数理解析研究所講究録
第 807 巻 1992 年 12-31

COMPLETELY INTEGRABLE FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

SHYUICHI IZUMIYA (泉屋周一)
Department of Mathematics, Hokkaido University, (北大理)
Sapporo 062, Japan

Abstract. This is a summary of recent papers ([5,8,9,10,11,12]). We consider some properties about completely integrable first order differential equations for real-valued functions. In the first part, we will give the general frame work and introduce the theory of Legendrian unfoldings in order to study this subject. We give a characterization of equations with classical complete solutions in terms of Legendrian unfoldings. We will consider the single equation case in the second part in where a characterization of complete integrability by the equation itself will be given. Furthermore, we will consider the holonomic case in the third part. We will give a classification of completely integrable holonomic systems by the equivalence relation due to Lie. By the aid of the classification, we will draw some pictures of graphs of complete integrals.

PART 1. GENERAL FRAMEWORK

We now consider the following fundamental problems in the geometric theory of differential equations :

A) Find out a good class of differential equations.

B) Characterize the above class in some sense.

C) Classify equations in the above class by a good equivalence relation.

In this note we stick to completely integrable first order partial differential equations for real-valued functions because this class has very nice general natures and it is a prototype of higher order cases.

In the classical theory, *a system of first order partial differential equations (or, briefly, an equation)* is written in the form $F_k(x_1,\dots,x_n,y,p_1,\dots,p_n)=0$ for $k=1,\dots,2n+1-r, r\geq n$. *A (classical) solution* of the equation is a smooth function $y=f(x_1,\dots,x_n)$ and $p_i=\frac{\partial f}{\partial x_i}(x)$. We usually assume that F_k are $(2n+1)$-variable smooth function and $\operatorname{rank}(\frac{\partial F_k}{\partial x_i},\frac{\partial F_k}{\partial z},\frac{\partial F_k}{\partial p_j})=2n+1-r$. We now define

$$\begin{aligned}\Sigma=&\{(x,y,p)|F_1(x,y,p)=\cdots=F_{2n+1-r}(x,y,p)=0 \text{ and}\\ &\operatorname{rank}(\frac{\partial F_k}{\partial p_j}(x,y,p)<\min(n,2n+1-r)\}.\end{aligned}$$

An (classical) solution of the equation is a smooth function $y=f(x_1,\dots,x_n)$ with $p_i=\frac{\partial f}{\partial x_i}$ which satisfy the relation $F_k=0$. We say that an $(r-n)$-parameter family of (classical) solutions $y=f(t_1,\dots,t_{r-n},x_1,\dots,x_n)$ of the equation is *a (classical) complete solution* if $\operatorname{rank}(\frac{\partial f}{\partial t_j},\frac{\partial^2 f}{\partial t_i\partial x_j})=r-n$.

The following theorem is one of the best results in the classical theory.

THEOREM 1.1. (CLASSICAL EXISTENCE THEOREM). *If the equation is involutory near a point (x_0, y_0, p_0) and $(x_0, y_0, p_0) \notin \Sigma$, then there exists a (classical) complete solution of the equation near (x_0, y_0, p_0).*

We say that the equation is involutory if $[F_j, F_k] = 0$ for $j, k = 1, \ldots, 2n+1-r$, where

$$[F, G] = F \cdot \frac{\partial G}{\partial z} - G \cdot \frac{\partial F}{\partial z} + \sum_{i=1}^{n} (\frac{\partial F}{\partial x_i} \cdot \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \cdot \frac{\partial G}{\partial x_i})$$
$$+ \sum_{i=1}^{n} p_i \cdot (\frac{\partial F}{\partial z} \cdot \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial z} \cdot \frac{\partial F}{\partial p_i}).$$

We now give a geometric framework of the theory of first order differential equations. Let $J^1(\mathrm{R}^n, \mathrm{R})$ be the 1-jet bundle of functions of n-variables. Since we only consider the local situation the 1-jet bundle $J^1(\mathrm{R}^n, \mathrm{R})$ may be considered as R^{2n+1} with a natural coordinate system $(x_1, \ldots, x_n, y, p_1, \ldots, p_n)$, where $(x_1, \ldots, x_n)$ is a coordinate system of R^n. We have the natural projection $\pi : J^1(\mathrm{R}^n, \mathrm{R}) \to \mathrm{R}^n \times \mathrm{R}$; $\pi(x, y, p) = (x, y)$.

An immersion germ $i : (L, q) \to J^1(\mathrm{R}^n, \mathrm{R})$ is said to be *a Legendrian immersion germ* if $\dim L = n$ and $i^*\theta = 0$, where $\theta = dy - \sum_{i=1}^{n} p_i dx_i$. The image of $\pi \circ i$ is called *a wave front set* of i. We say that $q \in L$ is *a Legendrian singular point* if $\operatorname{rank} d(\pi \circ i)_q < n$.

We now describe the geometric structure connected with first order differential equations. A first order differential equation is most naturally interpreted as being a closed subset of $J^1(\mathrm{R}^n, \mathrm{R})$. Unless the contrary is specifically stated, we use the following definition.

A system of first order differential equations (or, briefly an equation) is an r-dimensional submanifold $E \subset J^1(\mathrm{R}^n, \mathrm{R})$, where $n+1 \leq r \leq 2n$. If $r < 2n$, then E is said to be *overdetermined.* We also say that E is *maximally overdetermined (or holonomic)* if $r = n+1$.

By the philosophy of Lie, we may define the notion of solutions as follows. *An (abstract) solution* of E is a Legendrian immersion $i : L \to J^1(\mathrm{R}^n, \mathrm{R})$ such that $i(L) \subset E$.

Let $f : \mathrm{R}^n \to \mathrm{R}$ be a smooth function. Then $j^1 f : \mathrm{R}^n \to J^1(\mathrm{R}^n, \mathrm{R})$ is a Legendrian embedding. Hence, in our terminology, the (classical) solution of E is a smooth function f such that $j^1 f(\mathrm{R}^n) \subset E$.

