数理解析研究所講究録
第 802 巻 1992 年 128-136

Construction of Rectangular Designs

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Abstract. Some new methods of construction of rectangular designs which are 3-associate partially balanced incomplete block designs with a rectangular association scheme are presented. Such rectangular designs which may be new are tabulated under the range of r, k $\leq$ 10.

1. Introduction

Rectangular designs introduced by Vartak (1955) form a special class of 3-associate partially balanced incomplete block (PBIB) designs based on a rectangular association scheme. The rectangular association scheme is also called a 3-associate Kronecker product association scheme or a 3-associate extended group divisible association scheme. These designs have been studied by Hinkelmann (1964), Kageyama (1974), Raghavarao and Aggarwal (1974), Aggarwal (1975, 1977, 1983), Sinha et al. (1979), Kageyama and Tanaka (1981), Bhagwandas et al. (1985), Chang and Hinkelmann (1987), Suen (1989), and so on. These rectangular designs are useful as factorial experiments, having balance as well as orthogonality. Even so, any exhaustive list of such designs is not available. A recent review of constructional procedures for these designs is given by Gupta and Mukerjee (1989).

In this paper, some new methods of constructing rectangular PBIB designs from nested balanced incomplete block (BIB) designs, self-complementary BIB designs and semi-regular group divisible (GD) designs are given.

Definitions of nestedness, BIB and GD designs, and a difference set as well as a 3-associate PBIB design can be found in Preece (1967) and Raghavarao (1971).

2. Rectangular designs

Let there be $v = mn$ treatments arranged in a rectangle of m rows and n columns. With respect to each treatment, the first associates are the other n-1 treatments of the same row, the second associates are the other m-1 treatments of the same column, and the remaining (m-1)(n-1) treatments are third associates. For the rectangular association scheme, we get $n_1 = n-1$, $n_2 = m-1$, $n_3 = (m-1)(n-1)$. The treatments which are i-th associates are repeated λ_i times, i=1,2,3, in the design. We are interested in constructing rectangular designs with the concurrence number λ_3 bigger than λ_1 and λ_2, because when these designs are used as the m×n complete confounded factorial experiments, the loss of information on the main effects is small (Suen (1989)). However, the designs as in Theorem 2.4 have λ_3 smaller than λ_1, λ_2, but the construction is of combinatorial interest.

Let I_s be the identity matrix of order s and $J_{s\times t}$ be an s×t matrix with unit elements everywhere. $0_{s\times t}$ denotes an s×t zero matrix and $A\otimes B$ denotes the Kronecker product of matrices A and B. A' is the transpose of the matrix A.

2.1. Using nested BIB designs

Nested BIB designs have been studied by Preece (1967), and so on. These designs shall be used to construct rectangular designs.

Theorem 2.1. The existence of a nested BIB design with parameters

$$v',\ r';\ k_1' = 2k_2',\ k_2'\ ;\ b_1',\ b_2'\ ;\ \lambda_1',\ \lambda_2' \tag{2.1}$$

implies the existence of a rectangular design with parameters

$$v = 2v',\ b = 2b_2',\ r = r',\ k = 2k_2',\quad \lambda_1 = 0,\ \lambda_2 = \lambda_2',$$
$$\lambda_3 = \lambda_1' - \lambda_2';\quad m = v',\ n = 2.$$

Proof. A nested BIB design with parameters (2.1) has each block divided into two sub-blocks. Let N_1 be the incidence matrix for the treatments in the first half blocks and N_2 be the corresponding incidence matrix from the second half blocks. In this case, consider $I_2\otimes N_1 + (1_2 1_2' - I_2)\otimes N_2$ (= N, say) as the incidence matrix of a design, having the rectangular association scheme of the form

$$\begin{matrix} 1 & v'+1 \\ 2 & v'+2 \\ \vdots & \vdots \\ v' & 2v' \end{matrix} \quad . \qquad (2.2)$$

Here $\otimes$ denotes the Kronecker product and $\mathbf{1}_s = (1,\ldots,1)'$ of size s. Since N_1+N_2 and $[N_1:N_2]$ are BIB designs with concurrence numbers λ_1' and λ_2', respectively, our incidence matrix N can yield the required rectangular design. □

2.2. Using self-complementary BIB designs

A block design with parameters v, b, r and k is said to be self-complementary if $v = 2k$ (and then $b = 2r$).

