A Short Course on $b$-Functions and Vanishing Cycles

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§0. Introduction.

In this article, we use the notation appearing in [H] freely, and a $\mathcal{D}$-module means a left $\mathcal{D}$-module. Let $X$ be a complex manifold, $f$ a holomorphic function on $X$, and $\mathcal{M}$ a regular holonomic system on $X$. By Riemann-Hilbert (RH) correspondence, $\text{DR}(\mathcal{M})$ is a perverse sheaf. Hence its nearby cycle $p\psi_f(\text{DR}(\mathcal{M}))$ and vanishing cycle $p\phi_f(\text{DR}(\mathcal{M}))$ are perverse sheaves on $f^{-1}(0)$. If $f^{-1}(0)$ is a smooth hypersurface, again by RH correspondence there should be holonomic $\mathcal{D}_{f^{-1}(0)}$-modules $\mathcal{M}'$ and $\mathcal{M}''$ such that $p\psi_f(\text{DR}(\mathcal{M})) = \text{DR}(\mathcal{M}')$ and $p\phi_f(\text{DR}(\mathcal{M})) = \text{DR}(\mathcal{M}'')$. Malgrange [Ma] and Kashiwara [Kv] have given such $\mathcal{M}'$ and $\mathcal{M}''$ by using the notion of $V$-filtration. When $f^{-1}(0)$ is not smooth, the situation is reduced to the smooth case by the graph map of $f$. There are already excellent surveys [MS], [S] of this topic. This article may be considered as a very short version of [MS] or [S]. Although most proofs of assertions are omitted, those of Proposition 4.2 and 4.4 are exposed in order to convince readers that morphisms $T$, can, and $\text{var}$ correspond to the counterparts mentioned there.
In §1 we define $b$-functions and look at some examples. In §2 we define $V$-filtrations, which can be calculated by $b$-functions. We also look at some examples again. In §3 we state the stability under standard operations of the category of coherent $c\mathcal{D}$-modules which admit the canonical $V$-filtrations. In §4 we define moderate nearby cycles and moderate vanishing cycles, which turn out to be quasi-isomorphic to certain graded pieces of the canonical $V$-filtration. In §5 we recall nearby cycles and vanishing cycles, and state the main theorem (Theorem 5.1).

§1. $b$-Functions.

Let $X$ be a complex manifold and $f$ a holomorphic function on it. We set $\mathcal{D}_X[s] := \mathcal{D}_X \otimes_{\mathbb{C}} \mathbb{C}[s]$ where $s$ is an indeterminate central element. Let $\mathcal{I}_f$ denote the left ideal of $\mathcal{D}_X[s]$ consisting of all operators $P(s, x, D)$ in $\mathcal{D}_X[s]$ such that $P(s, x, D)f(x)^s = 0$ holds for a generic $x$. A $\mathcal{D}_X[s]$-module $\mathcal{N}_f := \mathcal{D}_X[s]/\mathcal{I}_f$ has a $\mathcal{D}_X$-linear endomorphism $t$ defined by $P(s)f^s \mapsto P(s+1)f^{s+1}$. Since we have $[t, s] = t$, $\mathcal{M}_f := \mathcal{N}_f/t\mathcal{N}_f$ is a $\mathcal{D}_X[s]$-module.

**Definition 1.1 [SSM], [Be]:** The minimal polynomial $b(s)$ of the multiplication by $s$ on $\mathcal{M}_f$ is said to be the $b$-function of $f$.

**Theorem 1.2 [Be], [Bj], [Kb].** The $\mathcal{D}_X$-module $\mathcal{M}_f$ is holonomic and the
b-function of $f$ locally exists.

Example 1.3 [Mi], [Y]: Let $X = \mathbb{C}^n$, $x_1, \ldots, x_n$ a coordinate system on $X$ and $D_i = \frac{\partial}{\partial x_i}$ $(1 \leq i \leq n)$. We assume $f$ to have an isolated singularity at the origin and $f(0) = 0$. We suppose that there exist $v = \sum_{i=1}^{n} r_i x_i D_i$, $r \in \mathbb{Z}_{>0}$, $r_1, \ldots, r_n \in \mathbb{Z}_{\geq 0}$ such that $v(f) = f$. The $b$-function of $f$ at a point where $df$ does not vanish is $s + 1$. Hence $s + 1$ is also a factor of the $b$-function $b(s)$ of $f$ at the origin. Since $vf^* = sf^*$, $\mathcal{M}_f$ is a singly generated $\mathcal{D}_X$-module. Let $\bar{\mathcal{M}}_f = (s + 1)\mathcal{M}_f$ and $\bar{b}(s)$ denote the minimal polynomial of $s$ on $\mathcal{M}_f$. Then we see that $b(s) = (s + 1)\bar{b}(s)$ and $\mathcal{M}_f = \mathcal{D}_X/\mathcal{D}_X f_1 + \cdots + \mathcal{D}_X f_n$ where $f_i = D_i(f)$. Let $v^*$ be the adjoint operator of $v$, i.e., $v^* = -\sum_{i=1}^{n} \frac{r_i}{r}(x_i D_i + 1)$. Then we see $\bar{b}(s) = \text{the minimal polynomial of } v^* \text{ on } \mathcal{O}_X/(f_1, \ldots, f_n)$. For a monomial $x^\alpha$ where $\alpha$ is a multi-index, we have $v^*(x^\alpha) = -\sum_{i=1}^{n} \frac{r_i}{r}(\alpha_i + 1)x^\alpha$. We define a set $R$ by $R = \left\{ \sum_{i=1}^{n} \frac{r_i}{r}(\alpha_i + 1) \right\}$ where $\{x^\alpha\}_\alpha$ is a basis for $\mathcal{O}_X/(f_1, \ldots, f_n)$. Then we obtain $b(s) = (s + 1)\prod_{\beta \in R}(s + \beta)$.