On the other hand, we can show that an (abstract) solution $i : L \to J^1(\mathrm{R}^n, \mathrm{R})$ is given by (at least locally) a jet extension $j^1 f$ of a smooth function f if and only if $\pi \circ i$ is a non-singular map. Thus the (abstract) solution has multi-valued near the Legendrian singular point. We also define the notion of singularities of equations. Let $E^r \subset J^1(\mathrm{R}^n, \mathrm{R})$ be an equation. Then $z \in E$ is said to be *a contact singular point* if $\theta(T_z E) = 0$. We also say that $z \in E$ is *a π-singular point* if $\operatorname{rank}(d\pi|E)_z < n+1$. We can easily show that if z is a contact singular point of E, then it is a π-singular point of E. Let $\Sigma(\pi|E)$ be the set of π-singular points and $\Sigma_c(E)$ be the set of contact singular points. We say that $D_E = \pi(\Sigma(\pi|E))$ is *a discriminant set* of the equation E.

Our purpose in this section is to establish the notion of (abstract) complete solutions. Let $y = (t_1, \ldots, t_{r-n}, x_1, \ldots, x_n)$ be the (classical) complete solution of E, then we have a jet extension

$$j^1_* f : \mathrm{R}^{r-n} \times \mathrm{R}^n \to J^1(\mathrm{R}^n, \mathrm{R})$$

which is defined by $j^1_* f(t,x) = j^1 f_t(x)$, where $f_t(x) = f(t,x)$. Then it is easy to show that $j^1_* f$ is an immersion. Since $dim\ E = r$, then $j^1_* f$ gives (at least locally) a parametrization of E and $j^1_* f(t \times \mathbf{R}^n)$ is a (classical) solution of E for any $t \in \mathbf{R}^{r-n}$. Thus there exists a foliation on E whose leaves are (classical) solutions. Thus we can generalize this notion to an abstract sense. We say that an equation $E \subset J^1(\mathbf{R}^n, \mathbf{R})$ is *completely integrable (or E has an (abstract) complete solution)* if there exists an n-dimensional completely integrable distribution $\mathcal{D}$ on E such that $\theta_z(\mathcal{D}_z) = 0$ for any $z \in E$.

By the Frobenius' theorem, we have the following proposition.

PROPOSITION 1.2 [8]. *Let $E^r \subset J^1(\mathbf{R}^n, \mathbf{R})$ be an equation. Then the following conditions are equivalent.*

(1) *E is completely integrable.*

(2) *For any $q \in E$, there exist a neighbourhood U of q in E and smooth functions*

$$\mu_1, \ldots, \mu_{r-n} \text{ on } U$$

such that

$$d\mu_1 \wedge \cdots \wedge d\mu_{r-n} \neq 0 \text{ on } U$$

and

$$\langle d\mu_1, \ldots, d\mu_{r-n} \rangle_{C^\infty(U)} \supset \langle \theta | U \rangle_{C^\infty(U)}$$

as $C^\infty(U)$-modules, where $C^\infty(U)$ denotes the ring of smooth functions on U.

(3) *For any $q \in E$, there exist a neighbourhood $V \times W$ of 0 in $\mathbf{R}^{r-n} \times \mathbf{R}^n$ and an embedding*

$$f : V \times W \to J^1(\mathbf{R}^n, \mathbf{R})$$

such that

$$f(0) = q, \ f(V \times W) \subset E$$

and

$$f | \{t\} \times W : \{t\} \times W \to J^1(\mathbf{R}^n, \mathbf{R})$$

is Legendrian embedding for any $t \in V$.

In this part an equation is defined to be an immersion $f : U \to J^1(\mathbf{R}^n, \mathbf{R})$ where U is an open subset of $\mathbf{R}^n$. By Proposition 1.2, we say that f is *completely integrable* if there exists a submersion

$$\mu = (\mu_1, \ldots, \mu_{r-n}) : U \to \mathbf{R}^{r-n}$$

such that

$$\langle d\mu_1, \ldots, \mu_{r-n} \rangle_{C^\infty(U)} \supset \langle f^* \theta \rangle_{C^\infty(U)}.$$

We call $\mu = (\mu_1, \ldots, \mu_{r-n})$ a *complete integral* of f and the pair

$$(\mu, f) : U \to \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})$$

is called *a first order differential equation with complete integral (or, briefly, an equation with complete integral).*

We can also define the above notions in terms of map germs : *An equation germ* is defined to be an immersion germ $f : (\mathbf{R}^r, 0) \to J^1(\mathbf{R}^n, \mathbf{R})$. We say that f is *completely integrable* if there exists a submersion germ

$$\mu = (\mu_1, \ldots, \mu_{r-n}) : (\mathbf{R}^r, 0) \to \mathbf{R}^{r-n}$$

such that

$$\langle d\mu_1, \ldots, d\mu_{r-n} \rangle_{\mathcal{E}_u} \supset \langle f^*\theta \rangle_{\mathcal{E}_u},$$

where $u = (u_1, \ldots, u_r)$ is the canonical coordinate of $(\mathbf{R}^r, 0)$ and $\mathcal{E}_u$ is the ring of function germs of u-variables at the origin. Then μ is called *a complete integral* of f and the pair

$$(\mu, f) : (\mathbf{R}^r, 0) \to \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})$$

is called *an equation germ with complete integral.* In order to understand the above notions, we give two examples.

EXAMPLE 1.3. We consider the following equations of two independent variables.

(1)
$$\begin{cases} (\dfrac{\partial y}{\partial x_1})^2 - x_1 = 0 \\ \dfrac{\partial y}{\partial x_2} = 0 \end{cases}$$

(2)
$$\begin{cases} y - (\dfrac{\partial y}{\partial x_1})^2 = 0 \\ \dfrac{\partial y}{\partial x_2} = 0. \end{cases}$$

We can exactly solve these equations, then complete solutions are

(1')
$$y = \pm \frac{2}{3} x_1^{\frac{3}{2}} + t$$

(2')
$$y = \frac{1}{4}(x_1 + t)^2,$$

where t is a parameter. The submanifold in $J^1(\mathbf{R}^n, \mathbf{R})$ which is defined by (1) is the image of the immersion $f : \mathbf{R}^3 \to J^1(\mathbf{R}^2, \mathbf{R})$ defined by $f(u_1, u_2, u_3) = (u_1^2, u_2, u_3, u_1, 0)$. If we consider a submersion $\mu : \mathbf{R}^3 \to \mathbf{R}$ defined by $\mu(u_1, u_2, u_3) = u_3 - \frac{2}{3}u_1^3$, we can easily check that $d\mu = f^*\theta$.

We also define (μ, f) by $f(u_1, u_2, u_3) = (u_1, u_2, u_3^2, u_3, 0)$ and $\mu(u_1, u_2, u_3) = u_3 - \frac{1}{2}u_1$, then the image of f is the submanifold in $J^1(\mathbf{R}^n, \mathbf{R})$ which is defined by the equation (2) and we have $f^*\theta = 2u_3 d\mu$.

