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The duality of Tsuchihashi cusp singularities

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Introduction

A cusp singularity is known as a normal surface singularity whose exceptional divisor of a suitable resolution is a cycle of nonsingular rational curves. In [N], Nakamura showed that each cusp singularity (V,p) has natural dual cusp singularity (V^*,p^*), and that invariants of these singularities have some dual relations.

One of these relations is the following:

Let $D_1+\cdots+D_r$ and $E_1+\cdots+E_s$ are the exceptional divisor of the resolution of V and V^*, respectively. We assume these are cycles of nonsingular rational curves. Then the following equality consisting of selfintersection numbers holds.

(1) $$D_1^2+\cdots+D_r^2+3r=-(E_1^2+\cdots+E_s^2+3s)\ .$$

On the other hand, higher dimensional cusp singularities are introduced by Tsuchihashi [T]. We established an equality which is a generalization of the equality for these cusp singularities.

Let N be a free $\mathbf{Z}$-module of rank $r<\infty$ and M the dual $\mathbf{Z}$-module. We assume that r is at least 2. We consider a pair (C,Γ) of an open convex cone C in $N_{\mathbf{R}} := N\otimes_{\mathbf{Z}}\mathbf{R}$ and a subgroup Γ of $\mathrm{Aut}(N)\simeq \mathrm{GL}(r,\mathbf{Z})$ with the following properties.

(1) For the closure $\bar{C}$ of C, $\bar{C}\cap(-\bar{C})=\{0\}$.

(2) $gC=C$ for every $g\in\Gamma$.

(3) The action of Γ on C is properly discontinuous and free.

(4) The quotient $(C/\mathbf{R}_+)/\Gamma$ is compact.

For such a pair (C,Γ), Tsuchihashi [T] constructed a complex analytic isolated singularity $V(C,\Gamma)$ by using the theory of toric varieties and called it a *cusp singularity.*

This cusp singulairty has a natural dual. Namely, let C^* be the interior of the cone $\{x \in M_{\mathbf{R}}\ ;\ \langle x,a\rangle \geq 0, \forall a \in C\}$ and $\Gamma^* := {}^t\Gamma$, where $M_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R}$ and $\langle\ ,\ \rangle : M_{\mathbf{R}} \times N_{\mathbf{R}} \to \mathbf{R}$ is the natural bilinear map. Then the pair (C^*,Γ^*) satisfies similar condition and hence defines a cusp singularity $V(C^*,\Gamma^*)$. We call $V(C^*,\Gamma^*)$ the dual cusp singularity of $V(C,\Gamma)$. Clearly, the dual of $V(C^*,\Gamma^*)$ is equal to $V(C,\Gamma)$.

The arithmetic genus defect χ_∞ and Ogata's zeta zero $Z(0)$ are numerical invariants defined for cusp singularities. Here note that our cusp singularities are called "Tsuchihashi singularities" in [SO], and the zeta function is defined by

$$Z(s) = \sum_{u \in (C\cap M)/\Gamma} \phi_C(u)^s\ ,$$

where $\phi_C(x)$ is the characteristic function of the cone C [SO, 4.2]. As it is mentioned in [SO, 4.2], this zeta function is slightly different from the one defined by the norm function in the case of self-dual homogeneous cones. However, the values at zero of these zeta functions are equal [SO, 4.2]. In this note, we denote this value by $Z(0)(C,\Gamma)$.

On the other hand, $\chi_\infty(p)$ for a cusp singularity p of dimension r is described explicitly as follows: We take a resolution of the singularity such that the exceptional set is a toric divisor $\cup_{i=1}^s D_i$ with simple normal crossing. Then $\chi_\infty(p)$ is equal to the intersection number

$$\left[\prod_{i=1}^s \frac{D_i}{1-\exp(-D_i)}\right]_r .$$

We get the following theorem [I5].

Theorem *The rational number $\chi_\infty(C^*,\Gamma^*)$ is equal to $(-1)^r Z(0)(C,\Gamma)$.*

This is a generalization of the equality (1) since it is written as

$$-12Z(0)(C,\Gamma) = -12\chi_\infty(C^*,\Gamma^*)$$

in our new notation.

For the convenience to understand the theorem, we will explain $Z(0)$ for $V(C,\Gamma)$ and χ_∞ for $V(C^*,\Gamma^*)$.

We introduce here some notations in this note.