Example 1.4: Let $X = \mathbb{C}^n$ and $f = x_1^{e_1} \cdots x_n^{e_n}$ where $e_i \in \mathbb{Z}_{\geq 0}$ $(1 \leq i \leq n)$. It is easy to check $D_1^{e_1} \cdots D_n^{e_n} f^{s+1} = \prod_{i=1}^{n} \prod_{k=1}^{e_i}(e_is + k)f^s$. On the other hand we suppose that there exist an operator $P(s) \in \mathcal{D}_X[s]$ and a nonzero
polynomial $b'(s) \in \mathbb{C}[s]$ such that $P(s)f^{s+1} = b'(s)f^s$. By the relative invariance under the action of $(\mathbb{C}^*)^n$, it is easy to see that there exists $Q(s) \in \mathbb{C}[x_1D_1, \ldots, x_nD_n, s]$ such that $P(s) = Q(s)D_1^{e_1} \cdots D_n^{e_n}$. Therefore we see that the $b$-function of $f$ at the origin is $\prod_{i=1}^{n} \prod_{k=1}^{e_i} (s + \frac{k}{e_i})$.

There are many other examples of $b$-functions which can be calculated. See [Y], for instance, and [SKKO] for $b$-functions of relative invariants of prehomogeneous spaces. More generally Kashiwara has proved in [K2] that for a holonomic $D_X$-module $\mathcal{M}$ and a section $u \in \mathcal{M}$ there exists locally an operator $P(s) \in D_X[s]$ and a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that $P(s)f^{s+1}u = b(s)f^su$. As an application, the holonomicity of $\mathcal{H}^i_{[X|f^{-1}(0)]}(\mathcal{M})$ has been proved there.

§2. $V$-Filtration.

First of all we introduce the lexicographical order in $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\sqrt{-1}$.

Let $Y$ be a smooth closed submanifold of $X$ of codimension one, $\mathcal{I}_Y$ the defining ideal of $Y$. For $k \in \mathbb{Z}$ we define

$$V_kD_X := \{ P \in D_X \mid P\mathcal{I}_Y \subset \mathcal{I}_Y^{i-k} \quad (\forall j \in \mathbb{Z}) \}$$

where $\mathcal{I}_Y^i = \mathcal{O}_X$ for $j \leq 0$. Then $\{ V_kD_X \}_{k \in \mathbb{Z}}$ is an exhaustive increasing
filtration. Let $t$ be a local equation of $Y$ and $D_t$ a local vector field such that $[D_t, t] = 1$. We have $t \in V_{-1}D_X$, $D_t \in V_1D_X$, $gr_0^Y D_X := V_0D_X/V_{-1}D_X = D_Y[tD_t]$ and $V_kD_X = \{ \sum_{k \geq j} a_{ij}(y, D_y) t^i D_t^j \}$.

**Definition 2.1** [Kv], [Ma]: Let $\mathcal{M}$ be a coherent $D_X$-module. An increasing filtration $\{ V_\alpha \mathcal{M} \}_{\alpha \in \mathbb{C}}$ satisfying the following conditions is called the canonical $V$-filtration.

1. $\mathcal{M} = \bigcup_{\alpha \in \mathbb{C}} V_\alpha \mathcal{M}$. Each $V_\alpha \mathcal{M}$ is a coherent $V_0D_X$-submodule.

2. $(V_iD_X)(V_\alpha \mathcal{M}) \subset V_{\alpha+i} \mathcal{M}$ ($\forall \alpha \in \mathbb{C}, \forall i \in \mathbb{Z}$).

3. $t(V_\alpha \mathcal{M}) = V_{\alpha-1} \mathcal{M}$ ($\forall \alpha < 0$).