In both cases, we can observe that $\pi \circ f(\mu^{-1}(t))$ is the graph of the solution in $\mathbf{R}^2 \times \mathbf{R}$.

LEMMA 1.4 [8]. *Let (μ, f) be an equation germ with complete integral. Then there exist unique elements*

$$h_1, \ldots, h_{r-n} \in C^\infty_{u_0}(R)$$

such that

$$f^*\theta = \sum_{i=1}^{r-n} h_i \cdot d\mu_i,$$

where $\mu(u) = (\mu_1(u), \ldots, \mu_{r-n}(u))$ *and* $C^\infty_{u_0}(R)$ *is the ring of smooth function germs at* u_0.

We now consider the 1-jet bundle $J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})$ and the canonical 1-form Θ on the space. Let $(t_1, \ldots, t_{r-n}, x_1, \ldots, x_n)$ be canonical coordinate system on $\mathbf{R}^{r-n} \times \mathbf{R}^n$ and

$$(t_1, \ldots, t_{r-n}, x_1, \ldots, x_n, y, q_1, \ldots, q_{r-n}, p_1, \ldots, p_n)$$

be corresponding coordinate system on $J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})$. Then the canonical 1-form is given by

$$\Theta = dy - \sum_{i=1}^{n} p_i \cdot dx_i - \sum_{i=1}^{r-n} q_i \cdot dt_i = \theta - \sum_{i=1}^{r-n} q_i \cdot dt_i.$$

We define the natural projection

$$\Pi : J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R}) \to (\mathbf{R}^{r-n} \times \mathbf{R}^n) \times \mathbf{R}$$

by $\Pi(t, x, y, q, p) = (t, x, y)$ We call the above 1-jet bundle *a unfolded 1-jet bundle.*

Define a map germ

$$\mathcal{L} : (R, u_0) \to J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})$$

by

$$\mathcal{L}(u) = (\mu(u), x \circ f(u), y \circ f(u), h(u), p \circ f(u)).$$

Then we can easily show that $\mathcal{L}$ is a Legendrian immersion germ. If we fix 1-forms Θ and θ, the Legendrian immersion germ $\mathcal{L}$ is uniquely determined by the Legendrian family (μ, f). We call $\mathcal{L}$ *a complete Legendrian unfolding associated with the Legendrian family* (μ, f). We remark that even in the one parameter case the notion of the Legendrian unfoldings is slightly different from the notion of extended Legendrian manifolds in the sense of Zakalyukin[16]. For example, we now consider a Legendrian immersion germ $\mathcal{L} : (\mathbf{R}^2, 0) \to J^1(\mathbf{R} \times \mathbf{R}, \mathbf{R})$ defined by $\mathcal{L}(u, v) = (u^2 + 3v^2, u, 2v^3, v, -2uv)$, it is the extended Legendrian immersion germ. But it is not a Legendrian unfolding because $u^2 + 3v^2$ is not a submersion. Let $\mathcal{L} : U \to J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})$ be a complete Legendrian unfolding associated to the Legendrian family (μ, f). Since $\mathcal{L}$ is uniquely determined by (μ, f), we denote $\ell_{(\mu,f)}$ instead of $\mathcal{L}$.

Conversely, let $\mathcal{L} : U \to J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})$ be a Legendrian immersion such that f is an immersion and μ is a submersion with $\Pi_1 \circ \mathcal{L} = (\mu, f)$, where $\Pi_1 : J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R}) \to J^1(\mathbf{R}^n, \mathbf{R})$ is the canonical projection. Then (μ, f) is an equation with complete integral and $\mathcal{L} = \ell_{(\mu,f)}$.

Some effects of the notion of Legendrian unfoldings on equations with complete integral are given in the following propositions.

PROPOSITION 1.5 [8]. *Let* $(\mu, f) : U \to \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})$ *be an equation with complete integral. Then* $f(\mu^{-1}(t))$ *is a (classical) solution for any* $t \in \mathbf{R}^{r-n}$ *if and only if* $\ell_{(\mu,f)}$ *is Legendrian non-singular.*

We say that an equation germ with complete integral is *regular* if $\ell_{(\mu,f)}$ is Legendrian non-singular.

PROPOSITION 1.6 [8]. *Let* $(\mu, f) : U \to \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})$ *be an equation with complete integral. For any* $u \in U$, *we denote*

$$\ell_{(\mu,f)}(u) = (\mu(u), x \circ f(u), y \circ f(u), h(u), p \circ f(u))$$

by the local coordinate of $J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})$. *Then* f *is contact singular at* $u_0 \in U$ *if and only if* $h(u_0) = 0$.

We now establish the notion of genericity of equation germs with complete integral. Let $U \subset \mathbf{R}^r$ be an open set. We denote by $\mathrm{Int}(U, \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R}))$ the set of equations with complete integral $(\mu, f) : U \to \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})$. We also define $L(U, J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R}))$ to be the set of complete Legendrian unfoldings $\ell_{(\mu,f)} : U \to J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})$.

These sets are topological spaces equipped with the Whitney C^∞-topology. A subset of

$$\mathrm{Int}(U, \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})) \text{ (respectively } L(U, J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})))$$

is said to be *generic* if it is an open dense subset in

$$\mathrm{Int}(U, \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})) \text{ (respectively } L(U, J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R}))).$$

The genericity of a property of germs are defined as follows. Let P be a property of equation germs with complete integral

$$(\mu, f) : (\mathbf{R}^r, 0) \to \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R})$$

(respectively, Legendrian unfoldings

$$\ell_{(\mu,f)} : (\mathbf{R}^r, 0) \to J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})).$$

For an openset $U \subset \mathbf{R}^r$, we define $\mathcal{P}(U)$ to be the set of

$$(\mu, f) \in \mathrm{Int}(U, \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R}))$$

(respectively,

$$\ell_{(\mu,f)} \in L(U, J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})))$$

such that the germ at x whose representative is given by (μ, f) (respectively $\ell_{(\mu,f)}$) has property P for any $x \in U$.

The property P is said to be *generic* if for some neighbourhood U of 0 in $\mathbf{R}^r$, the set $\mathcal{P}(U)$ is a generic subset in $\mathrm{Int}(U, \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R}))$ (respectively $L(U, J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R}))$).

