数理解析研究所講究録
第 807 巻 1992 年 119-125

On the global theory of singularities

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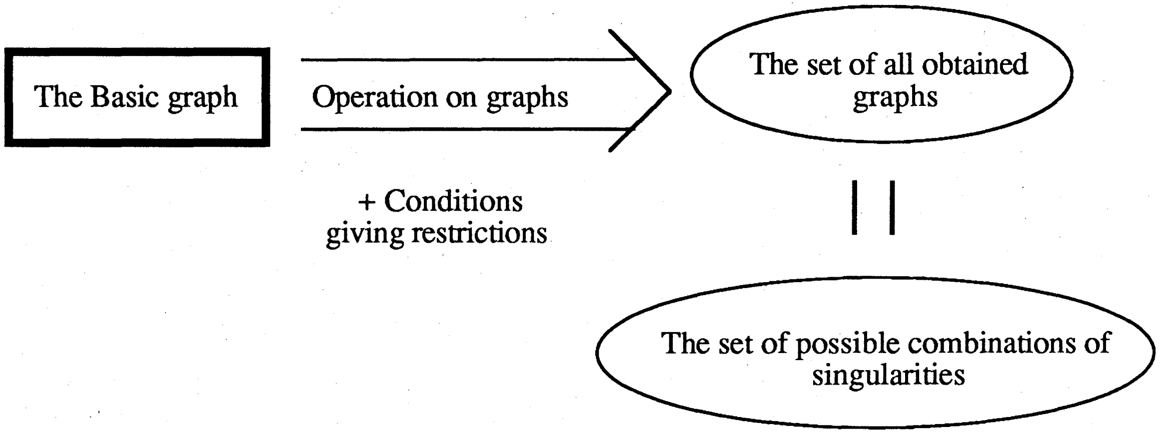
Recently I am considering on the global theory of singularities.

Perhaps what I have just stated reminds you of the Plücker formula for plane irreducible curves. As you know, it is an equality connecting integers defined as local invariants of singularities and integers defined as global invariants of varieties such as genus, degree, and class number. It restricts possible combinations of singularities. The data obtained from an actual combination of singularities satisfy the Plücker formula. However, given global invariants and a combination of local singularities satisfying it, there does not necessarily exist a plane curve with such data. For example, a plane sextic rational curve with unique singularity of type A_{20} (the singularity locally defined by $x^{21}+y^2=0$) never exists, though the data satisfy the Plücker formula:

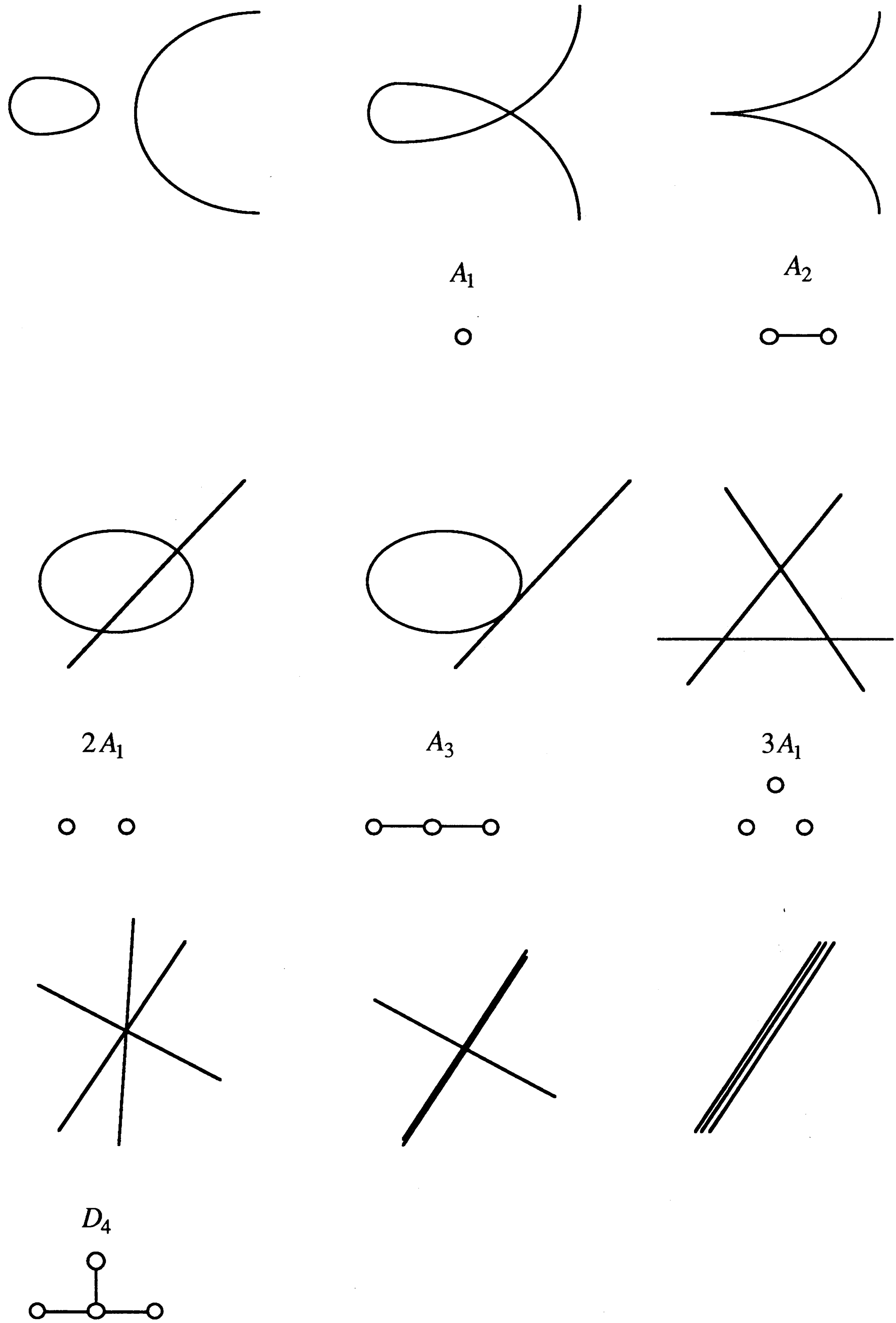
$$g=\frac{(d-1)(d-2)}{2}-\sum_{x\in Sing.}\delta_x.$$