4. The action of $(tD_t + 1 + \alpha)$ on $gr_\alpha^V \mathcal{M}$ ($\forall \alpha \in \mathbb{C}$) is nilpotent where $gr_\alpha^V \mathcal{M} = V_\alpha \mathcal{M}/V_{<\alpha} \mathcal{M}$ and $V_{<\alpha} \mathcal{M} = \bigcup_{\beta < \alpha} V_\beta \mathcal{M}$.

**Remarks 2.2:**

1. The definition of the canonical $V$-filtration does not depend on the choice of $t$ and $D_t$. The canonical $V$-filtration is unique if it exists.

2. Since the adjoint of $(tD_t + 1 + \alpha)$ is $-(tD_t - \alpha)$, the eigenvalue of $tD_t$ on $gr_\alpha^V \mathcal{N}$ is $\alpha$ for a right $D_X$-module $\mathcal{N}$.

3. $t : gr_\alpha^V \mathcal{M} \rightarrow gr_{\alpha-1}^V \mathcal{M}$ and $D_t : gr_{\alpha-1}^V \mathcal{M} \rightarrow gr_{\alpha}^V \mathcal{M}$ are bijective for
$\alpha \neq 0$.

**Definition 2.3:** We say that a coherent $D_X$-module $\mathcal{M}$ is specializable along $Y$ and we denote $\mathcal{M} \in B_Y$ if the following equivalent conditions are satisfied:

1. For any system of local generators $u_1, \ldots, u_l$ of $\mathcal{M}$ there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that $b(tD_t)u_i \in \sum_{j=1}^{l}(V_{-1}D_X)u_j$ $(1 \leq i \leq l)$.

2. $\mathcal{M}$ admits the canonical $V$-filtration with respect to $Y$ and there exists a finite set $A \subset \mathbb{C}$ such that $\{ \alpha \in \mathbb{C} | \text{gr}^V_\alpha \mathcal{M} \neq 0 \} \subset A + \mathbb{Z}$.

Let $\mathcal{M} \in B_Y$ and $u \in \mathcal{M}$. Then there exists a nonzero polynomial $b(s) \in \mathbb{C}[s]$ such that $b(tD_t)u \in (V_{-1}D_X)u$. The minimal polynomial among such is called the $b$-function of the section $u$. The canonical $V$-filtration of $\mathcal{M}$ is known to be given by $V_\alpha \mathcal{M} = \{ u \in \mathcal{M} | \text{all roots of the } b\text{-function of } u \text{ are greater than or equal to } -\alpha - 1 \}$.

**Proposition 2.4.** Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be an exact sequence of coherent $D_X$-modules. Then we have

1. $\mathcal{M} \in B_Y \iff \mathcal{M}', \mathcal{M}'' \in B_Y$.

2. The induced sequence $0 \to V_\alpha \mathcal{M}' \to V_\alpha \mathcal{M} \to V_\alpha \mathcal{M}'' \to 0$ is exact for
\[ \forall \alpha \in C \text{ if } \mathcal{M} \in B_Y. \]

(3) The induced sequence \[ 0 \rightarrow \text{gr}_\alpha^Y \mathcal{M}' \rightarrow \text{gr}_\alpha^Y \mathcal{M} \rightarrow \text{gr}_\alpha^Y \mathcal{M}'' \rightarrow 0 \] is exact for \( \forall \alpha \in C \text{ if } \mathcal{M} \in B_Y. \)

**Remark 2.5:** Let \( \mathcal{M} \in B_Y. \) Then \( \text{gr}_\alpha^Y \mathcal{M} \) is a coherent \( \text{gr}_0^Y \mathcal{D}_X = \mathcal{D}_Y [tD_t] \)-module for any \( \alpha \in C. \) Since the action of \( (tD_t+1+\alpha) \) is nilpotent on \( \text{gr}_\alpha^Y \mathcal{M}, \)

it is a coherent \( \mathcal{D}_Y \)-module.

**Example 2.6:** Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X \)-module with \( \text{Supp}(\mathcal{M}) \subset Y, \) and \( u \in \mathcal{M}. \) Then there exists \( i \in \mathbb{Z}_{>0} \) such that \( t^i u = 0. \) So we have \( \prod_{k=1}^{i} (tD_t + k)u = D_t^i t^i u = 0. \) Hence we obtain \( \mathcal{M} = \bigoplus_{i \in \mathbb{Z}_{>0}} \mathcal{M}_i \) where \( \mathcal{M}_i = \{ u \in \mathcal{M} | (tD_t + 1 + i) u = 0 \} \), and \( V_\alpha \mathcal{M} = \bigoplus_{i \leq \alpha} \mathcal{M}_i. \)

