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TOPOLOGICAL TRIVIALITY OF REAL ANALYTIC SINGULARITIES

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Introduction and Results

We consider a deformation F of a real analytic map germ $f:(\mathbf{R}^n,0)\to(\mathbf{R}^p,0)$ and are interested to know when such deformation F of f and deformation $F^{-1}(0)$ of the zero locus $f^{-1}(0)$ are topologically trivial.

Among other criterions for topological triviality, for a deformation of a holomorphic function, Lê and Ramanujam [L-R] and Timourian [T] showed that the constancy of Milnor number implies the topological triviality. In the real analytic case, this type of theorem is known only in very low dimensional cases (Damon and Gaffney [D-G]).

In this talk we present Lê-Ramanujam-Timourian type theorems for deformations of real analitic functions and deformations of isolated singularities of real analytic sets. We define a local algebra associated with a singularity and show that the constancy of dimendion of the algebra , in the function case togegher with the constancy of Milnor number, implies topological triviality.

Let $F(x,t):(\mathbf{R}^n\times[a,b],\{0\}\times[a,b])\to(\mathbf{R}^p,0)$ be a real analytic map defined in a neighbourhood of $\{0\}\times[a,b]$ in $\mathbf{R}^n\times[a,b]$. We assume that $F(0,t)\equiv 0$. We write $F_t:(\mathbf{R}^n,0)\to(\mathbf{R}^p,0)$ for the restricted map defined by $F_t(x)=F(x,t)$. We say that the zero locus $F^{-1}(0)$, or equivalently the family $\{F_t^{-1}(0)\}_{t\in[a,b]}$, is *toplogically trivial along* the interval $[a,b]$ if there exist a neighbourhood U of $\{0\}\times[a,b]$ in $\mathbf{R}^n\times[a,b]$ and a homeomorphism $h:(U,\{0\}\times[a,b])\to(U_a\times[a,b],\{0\}\times[a,b])$, where $U_a=\{x\in\mathbf{R}^n\mid(x,a)\in U\}$, such that

(1) h preserves the parameter t, i.e. h has the form $h(x,t)=(h_1(x,t),t)$,

(2) $$h(F^{-1}(0)\cap U)=(U_a\cap F_a^{-1}(0))\times[a,b]).$$

Now consider a deformation $F(x,t):(\mathbf{R}^n\times[a,b],\{0\}\times[a,b])\to(\mathbf{R},0)$ of a real analytic function. We say that F, or equivalently the family $F_t(x)_{t\in[a,b]}$, is *topologically trivial along* $[a,b]$ if there exist a neighbourhood U of $\{0\}\times[a,b]$ in

$\mathbf{R}^n \times [a,b]$ and a homeomorphism $h : (U_a \times [a,b], \{0\} \times [a,b] \to (U, \{0\} \times [a,b])$ such that

$$(1) \qquad h \text{ preserves the parameter } t,$$
$$(2) \qquad F \circ h(x,t) = F_a(x).$$

Let $\mathcal{A}_n$ denote the R-algebra of germs of real analytic functions at $0 \in \mathbf{R}^n$. For an analytic map-germ $f = (f_1, ..., f_p) : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, let (f) denote the ideal of $\mathcal{A}_n$ generated by the component functions $f_1, ..., f_p$, and let $\mathcal{A}_n/(f)$ be its quotient algebra.

In the case $n \leq p$, via an elementary argument of complex algebraic geometry, we have the following sufficient condition for $F^{-1}(0)$ to be topologically trivial.

Proposition. *Suppose $n \leq p$ and let $F(x,t) : (\mathbf{R}^n \times [a,b], \{0\} \times [a,b]) \to (\mathbf{R}^p, 0)$ be a real analytic map defined in a neighbourhood of $\{0\} \times [a,b]$ in $\mathbf{R}^n \times [a,b]$ such that $F(0,t) \equiv 0$. If $dim_R \mathcal{A}_n/(F_t) \equiv const < \infty$, then the family $\{F_t^{-1}(0)\}$ is topologically trivial along $[a,b]$, moreover $F^{-1}(0) \cap (U(0) \times [a,b]) = \{0\} \times [a,b]$ for a small neighbourhood $U(0)$ of 0 in $\mathbf{R}^n$*

This proposition will play an important role in the proof of the main theorems.

Now let's consider the other case $n > p$. Let $f = (f_1, ..., f_p) : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ be an analytic map-germ. Let $\rho : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ be a polynomial such that $\rho^{-1}(0) = \{0\}$. We call such a polynomial ρ a *control function* of $(\mathbf{R}^n, 0)$. An important and typical example of such ρ is $\rho = {x_1}^2 + \ldots + {x_n}^2$, where $(x_1, ..., x_n)$ is the standard coordinate system of $\mathbf{R}^n$. Consider the map-germ $(f, \rho) : (\mathbf{R}^n, 0) \to (\mathbf{R}^p \times \mathbf{R}, (0,0))$. Let $J(f, \rho)$ be the jacobian ideal of the map-germ (f, ρ) generated by the $(p+1) \times (p+1)$ minors of the jacobian matrix

$$\frac{D(f,\rho)}{D(x_1, ..., x_n)} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial x_1} & \cdots & \frac{\partial f_p}{\partial x_n} \\ \frac{\partial \rho}{\partial x_1} & \cdots & \frac{\partial \rho}{\partial x_n} \end{pmatrix}$$

Consider the ideal $(f, J(f, \rho))$ of $\mathcal{A}_n$ generated by the ideals (f) and $J(f, \rho)$ and consider the quotient algebra $\mathcal{A}_n/(f, J(f, \rho))$.