(In this case $g=0, d=6$ and $\delta_x=10$ for an A_{20}-singularity.)

Our point of view is different from this. We use graphs for the description instead of integers. (By a graph we mean a finite one-dimensional complex with some additional structure.) Moreover, we aim to give the necessary and sufficient condition for the description of possible combinations of singularities. The following figure explains our basic framework.



Here we consider a typical example to explain the above figure. We consider the case of cubic curves in the two-dimensional projective space $\mathbb{P}^2=\mathbb{P}^2(\mathbb{C})$ over the complex field $\mathbb{C}$. (Below we always assume that the ground field is the complex field $\mathbb{C}$.) It is easy to give the classification of plane cubic curves. We have the following 9 items. (Below we draw the figure of the set of real points, i.e. the intersection of the curve in $\mathbb{P}^2(\mathbb{C})$ and $\mathbb{P}^2(\mathbb{R})$, because we cannot draw the actual figure of the set of complex points.) The graphs beneath the figures are explained later.


A_1
A_2
$2A_1$
A_3
$3A_1$
D_4

The eighth item and the ninth have a multiple component, and they have degree 2 and 1 respectively as figures, if we ignore the multiplicity. We can exclude them from our consideration. Below we consider only 7 items without a multiple component.

Now, we know a series of singularities much closely related to Dynkin graphs. They are called ADE singularities, i.e., singularities related to Dynkin graphs of type A, D or E. (The explanation of this relation is one of the deepest aspects of modern achievement of mathematics.) The concept of Dynkin graphs is well-known because it plays the key role in the classification theory of semi-simple Lie groups. The local defining equations $f(x,y)=0$ of these singularities of dimension 1 are as follows:

$$A_k: x^{k+1}+y^2=0 \qquad (k=1,2,3,\cdots)$$
$$D_\ell: x^{\ell-1}+xy^2=0 \qquad (\ell=4,5,6.\cdots)$$
$$E_6: x^3+y^4=0$$
$$E_7: x^3+xy^3=0$$
$$E_8: x^3+y^5=0$$

(The above are equations of curve singularities. When we consider surface singularities we add a term z^2 to the above respective equation and we consider the singularity defined by $f(x,y)+z^2=0$. For example the surface singularity of type A_k is defined by $x^{k+1}+y^2+z^2=0$.)

The above seventh cubic curve has a unique singularity and it is of type D_4. We draw a Dynkin graph of type D_4 beneath the seventh curve. By the same method we can associate a Dynkin graph (possibly with several components) to each cubic curve. We have the empty graph, A_1, A_2, $2A_1$, A_3, $3A_1$ and D_4.

Here perhaps you can notice that the classification of cubic curves corresponds to subgraphs of D_4. 7 types of cubic curves have one-to-one correspondences with 7 kinds of subgraphs of D_4.

In the case of plane cubic curves the basic graph is the Dynkin graph of type D_4. The operation is to pick a subgraph. There is no condition giving restrictions. The set of all graph obtained from D_4 under this situation is equal to the set of possible combinations of singularities.

As suggested by the above example of cubic curves, the meaning of our basic framework above is as follows.

First we set up an appropriate range of objects we treat. For example we consider one of the followings ([5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [17], [18]):

1. All plane cubic curves without multiple components.
2. All plane quartic curves without multiple components.
3. All plane sextic curves with only ADE singularities as singularities.
4. All space cubic surfaces with only isolated singularities.
5. All space quartic surfaces with only ADE singularities as singularities.
6. All complete intersections with bidegree (2, 2) in the four dimensional projective space.

7. All complete intersections with bidegree (2, 3) in the four dimensional projective space. In the seventh case we assume that all singularities are ADE singularities.