**Example 2.7:** Let \( \mathcal{M} \) be a coherent \( \mathcal{D}_X \)-module. We assume \( Y \) to be non-characteristic for \( \mathcal{M}, \) i.e., \( \text{Ch}(\mathcal{M}) \cap T^*_Y X \subset T^*_X X. \) Then \( \mathcal{M} \in B_Y. \) The proof could be reduced to the case of \( \mathcal{D}_X v = \mathcal{D}_X / P \) with \( P \in V_N \mathcal{D}_X, \) \( \bar{P} = \bar{D}_t^N \in V_N \mathcal{D}_X / V_{N-1} \mathcal{D}_X \) and \( N \in \mathbb{Z}_{>0} \) where the bar indicates the canonical image. Since \( Pv = 0, \) in \( (V_N \mathcal{D}_X)v/(V_{N-1} \mathcal{D}_X)v \) we have \( \bar{P} \bar{v} = \bar{D}_t^N \bar{v} = 0, \) i.e., \( D_t^N v \in V_{N-1} \mathcal{D}_X v. \) Hence we obtain \( \prod_{k=0}^{N-1} (tD_t - k)v = t^N D_t^N v \in V_{-1} \mathcal{D}_X v. \)

In general, any root of the \( b \)-function of any section of \( \mathcal{M} \) is a nonnegative
integer. Therefore we see by Definition 2.1 (3)

\[ V_\alpha \mathcal{M} = \begin{cases} t^{-[\alpha]-1}\mathcal{M} & \alpha < -1 \\ \mathcal{M} & \alpha \geq -1 \end{cases} \]

where \([\alpha] = \max\{ n \in \mathbb{Z} | n \leq \alpha \}\).

Let \( f \) be a holomorphic function, \( \mathcal{M} \) a holonomic \( \mathcal{D}_X \)-module, \( u \in \mathcal{M} \) and \( Y = f^{-1}(0) \). Let \( i_f \) denote the graph of \( f : X \to X \times \mathbb{C} \) and \( t \) the coordinate of \( \mathbb{C} \) in \( X \times \mathbb{C} \). Then we see \( \mathcal{M} \in B_Y \iff i_{f*}\mathcal{M} \in B_{X \times \{0\}} \).

Furthermore there is the following correspondence under the isomorphism of \( \mathcal{D}_X[s, t] \) onto \( V_0(\mathcal{D}_X[t, D_t]) = \mathcal{D}_X[t, tD_t] \):

\[ s \mapsto -D_t t \]

\[ \mathcal{D}_X[s] f^s u \mapsto V_0(\mathcal{D}_X[t, D_t])u \otimes \delta(t-f) \]

\[ P(s)f^{s+1}u = b(s)f^s u \mapsto P(-D_t t)u \otimes \delta(t-f) = b(-D_t t)u \otimes \delta(t-f) \]

By Kashiwara's result recalled in §1, we obtain:

**Proposition 2.8.** All holonomic \( \mathcal{D}_X \)-modules belong to \( B_Y \).

**§3. Operations in \( B_Y \).**

**Proposition 3.1.** Let \( Y = \{ t = 0 \} \) and \( \mathcal{M} \in B_Y \). Then
\(1\) \(\mathcal{M}[t^{-1}] \in B_Y\).

\(2\) \(\mathcal{H}^j(\mathcal{M}^*) \in B_Y\) for \(\forall j\). Moreover for \(\forall j\) locally we have isomorphisms

\[
\text{gr}_\alpha^V(\mathcal{H}^j(\mathcal{M}^*)) \sim \mathcal{H}^j(\text{gr}_{-\alpha-1}^V(\mathcal{M}^*)) \quad (-1 < \alpha < 0)
\]

and

\[
\text{gr}_\beta^V(\mathcal{H}^j(\mathcal{M}^*)) \sim \mathcal{H}^j(\text{gr}_{\beta}^V(\mathcal{M}^*)) \quad (\beta = -1, 0).
\]

Under these isomorphisms the transpose \(t^*_i\) :

\[
\mathcal{H}^j(\text{gr}_{-1}^V(\mathcal{M}^*)) \rightarrow \mathcal{H}^j(\text{gr}_0^V(\mathcal{M}^*))
\]

corresponds to \(-D_t : \text{gr}_{-1}^V(\mathcal{H}^j(\mathcal{M}^*)) \rightarrow \text{gr}_0^V(\mathcal{H}^j(\mathcal{M}^*))\) and the transpose \(t_i^*\) :

\[
\mathcal{H}^j(\text{gr}_0^V(\mathcal{M}^*)) \rightarrow \mathcal{H}^j(\text{gr}_{-1}^V(\mathcal{M}^*))
\]

corresponds to \(t : \text{gr}_0^V(\mathcal{H}^j(\mathcal{M}^*)) \rightarrow \text{gr}_{-1}^V(\mathcal{H}^j(\mathcal{M}^*))\).

