, \ldots, b_{d-1}) \cdot (b_d)^{l-1} \). As \( b_{d-1}^l = b_d c \) by

our definition of the element \( c \), we have

\[ b_{d-1}^{l+1}(x - cu) = b_1 x_1 + \ldots + b_{d-1} x_{d-1} + b_d cv \; . \]

As \( [(b_1, \ldots, b_{d-1}) : b_{d-1}^{l+1}] = [(b_1, \ldots, b_{d-1}) : c] \) we get that

\[ x \in [(b_1, \ldots, b_{d-1}) : c] + (c) \; . \]

Now recall the direct system

\[ A/\Sigma(a) \rightarrow A/\Sigma(a^2) \rightarrow \ldots \rightarrow A/\Sigma(a^n) \rightarrow \ldots \rightarrow \text{nil}_m(A) \; , \]

and that all maps of this system are injective. Then consider the following commutative diagram:
where we put $b'_i := b_1', \ldots, b_{d-1}'$. Notice that the two vertical maps on both sides of this diagram are injective too. By (2) of (3.1) this diagram induces the next one.

$$
\begin{array}{c}
\Hom_A(k, A/\Sigma(a_{n-1})) 
\downarrow \quad a_{d-1} \quad \downarrow \quad a_d \\
\Hom_A(k, A/\Sigma(b_0', c)) 
\end{array}
\Rightarrow
\begin{array}{c}
\Hom_A(k, A/\Sigma(a_n)) 
\downarrow \quad a_d \\
\Hom_A(k, A/\Sigma(b_0', c)) 
\end{array}
$$

Choosing the integer $n$ large enough we can make all maps of the top row of this diagram bijective. Then the vertical map on the right of this diagram must be bijective too, i.e.,

$$
\Hom_A(k, A/\Sigma(b_0', c)) \xrightarrow{a_d} \Hom_A(k, A/\Sigma(a_n))
$$

and hence the map on the bottom of this diagram is surjective, namely the following sequence of $k$-vector spaces must be exact.

$$
0 \rightarrow \frac{\Sigma(b_0', c)}{[(b_0') : c] + (c)} \rightarrow \Hom_A(k, A/\Sigma(b_0', c)) \rightarrow \Hom_A(k, A/\Sigma(b_0', c)) \rightarrow 0
$$

Therefore we get the following result, which is an answer to our question described at the beginning of this section.

**Proposition (3.2).** Under the same notations described as above, one has the equality

$$
r(B_0'H') = \mu_{\ast}(K_\ast) + \sum_{i=1}^{d-1} \frac{(d - 1)}{i - 1} \cdot L \left( H^i_m(A) \right)
$$

if the integer $n$ is large enough. (More precisely, according to Proposition (3.8) of [7], this equality holds for all $n \geq 3$.)
4. Reduction step.

In this section, we shall discuss how we reduce our arguments to the case that the residue field $k$ of $A$ is algebraically closed.

Let $q$ be a parameter ideal of $A$, $R(q)$ the Rees algebra of $q$ and $\mathfrak{P}$ a homogeneous prime ideal of $R(q)$ with $\dim R(q)/\mathfrak{P} = 1$. Then by (2.1) the ideal $\mathfrak{P}$ contains the homogeneous prime ideal $m.R(q)$. We recall the following result given by S. Goto [3].

Lemma (4.1) (cf. (3.5) of [3]). Let $A$ be a Buchsbaum ring and let $\overline{k}$ denote the algebraic closure of the residue field $k$ of $A$. Then there exists a Buchsbaum local $A$-algebra $\overline{A}$ with maximal ideal $\overline{m}$ such that (1) $\overline{A}$ is $A$-flat, (2) $\overline{m} = m.A$, and (3) $\overline{k} \cong \overline{A}/\overline{m}$ as $k$-algebras.

Then it is clear that $q.A$ is still a parameter ideal of $\overline{A}$ and the ring $R(q) \otimes_A \overline{A}$ coincides with the Rees algebra of the parameter ideal $q.A$ of $\overline{A}$. We put $R := R(q)$ and $\overline{R} := R \otimes_A \overline{A}$. As $\dim \overline{R}/\mathfrak{P}.\overline{R} = 1$ there is a homogeneous prime ideal, say $\overline{\mathfrak{P}}$, of $\overline{R}$ such that $\overline{\mathfrak{P}} \supset \mathfrak{P}.\overline{R}$ and $\dim \overline{R}/\overline{\mathfrak{P}} = 1$. Then we claim the following.

Lemma (4.2). The fibre $\overline{R} \otimes_{\overline{R}/\overline{\mathfrak{P}}} (R/\mathfrak{P}.R)$ is a Gorenstein ring of dimension 0.

According to the theorem of [2], these lemmas imply the following.

Proposition (4.3). Under the same notations described as above, one has the equality $r(R/\mathfrak{P}) = r(\overline{R})$.

5. Proof of Theorem (1.1).

We use the same notations as in Section 3. Let $\mathfrak{P}$ be a homogeneous prime ideal of the Rees algebra $R(\mathfrak{b})$, where we put $\mathfrak{b} := b_1, \ldots, b_d$. 
such that $\dim \mathbb{R}(\mathfrak{b})/\mathfrak{P} = 1$. Recall that
\[
\text{Proj } \mathbb{R}(\mathfrak{b}) = \bigcup_{t=1}^{d} \text{Spec } \mathbb{A}[\frac{x}{b_t} | x \in (\mathfrak{b})].
\]

It has no loss of generality to assume that the given homogeneous prime ideal $\mathfrak{P}$ corresponds with the prime ideal $\mathfrak{N}$ of $\text{Spec } B^\prime$, where $B^\prime := \mathbb{A}[\frac{x}{b_d} | x \in (\mathfrak{b})]$. Then we have the following from (2.3) and (3.1) at once.