Theorem 2.2. The existence of a self-complementary BIB design with parameters

$$v' = 2k',\ b' = 2r',\ r',\ k',\ \lambda' \qquad (2.3)$$

implies the existence of a rectangular design with parameters

$$v = 2v',\ b = b',\ r = r',\ k = 2k',\ \lambda_1 = 0,\ \lambda_2 = \lambda',$$
$$\lambda_3 = r'-\lambda';\quad m = v',\ n = 2.$$

Proof. Let N be the incidence matrix of the BIB design with parameters (2.3). Then it follows that $[N' : \mathbf{1}_b\mathbf{1}_v' - N']'$ (= M, say) is the incidence matrix of the required rectangular design. The association scheme is the same as in (2.2). □

Remark 1. When $r' > 2\lambda'$, by using M in the proof of Theorem 2.2, the incidence matrix $[M:\mathbf{1}_{r'-2\lambda'}'\otimes(I_2\otimes\mathbf{1}_{v'})]$ yields a semi-regular GD design with parameters $v = 2v'$, $b = 4(r'-\lambda')$, $r = 2(r'-\lambda')$, $k = 2k'$, $\lambda_1 = 0$, $\lambda_2 = r'-\lambda'$, $m = v'$, $n = 2$, which is resolvable if the design with (2.3) is resolvable.

2.3. Using semi-regular GD designs

We consider a GD design with parameters $v' = mn$ (i.e. m groups of n treatments each), b', r', k', λ_1', λ_2', having the association scheme as

$$\begin{matrix} 1 & m+1 & \ldots & (n-1)m+1 \\ 2 & m+2 & \ldots & (n-1)m+2 \\ \vdots & \vdots & & \vdots \\ m & 2m & \ldots & nm \end{matrix} \quad .$$

Here, the treatments in the same row (column) are first (second) associates. In this case, we have

Theorem 2.3. The existence of a semi-regular GD design with parameters $v' = mn$, b', r', $k' = m$, $\lambda_1' = 0$, λ_2', having a set of n blocks as $(1,2,\ldots,m),(m+1,m+2,\ldots,2m),\ldots,((n-1)m+1,(n-1)m+2,\ldots,nm)$, implies the existence of a rectangular design with parameters

$$v = mn,\ b = b'-n,\ r = r'-1,\ k = k',\ \lambda_1 = 0,\ \lambda_2 = \lambda_2'-1,\ \lambda_3 = \lambda_2.$$

Proof. By deleting the set of n blocks described above from the solution of the original semi-regular GD design, we can get the required rectangular design. □

Remark 2. In Theorem 2.3, if the semi-regular GD design has q sets of n blocks $(1,\ldots,m),(m+1,\ldots,2m),\ldots,((n-1)m+1,\ldots,nm)$ each, then the deletion of p sets for $1 \le p \le q$ yields a rectangular design with parameters $v = mn$, $b = b'-np$, $r = r'-p$, $k = k'$, $\lambda_1 = 0$, $\lambda_2 = \lambda_2'-p$, $\lambda_3 = \lambda_2$. The case $q = 1$ is always possible to choose, because one set of n blocks as in Theorem 2.3 can be obtained by renaming treatments properly.

