

## Factorization of graphs with common transversal sets

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**Abstract.** In this paper,  $S_k$ -factorization algorithm of an edge-disjoint sum of two  $K_k$ -factors of  $K_v$  is given.

### 1. Introduction

Let  $G$  and  $H$  be graphs. A spanning subgraph  $F$  of  $G$  is called an  $H$ -factor if and only if each component of  $F$  is isomorphic to  $H$ . If  $G$  is expressible as an edge-disjoint sum of  $H$ -factors, then this sum is called an  $H$ -factorization of  $G$ . Let  $S_k$  be a *star* with  $k$  vertices. Let  $K_k$  and  $K_v$  be a *complete graph* with  $k$  vertices and  $v$  vertices, respectively.

### 2. Common transversal sets

We use common transversal sets on  $S_k$ -factorization of an edge-disjoint sum of two  $K_k$ -factors of  $K_v$ .

Consider a set  $N$  of  $v$  elements, where  $v=kt$ . Divide  $N$  into  $t$  subsets  $A_1, A_2, \dots, A_t$  so that they are mutually disjoint subsets of same size  $k$ . And divide  $N$  into another  $t$  subsets  $B_1, B_2, \dots, B_t$  so that they are mutually disjoint subsets of same size  $k$ . Let  $T$  be a  $t$ -element subsets of  $N$ . Then  $T$  is called a *common transversal set* of  $\{A_1, A_2, \dots, A_t\}$  and  $\{B_1, B_2, \dots, B_t\}$  when it holds that  $|T \cap A_j| = |T \cap B_j| = 1$ ,  $1 \leq j \leq t$ .

**Lemma 1.** Let  $\{A_1, A_2, \dots, A_t\}$  be a mutually disjoint partition of  $N$  and  $\{B_1, B_2, \dots, B_t\}$  be another mutually disjoint partition of  $N$ . Then there exists a common transversal set  $T$  of  $\{A_1, A_2, \dots, A_t\}$  and  $\{B_1, B_2, \dots, B_t\}$ .

### 3. $S_k$ -factorization algorithm of an edge-disjoint sum of two $K_k$ -factors of $K_v$

For  $k \geq 3$ , we have the following:

**Lemma 2.** *An edge-disjoint sum of two  $K_k$ -factors of  $K_v$  can be factorized into  $k$   $S_k$ -factors.*

*Proof.* Let  $F_1$  and  $F_2$  be edge-disjoint  $K_k$ -factors of  $K_v$ . And let  $G$  be a sum of  $F_1$  and  $F_2$ . Then  $V(F_1)=V(F_2)=V(G)=V(K_v)$ . Put  $F_1=K_k^{(1)} \cup K_k^{(2)} \cup \dots \cup K_k^{(t)}$  and  $F_2=K_k^{(t+1)} \cup K_k^{(t+2)} \cup \dots \cup K_k^{(2t)}$ . And let  $A_j=V(K_k^{(j)})$  and  $B_j=V(K_k^{(t+j)})$ ,  $1 \leq j \leq t$ . Then  $\{A_1, A_2, \dots, A_t\}$  is a mutually disjoint partition of  $V(G)$  and  $\{B_1, B_2, \dots, B_t\}$  is another mutually disjoint partition of  $V(G)$ . Let  $T_1$  be a common transversal set of  $\{A_1, A_2, \dots, A_t\}$  and  $\{B_1, B_2, \dots, B_t\}$ . Let  $T_j$  be a common transversal set of  $\{A_1-T, A_2-T, \dots, A_t-T\}$  and  $\{B_1-T, B_2-T, \dots, B_t-T\}$ , where  $T=T_1+T_2+\dots+T_{j-1}$  ( $2 \leq j \leq t$ ). Consider  $2k-2$  subgraphs  $G_j$  of  $G$  such as  $G_1=F_1$ ,  $G_2=F_2$ ,  $G_j=G_{j-2}-T_{j-2}$  ( $3 \leq j \leq 2k-2$ ), where  $T_{k+i}=T_{k-i+1}$  ( $1 \leq i \leq k-2$ ). Consider  $2k-2$  subgraphs  $H_j$  of  $G$  such as  $H_j=G_j-E(G_j-T_j)$  ( $1 \leq j \leq 2k-4$ ),  $H_{2k-3}=G_{2k-3}$ ,  $H_{2k-2}=G_{2k-2}$ .

Note that every component of  $G_j$  is a complete graph with  $(2k-j+1)/2$  vertices ( $j$ :odd) or  $(2k-j+2)/2$  vertices ( $j$ :even) and that every component of  $H_j$  is a star with  $(2k-j+1)/2$  vertices ( $j$ :odd) or  $(2k-j+2)/2$  vertices ( $j$ :even).

Then we can construct  $k$  edge-disjoint  $S_k$ -factors  $F_1', F_2', F_3, \dots, F_k$  of  $G$  as follows:

$$F_1'=H_1, F_2'=H_2, F_j=H_j \cup H_{2k-j+1} \quad (3 \leq j \leq k).$$

Therefore, it holds that  $G=F_1'(+ )F_2'(+ )F_3(+ )\dots(+ )F_k$ , which is an  $S_k$ -factorization.

□

**Note 1.** The symbol (+) is used to denote the sum of factors.

As a resolvable BIBD( $v, b, r, k, \lambda = 1$ ) is just a  $K_k$ -factorization of  $K_v$ , we have the following lemmas.

**Lemma 3.** *If there exists a resolvable BIBD( $v, b, r, k, \lambda = 1$ ) ( $r$ :even), then  $K_v$  has an  $S_k$ -factorization.*

**Lemma 4.** *If there exists a resolvable BIBD( $v, b, r, k, \lambda = 1$ ) ( $r$ :odd,  $m=v/k$ ), then  $K(m; k)$  has an  $S_k$ -factorization.*

**Note 2.**  $K(m; k)$  is a complete multipartite graph with  $m$  partite sets of  $k$  vertices each.

## References

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