**Lemma (5.1).**
1. $\mathfrak{N}$ is a maximal ideal of $B^\prime$ and $\mathfrak{N} \cap \mathbb{A} = \mathfrak{m}$.
2. $B^\prime/\mathfrak{m}.B^\prime$ is a Cohen-Macaulay local ring of dimension $d$.
3. $B^\prime/\mathfrak{m}.B^\prime$ is isomorphic to the polynomial ring over the residue field $k$ of $\mathbb{A}$ with $d - 1$ indeterminates, as $\mathbb{A}$-algebras.
4. $r(\mathbb{R}(\mathfrak{b})/\mathfrak{P}) = r(B^\prime/\mathfrak{N})$.

Now we are ready to prove Theorem (1.1).

By the reduction described in the preceding section, we may assume that the residue field $k$ of $\mathbb{A}$ is algebraically closed. Then applying Hilbert's Nullstellensatz to the extension field $B^\prime/P$ of $k$, we can find elements $c_1, \ldots, c_{d-1}$ in $\mathbb{A}$ not contained in maximal ideal $\mathfrak{m}$ of $\mathbb{A}$ such that

\[
N = (\frac{b_1}{b_d} - c_1, \ldots, \frac{b_{d-1}}{b_d} - c_{d-1}) + \mathfrak{m}.B^\prime.
\]

Put $b_1 := b_1 - c_1 b_d$, $b_2 := b_2 - c_2 b_d$, $\ldots$, $b_{d-1} := b_{d-1} - c_{d-1} b_d$. Then we have

\[
(b_1, \ldots, b_{d-1}, b_d) = (b_1', \ldots, b_{d-1}', b_d'),
\]

\[
B^\prime = \mathbb{A}[\frac{b_1'}{b_d}, \ldots, \frac{b_{d-1}'}{b_d}],
\]

\[
N = (\frac{b_1'}{b_d}, \ldots, \frac{b_{d-1}'}{b_d}) + \mathfrak{m}.B^\prime.
\]

We set $Q^\prime := (\frac{b_1'}{b_d}, \ldots, \frac{b_{d-1}'}{b_d}, c).B^\prime$. Thus we have
Lemma (5.2). (1) \( B'_{/Q'} B'_{/N} \cong A/[(b'_1, \ldots, b'_{d-1}) : c] + (c) \) as A-algebras.

(2) \([(b'_1, \ldots, b'_{d-1}) : c] + (c) = [(b_1, \ldots, b_{d-1}) : c] + (c)\).

By Lemma (5.2) we get that

\[
r(R(b)_{/B}) = r(B'_{/N}) = l_A(\text{Hom}_A(k, B'_{/Q'} B'_{/N}))
\]

\[
= l_A(\text{Hom}_A(k, A/[(b'_1, \ldots, b'_{d-1}) : c] + (c)))
\]

\[
= l_A(\text{Hom}_A(k, A/[(b_1, \ldots, b_{d-1}) : c] + (c)))
\]

\[
= l_A(\text{Hom}_A(k, B'_{/N} B'_{/N})) = r(B'_{/N})
\]

\[
= \mu_A(K_n) + \sum_{i=1}^{d-1} (d-1)_i l_A(\text{Hom}_A(A))
\]

6. Proof of Theorem (1.2).

We shall here prove Theorem (1.2). By Theorem (1.1) and Goto's blowing-up characterization of Buchsbaum rings, cf. (1.1) of [2], we get the implication \( (2) \implies (1) \) at once.

\( (1) \implies (2) \). Let \( q = (a_1, \ldots, a_d) \) be a parameter ideal of \( A \).

It is enough to show that \( r(R(q)_{/B}) = \mu_A(K_n) \) holds for \( B \in \text{Proj } R(q) \) with \( \text{dim } R(q)_{/B} = 1 \).

Passing the reduction described in Section 4, we may assume that the residue field \( k \) of \( A \) is algebraically closed. Recall that

\[
\text{Proj } R(q) = \coprod_{i=1}^{d} \text{Spec } A[\frac{x}{a_i} \mid x \in q]
\]

We may further assume that the given ideal \( B \) corresponds with a maximal ideal \( N \) of \( B \), where \( B := A[\frac{x}{a_d} \mid x \in q] \). As \( k \) is algebraically closed, we can find elements in \( A \), say \( e_1, \ldots, e_d \), such that \( N = (\frac{a_1}{a_d}, e_1, \ldots, \frac{a_{d-1}}{a_d}, \frac{a_{d-1}}{a_d} - e_{d-1}) + mB \), by Hilbert's Nullstellensatz. Put \( a_i' := a_i - e_i a_d \) for \( 1 \leq i < d \).
Then \((a'_1, \ldots, a'_{d-1}, a_d) = q\), namely \(a'_1, \ldots, a'_{d-1}, a_d\) is a system of parameters for \(A\) too. Moreover we have the followings:

\[
B = A\left[\frac{a'_1}{a_d}, \ldots, \frac{a'_{d-1}}{a_d}\right],
\]

\[
N = \left(\frac{a'_1}{a_d}, \ldots, \frac{a'_{d-1}}{a_d}\right) + m.B.
\]

Therefore we get that

\[
r(B_N) = l_A(\text{Hom}_A(k, A/(a'_1, \ldots, a'_{d-1}, a_d))) = \mu_A(K).
\]

This finishes the proof of Theorem (1.2).

References