Besides some open cones as C and C^*, cones are always closed convex rational polyhedral cones. Namely, a cone π in $N_{\mathbf{R}}$ is equal to $\mathbf{R}_0 n_1 + \cdots + \mathbf{R}_0 n_s$ for a finite subset $\{n_1,\cdots,n_s\}$ of the lattice N, where $\mathbf{R}_0 := \{c \in \mathbf{R}\ ;\ c \geq 0\}$. For a cone π in $N_{\mathbf{R}}$, the linear subspace $\pi + (-\pi)$ of $N_{\mathbf{R}}$ is denoted by $H(\pi)$. The interior of π as a subset of $H(\pi)$ is called the *relative interior* of π and is denoted by $\mathrm{rel.int}\,\pi$.

We denote $\sigma \prec \pi$ if σ is a face of a cone π. We denote by $F(\pi)$ the set of faces of π. π is said to be strongly convex if $\pi \cap (-\pi) = \{0\}$ or equivalently if the zero cone $\mathbf{0} := \{0\}$ is in $F(\pi)$.

A nonempty collection Φ of strongly convex cones in $N_{\mathbf{R}}$ is said to be a *fan* if (1) $\pi \in \Phi$ and $\sigma \prec \pi$ imply $\sigma \in \Phi$, and (2) if $\sigma, \tau \in \Phi$, then $\sigma \cap \tau$ is a common face of σ and τ. For a subset Ψ of a fan Φ and an element $\rho \in \Phi$, we denote $\Psi(\prec\rho) := \{\sigma \in \Psi\ ;\ \sigma \prec \rho\}$ and $\Psi(\rho\prec) := \{\sigma \in \Psi\ ;\ \rho \prec \sigma\}$. For an integer d we denote $\Psi(d) := \{\sigma \in \Psi\ ;\ \dim\sigma = d\}$.

For a subset $S \subset N_{\mathbf{R}}$, we denote $S^\perp := \{x \in M_{\mathbf{R}}\ ;\ \langle x,a\rangle = 0, \forall a \in S\}$ and $S^\vee := \{x \in M_{\mathbf{R}}\ ;\ \langle x,a\rangle \geq 0, \forall a \in S\}$. For a (closed convex) cone $\pi \subset N_{\mathbf{R}}$, $\pi^\vee \subset M_{\mathbf{R}}$ is called the dual cone of π. It is known that the correspondences $\sigma \mapsto \pi^\vee \cap \sigma^\perp$ define a bijection of $F(\pi)$ and $F(\pi^\vee)$ [O, Prop.A.6].

We use same notations for cones in the other real vector spaces with lattices.

1 The T-complexes

The notion of T-complexes was introduced in [I2] in order to describe the combinatorial structures of toric divisors. We briefly review the definition.

Let r be a positive integer and let $\mathcal{C}_r$ be the category of pairs $\alpha = (N(\alpha), c(\alpha))$ of free $\mathbf{Z}$-module $N(\alpha)$ of rank r and a strongly convex rational polyhedral cone $c(\alpha) \subset N(\alpha)_{\mathbf{R}}$. For two objects α, β of $\mathcal{C}_r$, a morphism $u : \alpha \to \beta$ consists of an isomorphism $u_{\mathbf{Z}} : N(\alpha) \longrightarrow N(\beta)$ such that $u_{\mathbf{R}}(c(\alpha))$ is a face of $c(\beta)$, where $u_{\mathbf{R}} := u_{\mathbf{Z}} \otimes 1_{\mathbf{R}}$. For a morphism u, we denote by $i(u)$ the source and by $f(u)$ the target, respectively, of u.

A subcategory Σ of $\mathcal{C}_r$ is said to be a *graph of cones* of dimension r if the objects and the morphisms in Σ are finite in number. The set of morphisms in Σ is denoted by mor Σ.

Let ρ be an object of a graph of cones Σ. We define graphs of cones $\Sigma(\rho\prec)$ and $\Sigma(\prec\rho)$ as follows: $\Sigma(\rho\prec)$ consists of the pairs $\beta' = (\beta, v)$ of $\beta \in \Sigma$ and $v \in \operatorname{mor}\Sigma$ with $i(v) = \rho$ and $f(v) = \beta$ for which we define $N(\beta') = N(\beta)$ and $c(\beta') = c(\beta)$. For $\beta' = (\beta, v)$ and $\gamma' = (\gamma, w)$ in $\Sigma(\rho\prec)$, a morphism $u' : \beta' \to \gamma'$ consists of $u : \beta \to \gamma$ with $u \circ v = w$. Similarly, $\Sigma(\prec\rho)$ consists of pairs $\alpha' = (\alpha, v)$ with $v \in \operatorname{mor}\Sigma$ of the source α and the target ρ.