Now we can state one of our two main theorems.

Theorem A. *Let $n > p$ and let $F(x,t) : (\mathbf{R}^n \times [a,b], \{0\} \times [a,b]) \to (\mathbf{R}^p, 0)$ be a real analytic map defined in a neighbourhood of $\{0\} \times [a,b]$ in $\mathbf{R}^n \times [a,b]$ such that $F(0,t) \equiv 0$. We set $F_t(x) = F(x,t)$. If for a control function ρ*

$$dim_{\mathbf{R}} \mathcal{A}_n/(F_t, J(F_t, \rho)) \equiv const < +\infty$$

, *then the family* $\{F_t^{-1}(0)\}_{t\in[a,b]}$ *is topologically trivial along* $[a,b]$.

Next we state a sufficient condition for a family of real analytic functions to be topologically trivial. For an analytic function germ $f:(\mathbf{R}^n,0)\to(\mathbf{R},0)$ let $\mu(f)$ denote the Milnor number of f:

$$\mu(f)=dim_{\mathbf{R}}\mathcal{A}_n/(\partial f/\partial x_1,\dots,\partial f/\partial x_n)$$

Theorem B. *Let* $F(x,t):(\mathbf{R}^n\times[a,b],\{0\}\times[a,b])\to(\mathbf{R},0)$ *be an analytic family of real analytic functions such that* $F(0,t)\equiv 0$. *If*

$$\mu(F_t)\equiv const<\infty \quad \text{and}$$
$$dim_{\mathbf{R}}\mathcal{A}_n/(F_t,J(F_t,\rho))\equiv const<\infty,$$

then the family $\{F_t\}$ *is topologically trivial along* $[a,b]$.

Complete proofs of Proposition and Theorems A and B are given in [F]. The proof of Proposition adopted in [F] was given by Shihoko Ishii at my request.

Remark 1. Damon and Gaffney [D-G] showed that when n=1,2,3 the Lê-Ramanujam-Timourian theorem holds also for the real case: *constancy of Milnor number implies constancy of topological types of analytic functions.*

Remark 2. The condition posed on f that $dim_{\mathbf{R}}\mathcal{A}_n/(f,J(f,\rho))<\infty$ is a generic condition in the strong sense that the set of analytic map-germs g with $dim_{\mathbf{R}}\mathcal{A}_n/(g,J(g,\rho))=\infty$ is ∞-codimensional in the set of all analytic map-germs $(\mathbf{R}^n,0)\to(\mathbf{R}^p,0)$. This fact will be proved in Section 5.

Remark 3. When $n=p$, for a map germ $f:(\mathbf{R}^n,0)\to\mathbf{R}^n,0)$, the algebra $\mathcal{A}_n/(f)$ contains an interesting topological information on f. Eisenbud and Levine gave a beautiful algebraic formula for the topological degree of f. If $dim_{\mathbf{R}}\mathcal{A}_n/(f)<\infty$, then the jacobian determinant Jf of f is non-zero in $\mathcal{A}_n/(f)$. Let $\varphi:\mathcal{A}_n/(f)\to\mathbf{R}$ be any linear function with $\varphi(Jf)>0$. Define a bilinear form $<\ ,\ >_\varphi$: $\mathcal{A}_n/(f)\times\mathcal{A}_n/(f)\to\mathbf{R}$ by $<\alpha,\beta>_\varphi=\varphi(\alpha\beta)$. Then we have

Theorem ([E-L]). *The toplological degree of* f *coincides with the signature of* $<\ ,\ >_\varphi$.

Remark 4. When $p=n-1$, the signature of the algebra $\mathcal{A}_n/(f,J(f,\rho))$, associated as follows, determines the topology of $f^{-1}(0)$ ([AFS],[AFN1]).

If $dim_{\mathbf{R}}\mathcal{A}_n/(f,J(f,\rho))<\infty$, then the zero locus $f^{-1}(0)$ consists of curves passing through $0\in\mathbf{R}^n$ and the number of branches of $f^{-1}(0)$ determines the topology of $f^{-1}(0)$. Since $p=n-1$, the ideal $J(f,\rho)$ is generated by the jacobian determinant of the map (f,ρ), which we denote by the same symbol $J(f,\rho)$. Then we have a new map germ $(f,J(f,\rho)):(\mathbf{R}^n,0)\to(\mathbf{R}^n,0)$.

Theorem ([AFS], [AFN1]). *If* $dim_{\mathbf{R}}\mathcal{A}_n/(f, J(f,\rho)) < \infty$, *then the number of branches of* $f^{-1}(0)$ *coincides with the topological degree of the map germ* $(f, J(f,\rho))$, *hence via the Eisenbud-Levine theorem it coincides with the signature of the associated bilinear form on* $\mathcal{A}_n/(f, J(f,\rho))$.

This theorem was extended to various directions by several authors ([AFN2], [D1],[D2],[M-S],[S]).

Because of this fact together with Theorems A and B, the author believes that one can extract various topological informations on $f^{-1}(0)$ from $\mathcal{A}_n/(f, J(f,\rho))$.

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