For the range of objects we consider, the basic graph is determined. For simpler cases the basic graph is unique. Some basic graphs can appear in the complicated cases.

Next the operation on graphs is defined. By the operation we can make new graphs from a given graph. At present we use three kinds of operations – to pick a subgraph, an elementary transformation, and a tie transformation. These three operations have many choices on the way of their process, and we can make considerably many kinds of graphs from a given one.

In addition we have some conditions giving restrictions on the process of operations.

Under the above conditions we consider the set of all graphs obtained starting from one of the basic graphs. A member of this set is not necessarily connected. The type of a connected component corresponds to the type of a singularity, and the number of connected components of each type indicates the number of corresponding singularities. In this way a graph is translated into the meaning of a combination of singularities, and the set of obtained graphs coincides with the set of possible combinations of singularities.

So far we have explained the superficial part of our theory. Indeed the greater part is hidden in the background. First, as the best weapon, we use the theory of periods of algebraic varieties. By this we translate the problem into the language of integral symmetric bilinear forms. It is known that interesting groups are associated with bilinear forms. So using the group theory, in particular using the theory of reflection groups, we translate it further into the language of graphs.

The above is our philosophy. In the case of sufficiently low dimension, low degree and low codimension, this philosophy works miraculously well and we can get the splendid description of possible combinations of singularities. In fact this part is the discovery of English mathematician Timms at the beginning of this century [6]. English mathematician Du Val wrote another paper giving interpretation of Timms' result from his view point little later [1].

We can imagine that we are looking at only a small part, and that there exists a hidden general principle in the background, since Timms' result is very beautiful. Therefore we can try to search how far our philosophy can be applied, and if possible we would be able to find out the hidden general principle. This is our objective. Today we have theory of periods of K3 surfaces, fully developed theory of integral bilinear forms, the theory of reflection groups including Weyl groups and the like. Using these theories we can go to a considerably deep point.

However, the world of actual algebraic varieties is much more complicated than what we expect. Therefore sometimes the vertical equality in the first figure does not exactly hold and a few exceptions appear. This is a really strange phenomenon actually occurring, though we have defined absolutely correct basic graphs, operations, restriction conditions etc. Our theory may have some defects, or there may be a faint break in the complete law because of the wisdom of the god.

Besides, at present, only ADE singularities can correspond to connected graphs. More complicated singularities have appeared in the objects of our study, but I do not know how we treat them exactly. We have some evidences showing that similar correspondences can be defined for such complicated singularities.

Now, even in the modern ages the theory of periods is not complete, and we cannot go further beyond a certain point. If we go beyond the cases related to K3 surfaces, we encounter difficulty. In the theory of periods there is little theory for characterizing points in the period domain corresponding to actual algebraic varieties. We know that we can generalize our philosophy and framework further, if only the theory for this characterization is developed. (If we cannot give a necessary and sufficient condition, a sufficiently good condition satisfied by points corresponding to actual algebraic varieties is enough for us. Perhaps to give a necessary and sufficient condition is an extremely difficult problem.)

At present on one hand I am making steady efforts to take complicated worse singularities in our theory and efforts to develope the theory to other aspects, and on the other hand I expect that I can find by chance a key to the hidden general principle from a different view point (for example a view point of the representation theory of Lie groups).

In the above explanation of our philosophy we have omitted some facts to help reader's understanding. Here we supplement them.

The most important point is that by the framework in the first figure we can treat not only combinations of singularities of global algebraic varieties but also other objects. We can treat the following two items by the present theory of ours:

A. Possible combinations of singular fibers of elliptic surfaces.

B. Possible combinations of singularities of local objects such as deformation fibers in the semi-universal deformation family of a fixed isolated singularity.

We explain the item A. Let $\Phi: X \to \mathbb{P}^1$ be an elliptic surface over $\mathbb{P}^1$. (X is a smooth compact complex surface. For any general point $x \in \mathbb{P}^1$ the inverse image $\Phi^{-1}(x)$ is a curve of genus 1.) We assume that Φ has no multiple fibers. It is known that the Euler number $e(X)$ of X is positive and a multiple of 12.

When $e(X) = 12, X$ is a rational surface, and we can describe possible combinations of singular fibers using our framework. In this case the basic graph is the Dynkin graph of type E_8 and we can apply elementary transformations twice as the operation. There is no restriction condition. We can associate the resulting Dynkin graphs with possible combinations of singular fibers ([2], [7]).

When $e(X) = 24$, X is a K3 surface. Also in this case we can develope our theory. Some partial results have already been obtained ([16]). I think that we can complete the theory in this case in future.

As for the item B, we know that in the cases corresponding to the following singularities possible combinations of singularities of deformation fibers can be described using our framework:

1. ADE singularities.
2. 3 kinds of hypersurface simple elliptic singularities ([2], [4], [7]).
3. A part of cusp singularities ([3]).
4. A part of 14 kinds of hypersurface triangle singularities.
5. 6 kinds of hypersurface quadrilateral singularities ([16]).

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