By the construction, we have a well-defined continuous mapping

$$(\Pi_1)_* : L(U, J^1(\mathbf{R}^{r-n} \times \mathbf{R}^n, \mathbf{R})) \to \mathrm{Int}(U, \mathbf{R}^{r-n} \times J^1(\mathbf{R}^n, \mathbf{R}))$$

defined by

$$(\Pi_1)_*(\ell_{(\mu,f)}) = \Pi_1 \circ \ell_{(\mu,f)} = (\mu, f).$$

The following theorem is fundamental in our theory.

THEOREM 1.7 [5,8,9]. *The continuous map*

$$(\Pi_1)_* : L(U, J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})) \to \mathrm{Int}(U, \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}))$$

is a homeomorphism.

This theorem asserts that the genericity of a property of equations with complete integral can be interpreted by the genericity of the corresponding property of Legendrian unfoldings. We now consider some generic properties as a consequence of the above theorem.

PROPOSITION 1.8 [8]. *For generic equation germ with complete integral*

$$(\mu, f) : (\mathbb{R}^r, 0) \to \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}),$$

the contact singular set $\Sigma(f_c)$ is empty or an n-dimensional submanifold.

We remark that even for generic equation germs with complete integral (μ, f), we cannot expect that the π-singular set $\Sigma(\pi \circ f)$ is an n-dimensional submanifold. In [5] we classified generic equation germs with complete integral in the case of $n = 1$ (i.e. ordinary differential equations). One of the normal form is given by

$$f(u,v) = (u, v^3 + uv^2, v^2 - 3v - 2u),$$
$$\mu(u,v) = \frac{1}{2}v^2 + u.$$

The π-singular set of this example is given by $\Sigma(\pi \circ f) = \{(u,v) | 3v^2 + 2uv\}$ and it is not a smooth submanifold. We can calculate that $f^*\theta = (3v + 2u)d\mu$, then we have $\Sigma(f_c) = \{(u,v) | 3v + 2u = 0\}$. Hence $\Sigma(f_c)$ is a smooth submanifold and it is a smooth component of $\Sigma(\pi \circ f)$.

On the other hand, we appreciate another normal form given by

$$f(u,v) = (u, v^2, v),$$
$$\mu(u,v) = v - \frac{1}{2}u.$$

In this case we can easily show that $f^*\theta = 2vd\mu$, then the corresponding Legendrian unfolding is given by $\ell_{(\mu,f)}(u,v) = (v - \frac{1}{2}u, u, v^2, 2v, v)$. It is clear that (μ, f) is a regular equation germ with complete integral. The π-singular set and the contact singular set of this equation is given by $\{(u,v) | v = 0\}$, then it is a smooth submanifold and the singular solution of f.

LEMMA 1.9 [8]. *For regular equations with complete integral (μ, f), we have $\Sigma(\pi \circ f) = \Sigma(f_c)$.*

The following theorem asserts that regular equation germs with complete integral are π-regular or have singular solutions.

THEOREM 1.10 [8]. *For regular equation germs with complete integral*

$$(\mu, f) : (\mathbb{R}^r, 0) \to \mathbb{R}^{r-n} \times J^1(\mathbb{R}^n, \mathbb{R}),$$

the π-singular set $\Sigma(\pi \circ f)$ is empty or an n-dimensional submanifold and the discriminant set D_f is an envelope of the family $\{\pi \circ f(\mu^{-1}(t))\}_{t \in (\mathbb{R}^{r-n}, 0)}$ consisting of graph of a classical complete solution of f.

REMARK. Of course, the set of all regular equations with complete integral is an open set in the space of equations with complete integral.

On the other hand, we can study generic properties of Legendrian unfoldings in terms of generating families.

Since $\mathcal{L}$ is a Legendrian immersion germ, there exists a generating family of $\mathcal{L}$ by the Arnol'd-Zakalyukin's theory ([1,15,16]). In this case the generating family is naturally constructed by the $(r-n)$-family of generating families associated with (μ, f).

Let

$$F : ((\mathbb{R}^{r-n} \times \mathbb{R}^n) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$$

be a function germ such that $d_2 F | 0 \times \mathbb{R}^n \times \mathbb{R}^k$ is non-singular, where

$$d_2 F(t, x, q) = (\frac{\partial F}{\partial q_1}(t, x, q), \ldots, \frac{\partial F}{\partial q_k}(t, x, q)).$$

It follows from the definition that $C(F) = d_2 F^{-1}(0)$ is a smooth r-manifold germ and

$$\pi_F : (C(F), 0) \to \mathbb{R}^{r-n}$$

is a submersion germ, where

$$\pi_F(t, x, q) = t.$$

Define map germs

$$\tilde{\Phi}_F : (C(F), 0) \to J^1(\mathbb{R}^n, \mathbb{R})$$

by

$$\tilde{\Phi}_F(t, x, q) = (x, F(t, x, q), \frac{\partial F}{\partial x}(t, x, q))$$

and

$$\Phi_F : (C(F), 0) \to J^1(\mathbb{R}^{r-n} \times \mathbb{R}^n, \mathbb{R})$$

by

$$\Phi_F(t, x, q) = (t, x, F(t, x, q), \frac{\partial F}{\partial t}(t, x, q), \frac{\partial F}{\partial x}(t, x, q)).$$

Since $\frac{\partial F}{\partial q_i} = 0$ on $C(F)$, we can easily show that

$$(\tilde{\Phi}_F)^* \theta = \sum_{i=1}^{r-n} \frac{\partial F}{\partial t_i} | C(F) \cdot dt_i | C(F).$$

By the definition, Φ_F is a Legendrian unfolding associated with the Legendrian family $(\pi_F, \tilde{\Phi}_F)$. By the same method of the theory of Arnol'd-Zakalyukin ([1,15,16]), we can show that the following proposition.

PROPOSITION 1.11 [8]. *All Legendrian unfolding germs are constructed by the above method.*

Then F is called *a generalized phase family of* Φ_F. We now consider ambiguity of the choice of generalized phase function germs. Let

$$F, G : ((\mathbf{R}^{r-n} \times \mathbf{R}^n) \times \mathbf{R}^k, 0) \to (\mathbf{R}, 0)$$

be generalized phase families. We say that F and G are *strictly $\mathcal{R}$-equivalent* if there exists a diffeomorphism germ

$$\Phi : ((\mathbf{R}^{r-n} \times \mathbf{R}^n) \times \mathbf{R}^k, 0) \to ((\mathbf{R}^{r-n} \times \mathbf{R}^n) \times \mathbf{R}^k, 0)$$

of the form $\Phi(t, x, q) = (t, x, \phi(t, x, q))$ such that $F \circ \Phi = G$. If we carefully read proofs of Lemmas 1 and 2 in (page 307 in [1]), we can find the following assertion.