**Proposition 3.2.** Let \(\mathcal{M}\) be a holonomic \(\mathcal{D}_X\)-module and \(i\) the inclusion of \(Y\) into \(X\). Then

\(1\) The restriction \(i^*\mathcal{M}\) is quasi-isomorphic to \(0 \rightarrow \text{gr}_{-1}^V \mathcal{M} \overset{D_t}{\rightarrow} \text{gr}_0^V \mathcal{M} \rightarrow 0\)

where the dot indicates the place of degree zero.

\(2\) For any \(\alpha \in \mathbb{C}\), \(\text{gr}_\alpha^V \mathcal{M}\) is a holonomic \(\mathcal{D}_Y\)-module.

**Proof:** (1) By Remark 2.2 (3) and Proposition 3.1 (2) we have

\[
i^*\mathcal{M}^* \xrightarrow{\sim} (0 \rightarrow \text{gr}_0^V(\mathcal{M}^*) \overset{i}{\rightarrow} \text{gr}_{-1}^V(\mathcal{M}^*) \rightarrow 0)
\]

\[
\xrightarrow{\sim} (0 \rightarrow (\text{gr}_0^V \mathcal{M})^* \overset{i^*}{\rightarrow} (\text{gr}_{-1}^V \mathcal{M})^* \rightarrow 0).
\]

Since \(i^*\mathcal{M} = (i^*\mathcal{M}^*)^*\), we obtain the assertion.
(2) We know that
\[ \mathcal{M} : \text{holonomic} \iff \mathcal{H}^j(\mathcal{M}^*) = 0 \text{ for } j \neq 0. \]

Hence by Proposition 3.1 (2) we obtain \( \mathcal{H}^j((\text{gr}_\alpha^Y \mathcal{M})^*) = 0 \) for \( j \neq 0 \). This means the holonomicity of \( \text{gr}_\alpha^Y \mathcal{M} \).

**Proposition 3.3.** Let \( g : X' \to X \) be a proper morphism of smooth manifolds. We suppose that \( Y' := g^{-1}(Y) \) is a smooth hypersurface and \( \mathcal{M} \in B_Y \), has a global good filtration. Then for any \( j \), we have \( \mathcal{H}^j(\mathbb{R}g_* \mathcal{M}) \in B_Y \) and the canonical \( V \)-filtration of \( \mathcal{M} \) induces the one for \( \mathcal{H}^j(\mathbb{R}g_* \mathcal{M}) \).

§4. Moderate Nearby Cycles and Moderate Vanishing Cycles.

Let \( Y \) be a smooth hypersurface defined by \( t : X \to \mathbb{C} \). For a coherent \( \mathcal{D}_X \)-module \( \mathcal{M} \in B_Y \), \( p \in \mathbb{Z}_{\geq 0} \) and \(-1 \leq \alpha < 0\), we define
\[ \mathcal{M}_{\alpha,p} := \bigoplus_{0 \leq k \leq p} \mathcal{M}[t^{-1}] \otimes e_{\alpha,k} \]
where \( e_{\alpha,k} = t^{\alpha+1}(\text{Log } t)^k/k! \). It is clear that for any \( \beta \in \mathbb{C} \)
\[ V_\beta \mathcal{M}_{\alpha,p} = \bigoplus_{0 \leq k \leq p} V_{\beta+\alpha+1}(\mathcal{M}[t^{-1}]) \otimes e_{\alpha,k}. \]
Then the monodromy $T = \exp(2\pi itD_t)$ induces a $\mathcal{D}_Y$-automorphism on $\mathcal{M}_{\alpha,p}$ by $T(m \otimes e_{\alpha,k}) = m \otimes T(e_{\alpha,k})$, and accordingly on $\text{gr}_{\beta}^V(\mathcal{M}_{\alpha,p})$.

**Definition 4.1:** For $-1 \leq \alpha \leq 0$, we define the moderate nearby cycle $\psi_{t,\alpha}^m(\mathcal{M})$ by

$$
\psi_{t,\alpha}^m(\mathcal{M}) := \lim_{p} \psi_{t,\alpha,p}^m(\mathcal{M})
$$

where $\psi_{t,\alpha,p}^m(\mathcal{M}) := i^*(\mathcal{M}_{\alpha,p})[-1]$.

By Proposition 3.2 we see

$$
\psi_{t,\alpha,p}^m(\mathcal{M}) \xrightarrow{\sim} (0 \rightarrow \text{gr}_{-1}^V(\mathcal{M}_{\alpha,p}) \xrightarrow{D_t} \text{gr}_{0}^V(\mathcal{M}_{\alpha,p}) \rightarrow 0).
$$

We remark that $T$ acts on $\psi_{t,\alpha}^m(\mathcal{M})$ as well.