2.4. Using difference sets

Theorem 2.4. If $4t-1$ is a prime or a prime power, then there exist rectangular designs with parameters

$$v = b = 3(4t-1),\ r = k = 4t-1,\ \lambda_1 = 2t-1,\ \lambda_2 = 2(t-1),\ \lambda_3 = t,\ m = 4t-1,\ n = 3; \tag{2.4}$$

$$v = b = 3(4t-1),\ r = k = 4t-2,\ \lambda_1 = 2t-1,\ \lambda_2 = 2(t-1),\ \lambda_3 = t-1,\ m = 4t-1,\ n = 3; \tag{2.5}$$

$$v = 3(4t-1),\ b = 12t,\ r = 4t,\ k = 4t-1,\ \lambda_1 = \lambda_2 = 2t-1,\ \lambda_3 = t,\ m = 4t-1,\ n = 3. \tag{2.6}$$

Proof. Let $R = GF(3) + GF(4t-1)$, a set of $3(4t-1)$ elements, be the direct sum of Galois fields of orders 3 and $4t-1$, respectively. The addition and multiplication are defined in the usual way, i.e.,

$$(x_1,x_2) + (y_1,y_2) = (x_1+y_1,x_2+y_2)$$
$$(x_1,x_2)(y_1,y_2) = (x_1y_1,x_2y_2)$$

where (x_1,x_2), $(y_1,y_2) \in R$. Let us consider the following array of $3(4t-1)$ elements of R given as: for $q = 4t-1$

$$\begin{matrix} (00), & (10), & (20) \\ (01), & (11), & (21) \\ \vdots & \vdots & \vdots \\ (0q), & (1q), & (2q). \end{matrix}$$

Two elements of R are first or second associates according as they belong to the same row or the same column of the array, otherwise they are third associates. Let $x = (2,a)$ where 2 and a are primitive roots of GF(3) and GF(4t-1), respectively. Then the initial blocks $\{(0,0),(1,1),x,x^2,\ldots,x^{4t-3}\}$ and $\{(1,1),x,x^2,\ldots,x^{4t-3}\}$ developed mod 3(4t-1), yield the required rectangular designs with parameters (2.4) and (2.5), respectively. Furthermore, by adding three blocks each consisting of 4t-1 elements of the three columns of the above array to a design with (2.4), we can get a rectangular design with parameters (2.6). In fact, the initial block of the rectangular design with parameters (2.4) consists of $(0,0),(1,1),(1,a^2),(1,a^4),\ldots,(1,a^{4t-4})$, $(2,a),(2,a^3),\ldots,(2,a^{4t-3})$. Since it follows that $(1,a^2,a^4,\ldots,a^{4t-4})$ and $(a,a^3,a^5,\ldots,a^{4t-3})$ are two difference sets of a BIB design with parameters $v' = b' = 4t-1$, $r' = k' = 2t-1$, $\lambda' = t-1$, the set of differences in the difference sets contain each element of the form $(0,b)$, $b \neq 0$, $2(t-1)$ times and each element of the form $(a,0)$, $a \neq 0$, occur $2t-1$ times, and the elements of the form (a,b), $a,b \neq 0$, occur t times. Thus we get the parameters (2.4). The parameters of the other designs follow in the similar way. □

Theorem 2.5. There exists a rectangular design with parameters

$$v = 8t,\ b = 2(4t-1),\ r = 4t-1,\ k = 4t,\ \lambda_1 = 0,\ \lambda_2 = 2t-1,$$
$$\lambda_3 = 2t,\ m = 4t,\ n = 2$$

when 4t-1 is a prime or a prime power.

Proof. It is known (cf. Dey (1986;page 136)) that there exists a BIB design with parameters $v = 4t$, $b = 2(4t-1)$, $r = 4t-1$, $k = 2t$, $\lambda = 2t-1$, when 4t-1 is a prime or a prime power. This design is generated by two initial blocks $(\infty,x^0,x^2,x^4,\ldots,x^{4(t-1)})$ and $(0,x,x^3,\ldots,x^{4t-3})$. Now let N_1 and N_2 be the incidence matrices of the subdesigns generated by the above two initial blocks respectively. Since $[N_1:N_2]$ and N_1+N_2 are a BIB design and a complete block design with concurrence numbers 2t-1 and 4t-1 respectively, the pattern

$$N^* = I_2 \otimes N_1 + (1_2 1_2' - I_2) \otimes N_2$$

gives the incidence matrix of a rectangular design with the required parameters. In fact, $\lambda_3 = 4t-1-(2t-1) = 2t$. □