PROPOSITION 1.12. *Let* $F, G : (\mathbf{R}^{r-n} \times \mathbf{R}^n) \times \mathbf{R}^k, 0) \to (\mathbf{R}, 0)$ *be generalized phase function germs such that* Image Φ_F = Image Φ_G *and*

$$\operatorname{rank} H(F|0 \times \mathbf{R}^k) = \operatorname{rank} H(G|0 \times \mathbf{R}^k) = 0,$$

where $H(f)$ is the Hessian matrix of f at the origin. Then F and G are strictly $\mathcal{R}$-equivalent.

In [6] it has been given an application of the theory of Legendrian unfoldings to the Cauchy problem of Hamilton-Jacobi equations.

PART 2. SINGLE EQUATIONS

In this part we will consider completely integrable single equations. We can get detailed informations about this subject. *An equation* is a submersion germ $F : (J^1(\mathbf{R}^n, \mathbf{R}), z_0) \to (\mathbf{R}, 0)$.

LEMMA 2.1 [7]. *Let $F = 0$ be an equation germ.*
1) *z_0 is a contact singular point of $F^{-1}(0)$ if and only if $F = F_{p_i} = F_{x_i} + p_i F_y = 0$ at z_0 for $i = 1, \ldots, n$.*
2) *z_0 is a π-singular point if and only if $F = F_{p_i} = 0$ at z_0 for $i = 1, \ldots, n$.*

In classical treatises on differential equations the notion of singular solutions is very confused. Even in the case of ordinary differential equations, we cannot find rigorous definitions of singular solutions since the first example was discovered (that is about 280 years ago). So, firstly, we must give a rigorous definition of singular solutions. A geometric solution $i : (L, q_0) \to (J^1(\mathbf{R}^n, \mathbf{R}), z_0)$ of $F = 0$ is called *a singular solution* of $F = 0$ if it satisfies the following condition :
(*) For any representative $\tilde{i} : U \to F^{-1}(0)$ of i and any open subset $V \subset U$, $\tilde{i}(V)$ is not contained in a leaf of any complete solutions of $F = 0$.

In classical treatises, the π-singular set $\Sigma_\pi(F)$ plays an important role. In some textbook ([2,3,4]), the singular solution is defined to be the π-singularset if it is a geometric solution. However, this property does not characterize the singular solutions. The following theorem indicates that the contact singular set $\Sigma_c(F)$ is much important.

THEOREM 2.2 [10,12]. *For an equation* $F : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}, 0)$ *and a geometric solution* $i : (L, q_0) \to (J^1(\mathbb{R}^n, \mathbb{R}), (x_0, y_0, p_0))$ *of* $F = 0$, *the following conditions are equivalent.*
(1) i *is a singular solution of* $F = 0$.
(2) *There exists a complete solution of* $F = 0$ *such that each leaves are transverse to* i.
(3) $\mathrm{Image}\, i \subseteq \Sigma_c(F)$.

We can also give a characterization of complete integrability as follows.

THEOREM 2.3 [10]. *For an equation* $F : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}, 0)$, *the following are equivalent.*
(1) $F = 0$ *is completely integrable.*
(2) $\Sigma_c(F) = \emptyset$ *or* $\Sigma_c(F)$ *is an* n*-dimensional submanifold.*

By the definition, if $\Sigma_c(F)$ is an n-dimensional submanifold, it is automatically a geometric solution of $F = 0$. Then we have the following corollary of Theorems 2.2 and 2.3.

COROLLARY 2.4 [10]. *An equation* $F : (J^1(\mathbb{R}^n, \mathbb{R}), z_0) \to (\mathbb{R}, 0)$ *has a singular solution if and only if* $\Sigma_c(F)$ *is an* n*-dimensional submanifold. Moreover,* $\Sigma_c(F)$ *is the singular solution of* $F = 0$.

We now give some examples to understand these situations.

EXAMPLES 2.5. 1)The Clairaut equation The following is the classical example of an equation with singular solution : $y = \sum_{i=1}^n x_i p_i + f(p_1, \ldots, p_n)$, where f is a smooth function. The complete solution is given by $y = \sum_{i=1}^n x_i t_i + f(t_1, \ldots, t_n)$ and the singular solution is the envelope of graphs of complete solution.
2) The dual of the Clairaut equation Consider the equation : $y = f(x_1, \ldots, x_n)$. This equation is given by the Legendre transform (see [7]) of the Clairaut equation. The complete solution is given by $\{(t, f(t), u) | (t, u) \in \mathbb{R}^n \times \mathbb{R}^n\}$, where $t = (t_1, \ldots, t_n)$ is the parameter.
The singular solution is given by $\Sigma_c(F) = \{(x, f(x), f_x(x)) | x \in \mathbb{R}^n\}$. We can observe that $F^{-1}(0) = \Sigma_\pi(F) \supset \Sigma_c(F)$.
3) Consider the following equation : $y - 2p^3 = 0$. We can show that $\Sigma_\pi(F) = \Sigma_c(F) = \{(x, 0, 0) | x \in \mathbb{R}\}$ which is a singular solution. We also have a complete solution $s : (\mathbb{R} \times \mathbb{R}, 0) \to J^1(\mathbb{R}, \mathbb{R})$ given by $s(u, t) = (3u^2 + t, 2u^3, u)$, where t is the parameter. In this case the singular solution is a locus of cusps of the complete solution (not an envelope!).
4) Consider the following equation : $y - p^3 - xp^2 = 0$. We can show that $\Sigma_\pi(F) = \{(x, y, p) | y - p^3 - xp^2 = 0 \text{ and } 3p^2 - 2xp = 0\}$ and $\Sigma_c(F) = \{(x, 0, 0)\}$. It follows that $\Sigma_c(F)$ is the singular solution.

On the other hand, we also give a characterization of first order partial differential equations (briefly, equations) with (classical) complete solutions. Roughly speaking, this class of equations is equal to a class of equations with singular solutions which will be called Clairaut type equations. In [7] it has been shown that equations with singular solution are not generic in the space of single equations. However, this class of equations is quite interesting. One of the typical examples of equations with singular solutions is the (classical) Clairaut equation which has a (classical) complete solution consisting of hyperplanes. Moreover, the graph of the singular solution is an envelope of the family of graphs of the complete solution. But, we have no reasons why the complete solution must consist of hyperplanes.