**Proposition 4.2.** For $\mathcal{M} \in B_Y$ and $-1 \leq \alpha < 0$, we have a quasi-isomorphism $\text{gr}_{\alpha}^V \mathcal{M} \xrightarrow{\sim} \psi_{t,\alpha}^m(\mathcal{M})$. Here the action of $T$ on $\psi_{t,\alpha}^m(\mathcal{M})$ corresponds to that of $\exp(-2\pi itD_t)$.

**Proof:** Since $V_{<0} \mathcal{M} = V_{<0}(\mathcal{M}[t^{-1}])$, we have

$$
\text{gr}_{-1}^V(\mathcal{M}_{\alpha,p}) = \bigoplus_{0 \leq k \leq p} \text{gr}_{\alpha}^V(\mathcal{M}[t^{-1}]) \otimes e_{\alpha,k} = \bigoplus_{0 \leq k \leq p} \text{gr}_{\alpha}^V(\mathcal{M}) \otimes e_{\alpha,k}.
$$

As $\mathcal{M}_{\alpha,p} = \mathcal{M}_{\alpha,p}[t^{-1}]$, we know $\mathcal{H}^0(\psi_{t,\alpha,p}^m(\mathcal{M})) = \text{Ker}(D_t) = \text{Ker} (tD_t : \text{gr}_{-1}^V(\mathcal{M}_{\alpha,p}) \rightarrow \text{gr}_{-1}^V(\mathcal{M}_{\alpha,p}))$. Since $tD_t (m \otimes e_{\alpha,k}) = [(tD_t + \alpha + 1)m] \otimes$
$e_{\alpha,k} + m \otimes e_{\alpha,k-1}$, we see $\sum_{k=0}^{p} m_k \otimes e_{\alpha,k} \in \text{Ker}(tD_t) = \mathcal{H}^{0}(\psi_{t,\alpha,p}(\mathcal{M})) \iff (tD_t + \alpha + 1)m_k + m_{k+1} = 0 \ (0 \leq k \leq p - 1) \iff m_k = [-(tD_t + \alpha + 1)]^k m_0 \ (0 \leq k \leq p)$. Hence for $p$ such that $(tD_t + \alpha + 1)^p = 0$ on $\text{gr}^V_{\alpha}(\mathcal{M})$, the morphism $\text{gr}^V_{\alpha}(\mathcal{M}) \ni m_0 \mapsto \sum_{k=0}^{p} [-(tD_t + \alpha + 1)]^k m_0 \otimes e_{\alpha,k} \in \mathcal{H}^{0}(\psi_{t,p}(\mathcal{M}))$ is isomorphic.

Let $x = \sum_{k=0}^{p} [-(tD_t + \alpha + 1)]^k m_0 \otimes e_{\alpha,k} \in \text{Ker}(tD_t)$. Then we have $0 = (tD_t)x = \sum_{k=0}^{p} [-(tD_t + \alpha + 1)]^k (tD_t)m_0 \otimes e_{\alpha,k} + \sum_{k=0}^{p} [-(tD_t + \alpha + 1)]^k m_0 \otimes (tD_t)e_{\alpha,k}$, and thus $\sum_{k=0}^{p} [-(tD_t + \alpha + 1)]^k m_0 \otimes (2\pi itD_t)e_{\alpha,k} = \sum_{k=0}^{p} [-(tD_t + \alpha + 1)]^k ((-2\pi i tD_t)m_0) \otimes e_{\alpha,k}$. Hence the monodromy $T$ corresponds to $\exp(-2\pi itD_t)$.

Since $t$ induces an isomorphism $\text{gr}^V_{0}(\mathcal{M}_{0,p}) \sim \text{gr}^V_{-1}(\mathcal{M}_{0,p})$, we see $\mathcal{H}^{1}(\psi_{t,\alpha,p}(\mathcal{M})) = \text{Coker}(D_t) = \text{Coker}(D_t : \text{gr}^V_{0}(\mathcal{M}_{0,p}) \rightarrow \text{gr}^V_{0}(\mathcal{M}_{0,p}))$. For $\sum_{k=0}^{p} m_k \otimes e_{\alpha,k} \in \bigoplus_{0 \leq k \leq p} \text{gr}^V_{\alpha+1}(\mathcal{M}[t^{-1}]) \otimes e_{\alpha,k} = \text{gr}^V_{0}(\mathcal{M}_{0,p})$, we have $D_t(\sum_{k=0}^{p} m_k \otimes e_{\alpha,k}) = \sum_{k=0}^{p} ((D_t + \alpha + 1)m_k \otimes e_{\alpha,k} + m_k \otimes e_{\alpha,k-1}) = \sum_{k=0}^{p} m'_k \otimes e_{\alpha,k}$ where $m'_k = (D_t + \alpha + 1)m_k + m_{k+1}$. Hence for $l$ such that $(D_t + \alpha + 1)^l = 0$ on $\text{gr}^V_{\alpha+1}(\mathcal{M}[t^{-1}])$, we have $m \otimes e_{\alpha,k} = D_t(\sum_{i=1}^{l} [-(D_t + \alpha + 1)]^{i-1} m \otimes e_{\alpha,k+i})$ and thus $\mathcal{H}^{l}(\psi_{t,\alpha}(\mathcal{M})) = 0$.