Theorem 2.6. There exists a rectangular design with parameters

$$v = b = 2(4t+1),\ r = k = 4t,\ \lambda_1 = 0,\ \lambda_2 = 2t-1,\ \lambda_3 = 2t,$$
$$m = 4t+1,\ n = 2$$

when 4t+1 is a prime or a prime power.

Proof. Let N_1 and N_2 be the incidence matrices of the designs generated by the initial blocks $(x^0,x^2,\ldots,x^{4t-2})$ and $(x,x^3,\ldots,$

x^{4t-1}). Then $[N_1:N_2]$ is the incidence matrix of a BIB design with parameters $v = 4t+1$, $b = 2(4t+1)$, $r = 4t$, $k = 2t$, $\lambda = 2t-1$, and N_1+N_2 is a BIB design with $\lambda = 4t-1$. Hence it follows that the pattern $I_2 \otimes N_1 + (1_2 1_2' - I_2) \otimes N_2$ provides a rectangular design with the required parameters. In fact, $\lambda_3 = 4t-1-(2t-1) = 2t$. □

Table 1 gives a list of rectangular designs with $r, k \le 10$ which can be constructed by Theorems 2.1 to 2.6. In Source, for example, R-Ser. 1 means a BIB design of Series 1 in Raghavarao (1971; Table 5.10.1), while P-No. 1 means a nested BIB design of No. 1 in Preece (1967; Table 3). Further, the GD designs in the table (denoted by SR 6, SR 7, etc.) are the designs listed by Clatworthy (1973). In the design of No.7, we interchange treatments 4 and 10, and also 6 and 8, before application of Theorem 2.3 to SR 11.

Table 1. Rectangular designs with $r, k \le 10$

No.	v	m	n	b	r	k	λ_1	λ_2	λ_3	Theorem	Source
1	6	2	3	6	2	2	0	0	1	2.3	SR 6
2	6	2	3	15	5	2	0	1	2	2.3	SR 7
3	6	2	3	24	8	2	0	2	3	2.3	SR 8
4	6	2	3	21	7	2	0	1	3	Remark 2	SR 8, p=2
5	6	2	3	18	6	2	0	0	3	Remark 2	SR 8, p=3
6	8	2	4	12	3	2	0	0	1	2.3	SR 9
7*	10	2	5	20	4	2	0	0	1	2.3	SR 11
8	12	2	6	30	5	2	0	0	1	2.3	SR 13
9	14	2	7	42	6	2	0	0	1	2.3	SR 14
10	16	2	8	56	7	2	0	0	1	2.3	SR 15
11	18	2	9	72	8	2	0	0	1	2.3	SR 16
12	20	2	10	90	9	2	0	0	1	2.3	SR 17
13	9	3	3	6	2	3	0	0	1	2.3	SR 23
14	15	3	5	20	4	3	0	0	1	2.3	SR 28
15	21	3	7	42	6	3	0	0	1	2.3	SR 31
16	24	3	8	56	7	3	0	0	1	2.3	SR 32
17	27	3	9	72	8	3	0	0	1	2.3	SR 33
19	8	4	2	6	3	4	0	1	2	2.2 2.5(t=1)	R-Ser. 1
20	8	4	2	18	9	4	0	4	5	2.3	SR 40