An equation $F = 0$ is said to be *Clairaut type* if there exist smooth function germs $B_{ji}, A_i : (J^1(\mathbb{R}^n, \mathbb{R}, (x_0, y_0, p_0)) \to \mathbb{R}$ for $i, j = 1, \dots, n$ such that

$$F_{x_i} + p_i F_y = \sum_{j=1}^{n} B_{ji} F_{p_j} + A_i F \qquad (i = 1, \dots, n)$$

and satisfy that

$$(1) \qquad B_{ji} = B_{ij}$$

$$(2) \qquad \frac{\partial B_{jk}}{\partial x_i} + p_i \frac{\partial B_{jk}}{\partial y} + \sum_{\ell=1}^{n} B_{\ell i} \frac{\partial B_{jk}}{\partial p_\ell} = \frac{\partial B_{ji}}{\partial x_k} + p_k \frac{\partial B_{ji}}{\partial y} + \sum_{\ell=1}^{n} B_{\ell k} \frac{\partial B_{ji}}{\partial p_\ell}$$

at any $(x, y, p) \in (F^{-1}(0), (x_0, y_0, p_0))$ for $i, j, k = 1, \dots, n$.

THEOREM 2.6 [11]. *For an equation germ $F = 0$, the following are equivalent.*
(1) $F = 0$ is the Clairaut type equation.
(2) $F = 0$ has a (classical) complete solution.

In this case, if $\Sigma_\pi(F) \neq \emptyset$, then $\Sigma_\pi(F)$ is a geometric solution (i.e. the singular solution) of $F = 0$ and the discriminant set D_F is the envelope of the family of graphs of the complete solution.

By the classical existence theorem, if $F = 0$ is a π-regular equation, then there exists a (classical) complete solution. Then we can assert that π-regular equation is Clairaut type by the above theorem.

EXAMPLES 2.7. 1) Of course, one of examples of Clairaut type equations is the (classical) Clairaut equation. The Clairaut equation is given by $y = \sum_{i=1}^{n} x_i p_i + f(p_1, \dots, p_n)$, where f is a smooth function. The complete solution is given by $y = \sum_{i=1}^{n} x_i t_i + f(t_1, \dots, t_n)$ and we can easily verify that $F_{x_i} + p_i F_y = 0$ for $i = 1, \dots, n$.
2) The second example is an equation for "Free particle" in the n-dimensional space. Consider the following equation ; $y^2 + \sum_{i=1}^{n} p_i^2 - 1 = 0$. Then we have

$$\Sigma_\pi(F) = \{(x_1, \dots, x_n, \pm 1, 0, \dots, 0) | (x_1, \dots, x_n) \in \mathbb{R}^n\}.$$

We can calculate that $F_{x_i} + p_i F_y = y F_{p_i}$ for $i = 1, \dots, n$. Then we have $B_{ij} = \pm(1 - \sum_{k=1}^{n} p_k^2)^{\frac{1}{2}}$ and $A_i = \frac{2p_i}{y \pm (1 - \sum_{k=1}^{n} p_k)^{\frac{1}{2}}}$, where $\pm$ corresponds to the point $(0, \pm 1, 0)$. The complete solution is given by $y = \pm\cos(\frac{1}{(\sum_{i=1}^{n} (1+t_i)^2)^{\frac{1}{2}}} \sum_{i=1}^{n} (x_i + t_i x_i))$, which is defined on $(\mathbb{R}^n \times \mathbb{R}^n, (0, 0))$.

PART 3. HOLONOMIC SYSTEMS

In this part we will give a generic classification of completely integrable holonomic systems of equations. by the equivalence relation under the group of point transformations in the sense of Sophus Lie. *A point transformation* ϕ on $\mathbb{R}^n \times \mathbb{R}$ is, by definition, a diffeomorphism of $\mathbb{R}^n \times \mathbb{R}$ onto itself.

To define a lift of ϕ, we give a contact manifold which is a fiberwise compactification of $J^1(\mathbb{R}^n, \mathbb{R})$. Let $\tilde{\pi} : PT^*(\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ be a projective cotangent bundle over $\mathbb{R}^n \times \mathbb{R}$. There exists a canonical contact structure on $PT^*(\mathbb{R}^n \times \mathbb{R})$ and if we adopt the homogeneous coordinate $(x_1, \ldots, x_n, y, [\xi_1; \ldots; \xi_n; \eta])$, the affine coordinate neighbourhood which is defined by $\eta \neq 0$ is contact diffeomorphic to $J^1(\mathbb{R}^n, \mathbb{R})$. Since we only consider local situations, we may regard point transformation as a diffeomorphism germ $\phi : (\mathbb{R}^n \times \mathbb{R}, (x_0, y_0)) \to (\mathbb{R}^n \times \mathbb{R}, (x_1, y_1))$. Then we have a canonical contact lift $\hat{\phi} : (PT^*(\mathbb{R}^n \times \mathbb{R}), z_0) \to (PT^*(\mathbb{R}^n \times \mathbb{R}), z_1)$ of ϕ.

Following Lie, the most natural equivalence relation among equation germs is given by point transformations. Here a holonomic system is an immersion germ $f : (\mathbb{R}^{n+1}, 0) \to J^1(\mathbb{R}^n \mathbb{R})$. Let $f,\ g : (\mathbb{R}^{n+1}, 0) \to J^1(\mathbb{R}^n, \mathbb{R})$ be holonomic systems. We say that f and g are *equivalent as equations* if there exist a diffeomorphism germ $\psi : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^r, 0)$ and a point transformation $\phi : (\mathbb{R}^n \times \mathbb{R}, \pi(z_0)) \to (\mathbb{R}^n \times \mathbb{R}, \pi(z_1))$ such that the lift $\hat{\phi}$ of ϕ satisfies that $\hat{\phi}(z_0) = z_1$ and $\hat{\phi} \circ f = g \circ \psi$, where $z_0 = f(0)$ and $z_1 = g(0)$.