**Definition 4.3:** We define the moderate vanishing cycle $\phi_{t,0}^{m_{0}}(\mathcal{M})$ to be the inductive limit of the mapping cone $\phi_{t,0,p}^{m}(\mathcal{M})$ of the natural morphism...
\[ i^* \mathcal{M}[-1] \to i^* \mathcal{M}_{-1,p}[-1] = \psi_{t,-1,p}^m(\mathcal{M}), \text{ i.e.,} \]

\[ \phi_{t,0,p}^m = (0 \to \text{gr}^{-1}_{-1} \mathcal{M} \xrightarrow{j \oplus -D_t} \text{gr}^{-1}_{-1} \mathcal{M}_{-1,p} \oplus \text{gr}^V_0 \mathcal{M} \xrightarrow{D_t + j} \text{gr}^V_0 \mathcal{M}_{-1,p} \to 0) \]

where \( j \) is the natural morphism \( \mathcal{M} \to \mathcal{M}_{-1,p} = \bigoplus_{0 \leq k \leq p} \mathcal{M}[t^{-1}] \otimes e_{-1,k} \).

We define morphisms \( \text{can} : \psi_{t,-1}^m(\mathcal{M}) \to \phi_{t,0}^m(\mathcal{M}) \) and \( \text{var} : \phi_{t,0}^m(\mathcal{M}) \to \psi_{t,-1}^m(\mathcal{M}) \) by the morphisms \( \text{id} : \psi_{t,-1}^m(\mathcal{M}) \to \phi_{t,0}^m(\mathcal{M}) \) and \( T - \text{id} : \phi_{t,0}^m(\mathcal{M}) \to \psi_{t,-1}^m(\mathcal{M}) \) respectively.

**Proposition 4.4.** For \( \mathcal{M} \in B_Y \), we have a quasi-isomorphism \( \text{gr}^V_0 \mathcal{M} \xrightarrow{\sim} \phi_{t,0}^m(\mathcal{M}) \). Moreover can corresponds to \( D_t : \text{gr}^{-1}_{-1} \mathcal{M} \to \text{gr}^V_0 \mathcal{M} \) and \( \text{var} \) to \( \frac{\exp(-2\pi itD_t) - 1}{itD_t} \).

**Proof:** Let \( x = \sum_{k=0}^p m_k \otimes e_{-1,k} + n_0 \in \text{gr}^{-1}_{-1} \mathcal{M}_{-1,p} \oplus \text{gr}^V_0 \mathcal{M} = \bigoplus_{k=0}^p \text{gr}^{-1}_{-1} \mathcal{M} \otimes e_{-1,k} \oplus \text{gr}^V_0 \mathcal{M} \). Then we can check

\[
x \in \text{Ker}(D_t + j) \iff \begin{cases} m_1 = -tD_t m_0 - tn_0 \\ m_{k+1} = -tD_t m_k \quad (k \geq 1). \end{cases}
\]

Hence we obtain an isomorphism \( \text{gr}^{-1}_{-1} \mathcal{M} \oplus \text{gr}^V_0 \mathcal{M} \xrightarrow{\sim} \text{Ker}(D_t + j) \) defined by \( m_0 + n_0 \mapsto \sum m_k \otimes e_{-1,k} + n_0 \) with (*)}. So we see \( \text{gr}^V_0 \mathcal{M} \xrightarrow{\sim} \mathcal{H}^0(\phi_{t,0}^m) \). Since \( m \equiv D_t m \mod \text{Im}(j \oplus -D_t) \) for \( m \in \text{gr}^{-1}_{-1} \mathcal{M} \), the morphism \( \text{can} \) corresponds to \( D_t : \text{gr}^{-1}_{-1} \mathcal{M} \to \text{gr}^V_0 \mathcal{M} \). The element \( \sum_{k \geq 1} (-tD_t)^{k-1}(-tn) \otimes e_{-1,k} + \)
$n \in \text{Ker}(D_t + j)$ corresponds to $n \in \text{gr}^V_0 M$. Since the coefficient of $(T - id)(\sum_{k \geq 1}(-tD_t)^{k-1}(-tn) \otimes e_{-1,k})$ at $e_{-1,0}$ is $\sum_{k \geq 1}(2\pi i)^k \frac{(-tD_t)^{k-1}}{k!}(-tn)$, the morphism $\text{var}$ corresponds to $[\frac{\exp(-2\pi itD_t)-1}{itD_t}]t : \text{gr}^V_0 M \rightarrow \text{gr}^V_{-1} M$.