For each $\beta' = (\beta, v) \in \Sigma(\rho\prec)$, we define $\beta'[\rho]$ by $N(\beta'[\rho]) := N(\beta)[v_{\mathbf{R}}(c(\rho))]$ and $c(\beta'[\rho]) := c(\beta)[v_{\mathbf{R}}(c(\rho))]$, and for each $u' : \beta' \to \gamma' \in \operatorname{mor}\Sigma(\rho\prec)$, we define $u'[\rho]_{\mathbf{Z}} : N(\beta'[\rho]) \longrightarrow N(\gamma'[\rho])$ to be the isomorphism induced by $u'_{\mathbf{Z}}$. Then we get a graph of cones $\Sigma[\rho]$ of dimension $r - \dim\rho$ which is equivalent to $\Sigma(\rho\prec)$ as categories.

For a finite fan Δ of $N_{\mathbf{R}}$, any subset Σ of Δ is regarded as a graph of cones by defining $N(\alpha) := N$ and $c(\alpha) := \alpha$ for each $\alpha \in \Sigma$ and defining that a morphism $u : \alpha \to \beta$ for $\alpha, \beta \in \Sigma$ is in mor Σ if and only if $u_{\mathbf{Z}} = 1_N$.

A free cone $\alpha = (N(\alpha), c(\alpha))$ is said to be nonsingular if $c(\alpha)$ is a nonsingular cone of $N(\alpha)_{\mathbf{R}}$, i.e., $c(\alpha) = \mathbf{R}_0 x_1 + \cdots + \mathbf{R}_0 x_{d(\alpha)}$ for a basis $\{x_1, \cdots, x_{r(\alpha)}\}$.

A graph of cones Σ of dimension r is called a *T-complex*, if it satisfies the following conditions.

(1) Σ is nonempty and connected.

(2) The graph of cones $\Sigma(\prec\rho)$ is isomorphic to $F(\rho) \setminus \{0\}$ for every $\rho \in \Sigma$, where $F(\rho)$ is the fan consisting of the faces of ρ.

(3) For each $\rho \in \Sigma$, the graph of cones $\Sigma[\rho]$ is isomorphic to a complete fan of $N(\rho)[\rho]_{\mathbf{R}}$.

A T-complex Σ is said to be *nonsingular* if it consists of nonsingular free cones.

We define the support $|\Sigma|$ of a T-complex Σ as the disjoint union

$$\coprod_{\alpha\in\Sigma} (c(\alpha) \setminus \{0\})$$

modulo the equivalence relation generated by $a \sim u_{\mathbf{R}}(a)$ for $u : \alpha \to \beta \in \operatorname{mor}\Sigma$ and $a \in c(\alpha) \setminus \{0\}$.

A *morphism* $\varphi : \Sigma' \longrightarrow \Sigma$ of T-complexes consists of a functor $\bar{\varphi} : \Sigma' \longrightarrow \Sigma$ and a collection $\{\varphi^\alpha\ ;\ \alpha \in \Sigma\}$ of injective $\mathbf{Z}$-homomorphisms $\varphi^\alpha : N(\alpha) \longrightarrow N(\bar{\varphi}(\alpha))$ such that $\varphi^\alpha_{\mathbf{R}}(\operatorname{rel.int} c(\alpha)) \subset \operatorname{rel.int} c(\bar{\varphi}(\alpha))$ and the diagram

$$\begin{array}{ccc} N(\alpha) & \xrightarrow{u_{\mathbf{Z}}} & N(\beta) \\ \varphi^{\alpha}\downarrow & & \downarrow\varphi^{\beta} \\ N(\bar{\varphi}(\alpha)) & \xrightarrow{\bar{\varphi}(u)_{\mathbf{Z}}} & N(\bar{\varphi}(\beta)) \end{array}$$

is commutative for every $u : \alpha \to \beta \in \operatorname{mor}\Sigma$.

A morphism $\varphi : \Sigma' \longrightarrow \Sigma$ of T-complexes induces a map $|\varphi| : |\Sigma'| \longrightarrow |\Sigma|$ of the supports. The morphism φ is said to be a *subdivision* if the all φ^{α}'s are isomorphic and $|\varphi|$ is a bijection.