We now consider a holonomic system with complete integral $(\mu, f) : (\mathbb{R}^{n+1}, 0) \to \mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R})$. This leads us to the following definition. Let (μ, g) be a pair of a map germ $g : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$ and a submersion germ $\mu : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}, 0)$. Then the diagram

$$(\mathbb{R}, 0) \xleftarrow{\mu} (\mathbb{R}^{n+1}, 0) \xrightarrow{g} (\mathbb{R}^n \times \mathbb{R}, 0)$$

or briefly (μ, g), is called *a (holonomic) integral diagram* if there exists a holonomic system$f : (\mathbb{R}^{n+1}, 0) \to J^1(\mathbb{R}^n, \mathbb{R})$ such that (μ, f) is an equation germ with complete integral and $\pi \circ f = g$, and we say that the integral diagram (μ, g) is *induced by* f. Furthermore we introduce an equivalence relation among integral diagrams. Let (μ, g) and (μ', g') be integral diagrams. Then (μ, g) and (μ', g') are *equivalent* (*respectively, strictly equivalent*) if the diagram

$$\begin{tikzcd}
(\mathbb{R}, 0) \arrow[d, "\kappa"'] & (\mathbb{R}^{n+1}, 0) \arrow[l, "\mu"'] \arrow[r, "g"] \arrow[d, "\psi"'] & (\mathbb{R}^n \times \mathbb{R}, 0) \arrow[d, "\phi"] \\
(\mathbb{R}, 0) & (\mathbb{R}^{n+1}, 0) \arrow[l, "\mu'"] \arrow[r, "g'"'] & (\mathbb{R}^n \times \mathbb{R}, 0)
\end{tikzcd}$$

commutes for some diffeomorphism germs κ, ψ and ϕ (respectively, $\kappa = id_{\mathbb{R}}$).

We can assert the following theorem which reduces the equivalence problem for completely integrable holonomic systems to that of for the corresponding induced integral diagrams.

THEOREM 3.1 [9]. *Let (μ, f) and $(\mu', f') : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R} \times J^1(\mathbb{R}^n, \mathbb{R}), 0 \times v)$ be equations with complete integral such that the set of critical points of $\pi \circ f$ and $\pi \circ f'$ are closed sets without interior points. Then the followings are equivalent:*

(1) *f and f' are equivalent as equations.*
(2) *$(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are equivalent as integral diagrams.*

Our classifications are the following :

THEOREM 3.2 [9]. *Suppose that $1 \leq n \leq 3$. For generic holonomic equation germ with complete integral $(\mu, f) : (\mathbf{R}^{n+1}, 0) \to \mathbf{R} \times J^1(\mathbf{R}^n, \mathbf{R})$, the integral diagram $(\mu, \pi \circ f)$ is strictly equivalent to one of germs in the following list :*
I) $n = 1$;

(1) $$\mu = u_2 \qquad g = (u_1, u_2)$$

(2) $$\mu = \frac{2}{3}u_1^3 + u_2 \qquad g = (u_1^2, u_2)$$

(3) $$\mu = u_2 - \frac{1}{2}u_1 \qquad g = (u_1, u_2^2)$$

(4) $$\mu = \frac{3}{4}u_1^4 + \frac{1}{2}u_1^2 u_2 + u_2 + \alpha \circ g \qquad g = (u_1^3 + u_2 u_1, u_2)$$

(5) $$\mu = u_2 + \alpha \circ g \qquad g = (u_1, u_2^3 + u_1 u_2)$$

(6) $$\mu = -3u_2^2 + 4u_1 u_2 + u_1 + \alpha \circ g \qquad g = (u_1, u_2^3 + u_1 u_2^2).$$

II) $n = 2$;

(1) $$\mu = u_3 \qquad g = (u_1, u_2, u_3)$$

(2) $$\mu = \frac{2}{3}u_1^3 + u_2 + u_3 \qquad g = (u_1^2, u_2, u_3)$$

(3) $$\mu = u_3 - \frac{1}{2}u_1 \qquad g = (u_1, u_2, u_3^2)$$

(4) $$\mu = \frac{3}{4}u_1^4 + \frac{1}{2}u_1^2 u_2 + u_3 \qquad g = (u_1^3 + u_2 u_1, u_2, u_3)$$

(5) $$\mu = u_3 \qquad g = (u_1, u_2, u_3^3 + u_1 u_3)$$

(6) $$\mu = -3u_3^2 + 4u_1u_3 + u_2$$
$$g = (u_1, u_2, u_3^3 + u_1u_3^2)$$

(7) $$\mu = \frac{4}{5}u_1^5 + \frac{1}{2}u_2u_1^2 + \frac{2}{3}u_3u_1^3 + u_3 + \alpha \circ g$$
$$g = (u_1^4 + u_2u_1 + u_3u_1^2, u_2, u_3)$$

(8) $$\mu = 2u_2^3 \pm 2u_1^2u_2 + u_3 + \alpha \circ g$$
$$g = (u_1u_2 + u_1u_3, u_1^2 \pm 3u_2^2, u_3)$$

(9) $$\mu = u_3 + \alpha \circ g$$
$$g = (3u_1^2 + u_2u_3, u_2, \pm u_3^2 + 2u_1^3 + u_3u_2)$$

(10) $$\mu = u_3 + \alpha \circ g$$
$$g = (u_1, u_2, \pm u_3^4 + u_1u_3 + u_2u_3^2)$$

(11) $$\mu = -4u_3^3 + \frac{9}{2}u_1u_3^2 + 6u_2u_3 + u_1 + \alpha \circ g$$
$$g = (u_1, u_2, u_3^4 + u_1u_3^3 + u_2u_3^2).$$

III) $n = 3$;

(1) $$\mu = u_4$$
$$g = (u_1, u_2, u_3, u_4)$$

(2) $$\mu = \frac{2}{3}u_1^3 + u_2 + u_3 + u_4$$
$$g = (u_1^2, u_2, u_3, u_4)$$

(3) $$\mu = u_4 - \frac{1}{2}u_1$$
$$g = (u_1, u_2, u_3, u_4^2)$$

(4) $$\mu = \frac{3}{4}u_1^4 + \frac{1}{2}u_1^2u_2 + u_3 + u_4$$
$$g = (u_1^3 + u_2u_1, u_2, u_3, u_4)$$

(5) $$\mu = u_3$$
$$g = (u_1, u_2, u_3, u_4^3 + u_1u_4 + u_2 + u_3)$$

(6) $$\mu = -3u_4^2 + 4u_1u_4 + u_2$$
$$g = (u_1, u_2, u_3, u_4^3 + u_1u_4^2)$$

(7) $$\mu = \frac{4}{5}u_1^5 + \frac{1}{2}u_2u_1^2 + \frac{2}{3}u_3u_1^3 + u_4$$
$$g = (u_1^4 + u_2u_1 + u_3u_1^2, u_2, u_3, u_4)$$