§5. Nearby Cycles and Vanishing Cycles.

Let $f$ be a nonconstant holomorphic function on $X$, $i$ the inclusion of $f^{-1}(0)$ into $X$ and $K \in D_c^b(C_X)$. Let $\tilde{C}^\infty$ denote the universal covering of $C^\infty$ and $p$ the natural map $\tilde{X}^\infty := X \times_{C} \tilde{C}^\infty \rightarrow X$. Following [SGA7] we define the nearby cycle $\psi_f(K)$ by $\psi_f(K) := i^{-1}R_p_*p^{-1}K \in D^b_c(C_{f^{-1}(0)})$. The natural morphism $K \rightarrow R_p_*p^{-1}K$ induces a morphism $i^{-1}K \rightarrow \psi_f(K)$, whose mapping cone $\phi_f(K) \in D^b_c(C_{f^{-1}(0)})$ is called the vanishing cycle. By the definition of $\phi_f(K)$ we have the canonical morphism $\text{can} : \psi_f(K) \rightarrow \phi_f(K)$. Associated to the canonical generator of $\pi_1(C^\infty)$ the monodromy automorphism $T$ acts on $\psi_f(K)$ and $\phi_f(K)$. Since $(T - id)|_{-1}K = 0$, $T - id$ induces the variation $\text{var} : \phi_f(K) \rightarrow \psi_f(K)$. For $\lambda \in C^\infty$ we define a subcomplex $\psi_{f,\lambda}(K)$ of $\psi_f(K)$ by

$$\psi_{f,\lambda}(K) := \{ x \in \psi_f(K) \mid (T - \lambda id)^m x = 0 \ (m > 0) \}.$$ 

Since $\psi_f(K) \in D^b_c(C_{f^{-1}(0)})$, we have a quasi-isomorphism $\bigoplus_{\lambda \in C^\infty} \psi_{f,\lambda}(K) \xrightarrow{\sim} \psi_f(K)$. Similarly we have $\bigoplus_{\lambda \in C^\infty} \phi_{f,\lambda}(K) \xrightarrow{\sim} \phi_f(K)$ as well. For
convenience we set $p\psi_f(K) := \psi_f(K)[-1]$ and $p\phi_f(K) := \phi_f(K)[-1]$.

Let $Y$ be a smooth hypersurface of $X$ defined by $t = 0$. When $\mathcal{M}$ is a regular holonomic $D_X$-module, we have quasi-isomorphisms

$$
\text{DR}(\psi_{t,\alpha}^m(\mathcal{M})) \sim_{\text{qis}} p\psi_{t,e^{2\pi i\alpha}}(\text{DR}(\mathcal{M})) \quad (-1 \leq \alpha < 0)
$$

$$
\text{DR}(\phi_{t,0}^m(\mathcal{M})) \sim_{\text{qis}} p\phi_{t,1}(\text{DR}(\mathcal{M}))
$$

(see [SGA7]). Hence we obtain:

**Theorem 5.1 [Kv], [Ma].** For a regular holonomic $D_X$-module $\mathcal{M}$, we have

$$
\text{DR}(\text{gr}_\alpha^V\mathcal{M}) \sim_{\text{qis}} \begin{cases} 
p\psi_{t,e^{2\pi i\alpha}}(\text{DR}(\mathcal{M})) & (-1 \leq \alpha < 0) 
\end{cases}
$$

Moreover under the above quasi-isomorphisms we have the following correspondences:

$$
\exp(-2\pi itD_t) \leftrightarrow T
$$

$$
D_t : \text{gr}_{-1}^V\mathcal{M} \rightarrow \text{gr}^V_0\mathcal{M} \leftrightarrow \text{can} : p\psi_{t,1}(\mathcal{M}) \rightarrow p\phi_{t,1}(\mathcal{M})
$$

$$
\frac{[\exp(-2\pi itD_t) - 1]}{tD_t} \leftrightarrow \var : p\phi_{t,1}(\mathcal{M}) \rightarrow p\psi_{t,1}(\mathcal{M}).
$$
Corollary 5.2. For a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$, we have

$$
\phi_t(D_X(\text{DR}(\mathcal{M}))) \stackrel{\sim}{\longrightarrow} D_Y \phi_t(\text{DR}(\mathcal{M}))
$$

$$
\psi_t(D_X(\text{DR}(\mathcal{M}))) \stackrel{\sim}{\longrightarrow} D_Y \psi_t(\text{DR}(\mathcal{M})).
$$

References