21	10	5	2	10	4	4	0	1	2	2.1 2.6(t=1)	P-No. 1
22	12	4	3	15	5	4	0	1	2	2.3	SR 42
23	12	6	2	30	10	4	0	2	4	2.1	P-No. 7
24	12	4	3	24	8	4	0	2	3	2.3	SR 43
25	16	4	4	12	3	4	0	0	1	2.3	SR 44
26	16	8	2	28	7	4	0	1	2	2.1	P-No. 3(i)(ii)
27	20	4	5	20	4	4	0	0	1	2.3	SR 45
28	28	4	7	42	6	4	0	0	1	2.3	SR 48
29	32	4	8	56	7	4	0	0	1	2.3	SR 49
30	36	4	9	72	8	4	0	0	1	2.3	SR 50
31	15	5	3	15	5	5	0	1	2	2.3	SR 56
32	15	5	3	24	8	5	0	2	3	2.3	SR 57
33	20	5	4	28	7	5	0	1	2	2.3	SR 59
34	25	5	5	20	4	5	0	0	1	2.3	SR 60
35	35	5	7	42	6	5	0	0	1	2.3	SR 62
36	40	5	8	56	7	5	0	0	1	2.3	SR 63
37	45	5	9	72	8	5	0	0	1	2.3	SR 64
38	12	6	2	14	7	6	0	3	4	2.3	SR 69
39	12	6	2	10	5	6	0	2	3	2.2	R-Ser. 7
40	14	7	2	14	6	6	0	2	3	2.1	P-No. 2
41	18	6	3	15	5	6	0	1	2	2.3	SR 72
42	18	6	3	24	8	6	0	2	3	2.3	SR 73
43	18	9	2	24	8	6	0	2	3	2.1	P-No. 4(ii)
44	21	7	3	21	6	6	3	2	1	2.4	
45	24	5	4	28	7	6	0	1	2	2.3	SR 74
46	42	6	7	42	6	6	0	0	1	2.3	SR 77
47	48	6	8	56	7	6	0	0	1	2.3	SR 78
48	54	6	9	72	8	6	0	0	1	2.3	SR 79
49	14	7	2	14	7	7	0	3	4	2.3	SR 82
50	21	7	3	24	8	7	0	2	3	2.3	SR 85
51	21	7	3	24	8	7	3	3	2	2.4	
52	28	7	4	28	7	7	0	1	2	2.3	SR 86
53	49	7	7	42	6	7	0	0	1	2.3	SR 87
54	56	7	8	56	7	7	0	0	1	2.3	SR 88
55	63	7	9	72	8	7	0	0	1	2.3	SR 89
56	16	8	2	14	7	8	0	3	4	2.2 2.5(t=2)	R-Ser. 15
57	18	9	2	18	8	8	0	3	4	2.1 2.6(t=2)	P-No. 4(i)
58	24	8	3	24	8	8	0	2	3	2.3	SR 94

59	32	8	4	28	7	8	0	1	2	2.3	SR 95
60	64	8	8	56	7	8	0	0	1	2.3	SR 97
61	72	8	9	72	8	8	0	0	1	2.3	SR 98
62	18	9	2	12	7	9	0	3	4	2.3	SR 100
63	27	9	3	24	8	9	0	2	3	2.3	SR 102
64	81	9	9	72	8	9	0	0	1	2.3	SR 105
65	20	10	2	18	9	10	0	4	5	2.2	R-Ser. 26
66	22	11	2	22	10	10	0	4	5	2.1	P-No. 8
67	33	11	3	33	10	10	5	4	2	2.4	

Remark 3. Since BIB designs of R-Ser. 1 and 15 and copies of BIB designs of R-Ser. 7 and 26 are resolvable ones with $r' > 2\lambda'$, by Remark 1, we can get some resolvable semi-regular GD designs.

Additional remark. In Table 1, we do not list designs constructed easily by the Kronecker product of incidence matrices of two BIB designs. Another fundamental method of constructing rectangular designs is by some patterned approach of using incidence matrices of BIB designs. Mostly, the approach will produce designs of relatively large values of parameters and with small values of m. Hence they are not discussed here.

Acknowledgement

This was prepared while the first author visited Department of Mathematics, Hiroshima University, in September 1991. The financial support provided by Indian National Science Academy, New Delhi and Japan Society for Promotion of Science, Tokyo is gratefully acknowledged.

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