(8) $$\mu = 2u_2^3 \pm 2u_1^2u_2 + u_3u_1^2 + u_4$$
$$g = (u_1u_2 + u_1u_3, u_1^2 \pm 3u_2^2, u_3, u_4)$$

(9) $$\mu = u_4$$
$$g = (3u_1^2 + u_3u_4, u_2, u_3, 2u_1^3 + u_4^2 + u_2u_4)$$

$$\text{(10)} \qquad \mu = u_4$$
$$g = (u_1, u_2, u_3, u_4^4 + u_1u_4 + u_2u_4^2)$$
$$\text{(11)} \qquad \mu = -4u_4^3 + \frac{9}{2}u_1u_4^2 + 6u_2u_4 + u_3$$
$$g = (u_1, u_2, u_3, u_4^4 + u_1u_4^3 + u_1u_4^2).$$
$$\text{(12)} \qquad \mu = \frac{5}{6}u_1^6 + \frac{1}{2}u_2u_1^2 + \frac{2}{3}u_3u_1^3 + \frac{3}{4}u_4u_1^4 + u_4 + \alpha \circ g$$
$$g = (u_1^5 + u_2u_1 + u_3u_1^2 + u_4u_1^3, u_2, u_3, u_4)$$
$$\text{(13)} \qquad \mu = 3u_2^4 \pm 2u_1^2u_2 + u_4u_2^2 + u_4 + \alpha \circ g$$
$$g = (u_1u_2 + u_1u_3, u_1^2 \pm 4u_2^3 + 2u_4u_2, u_3, u_4)$$
$$\text{(14)} \qquad \mu = u_4 + \alpha \circ g$$
$$g = (u_1, u_2, u_3, u_4^5 + u_1u_4 + u_2u_4^2 + u_3u_4^3)$$
$$\text{(15)} \qquad \mu = -5u_4^4 + \frac{16}{3}u_1u_4^3 + 6u_2u_4^2 + 8u_3u_4 + u_1 + \alpha \circ g$$
$$g = (u_1, u_2, u_3, u_4^5 + u_1u_4^4 + u_2u_4^3 + u_3u_4^2).$$

Here, α are C^∞-function germs and the notion of genericity of germs with complete integral will be defined in §3.

Since diagrams (μ, g) are integral diagrams, then α must satisfy some conditions, however, we do not argue about such conditions here.

One of the advantages of our theory is to draw the phase portraits of equations even if we cannot solve exactly. For example, in the case $n = 2$; (4) in the above list , we have

$$g = (u_1, u_2, u_3^2)$$
$$\mu = \frac{3}{4}u_1^4 + \frac{1}{2}u_1^2u_2 + u_3.$$

We can easily get

$$g(\mu^{-1}(c)) = (u_1^3 + u_2u_1, u_2, c - (\frac{3}{4}u_1^4 + \frac{1}{2}u_1^2u_2 + u_3))$$

for any $c \in \mathbf{R}$. This is a family of swallowtail. Furthermore, the discriminant set is the critical value set of g in this case, so that, it is the image of $G(v_1, v_2) = (-2v_1^3, -3v_1^2, v_3))$ which is the cuspidal edge. If we can draw both pictures on the same screen by a computer, we can draw in Fig.1.

The family of swallowtail should be moved along the cuspidal edge if we change the parameter. In fact, I asked to Dr. Richard Morris in Liverpool University to draw the picture by IRIS computer, then we could get the animation which describe this situation.

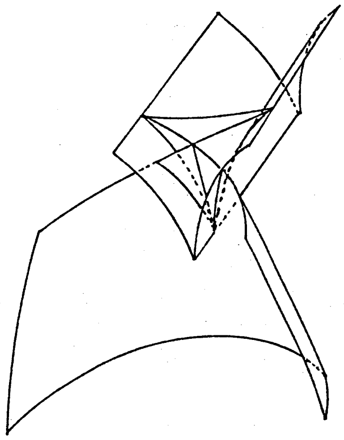

Fig.1

The pictures of other type perestroikas of the graph of solutions is in the appendix.

REFERENCES

1. V. I. Arnol'd, S. M. Gusein-Zade and A. N. Varchenko, "Singularities of Differentiable Maps vol. I," Birkhäuser, 1986.
2. C. Carathèodory, "Calculus of Variations and Partial Differential Equations of First Order, Part I, Partial Differential Equations of First Order,," Holden-Day, 1965.
3. R.Courant and D. Hilbert, "Methods of mathematical physics I, II," Wiley, 1962.
4. A. R. Forsyth, "Theory of differential equations, Part *III* partial differential equations," Cambridge Univ. Press, London, 1906.
5. A. Hayakawa, G. Ishikawa, S. Izumiya and K. Yamaguchi, *Classification of generic integral diagram and first order ordinary differential equations*, preprint.
6. S. Izumiya, *Geometric singularities for Hamilton-Jacobi equation*, Advanced Studies in Pure Math. (1991) (to appear).
7. S. Izumiya, *Generic properties of first order partial differential equations*, Proceedings of Topology conference in Hawaii, Birkhäuser. (to appear)
8. S. Izumiya, *The theory of Legendrian unfoldings and first order differential equations*, preprint.
9. S. Izumiya, *Completely integrable holonomic systems of first order differential equations*, preprint.
10. S. Izumiya, *Singular solutions of first order differential equations*, preprint.
11. S. Izumiya, *The Clairaut type equation*, preprint.
12. S. Izumiya, *What is the Clairaut euqation ?*, preprint.
13. M. Kossowski, *First order partial differential equations with singular solution*, Indiana Univ. Math. Jour. **35** (1986), 209–223.
14. V. V. Lychagin, *Local classification of non-linear first order partial differential equations*, Russian Math. Surveys **30** (1975), 105–175.
15. V. M. Zakalyukin, *Lagrangian and Legendrian singularities*, Funct. Anal. Appl. **10** (1976), 23–31.
16. V. M. Zakalyukin, *Reconstructions of fronts and caustics depending on a parameter and versality of mappings*, Journal of Soviet Math. **27** (1983), 2713–2735.

Appendix : Pictures of the perestroikas of graphs of solutions