Example 1.1 Let $\tilde{\Sigma}$ be a fan of $N_{\mathbf{R}}$ such that $C_1 := |\tilde{\Sigma}| \setminus \{0\}$ is an open cone and $\tilde{\Sigma}$ is locally finite at each point of C_1. Assume that a subgroup $\Gamma_1 \subset \operatorname{Aut}(N)$ induces a free action on $\tilde{\Sigma} \setminus \{\mathbf{0}\}$ and the quotient is finite. Let Σ be the set of representatives of the free quotient. For each $\alpha \in \Sigma$, we set $N(\alpha) := N$ and $c(\alpha) := \alpha$ and we define

$$\operatorname{mor}\Sigma := \{u : \alpha \to \beta \, ; \, u_{\mathbf{Z}} \in \Gamma_1\} \, .$$

Then Σ is a T-complex. Let $\tilde{\Sigma}'$ be a Γ_1-equivariant subdivision of $\tilde{\Sigma}$ and Σ' the T-complex obtained from $\tilde{\Sigma}'$ by the action of Γ_1. Then Σ' is a subdivision of Σ, since both $|\Sigma'|$ and $|\Sigma|$ are naturally bijective to the quotient C_1/Γ_1.

Let (C, Γ) be the pair which defines a cusp singularity.

We take a Γ-invariant nonsingular fan $\tilde{\Xi} \cup \{\mathbf{0}\}$ of $N_{\mathbf{R}}$ with the support $C \cup \{0\}$ which is locally finite at each point of C. Simmilarly, we take Γ^*-invariant nonsingular fan $\tilde{\Delta} \cup \{\mathbf{0}\}$ of $M_{\mathbf{R}}$ with the support $C^* \cup \{0\}$ which is locally finite at each point of C^*.

Here we assume $\mathbf{0} \notin \tilde{\Xi}$ and $\mathbf{0} \notin \tilde{\Delta}$ for the convenience of the notations.

Then these are the cases of the above example, and we get T-complexes $\Xi = \tilde{\Xi}/\Gamma$ and $\Delta = \tilde{\Delta}/\Gamma^*$.

2 The invariants $Z(0)$ and χ_∞

For a graph of cones Φ, the set of morphisms in Φ is denoted by mor Φ. This is a finite set by definition. For a covariant functor

$$A : \mathcal{C} \to (\text{Additive groups}) ,$$

we denote by A_Φ the restriction of A to Φ. In other words, A_Φ is the finite system of the additive groups $(A(\alpha))_{\alpha\in\Phi}$ and the homomorphisms $(A(u) : A(i(u)) \to A(f(u)))_{u\in \mathrm{mor}\,\Phi}$. The inductive limit ind lim A_Φ of the system A_Φ is described as the cokernel

$$\bigoplus_{u\in \mathrm{mor}\,\Phi} A(i(u)) \underset{q}{\overset{p}{\rightrightarrows}} \bigoplus_{\alpha\in\Phi} A(\alpha) \longrightarrow \mathrm{ind\,lim}\, A_\Phi ,$$

where p consists of the identities $1_{A(i(u))} : A(i(u)) \to A(i(u)) \subset \bigoplus_{\alpha\in\Phi} A(\alpha)$ and q consists of the homomorphisms $A(u) : A(i(u)) \to A(f(u)) \subset \bigoplus_{\alpha\in\Phi} A(\alpha)$.

For a nonsingular free cones $\alpha = (N(\alpha), c(\alpha))$, we set gen $\alpha := \{x_1, \cdots, x_{d(\alpha)}\}$ and $x(\alpha) := \prod_{x\in \mathrm{gen}\,\alpha} x \in \mathrm{S}^{d(\alpha)}(N(\alpha)_{\mathbf{Q}})$, where S^d means the d-th symmetric power over the rational number field $\mathbf{Q}$. We denote by $\mathcal{C}^{\mathrm{n.s.}}$ the subcategory of $\mathcal{C}$ consisting of nonsingular free cones.

A functor $D^0 : \mathcal{C}^{\mathrm{n.s.}} \to$ ($\mathbf{Q}$-vector spaces) is defined by

$$D^0(\alpha) := \{f/x(\alpha) \,;\, f \in \mathrm{S}^{d(\alpha)}(N(\alpha)_{\mathbf{Q}})\} .$$

For $u : \alpha \to \beta$, $D^0(u) : D^0(\alpha) \to D^0(\beta)$ is defined to be the natural injection induced by the isomorphism $u_{\mathbf{Q}} : N(\alpha)_{\mathbf{Q}} \to N(\beta)_{\mathbf{Q}}$. Note that gen α is mapped into gen β by $u_{\mathbf{Z}}$.

Let $\mathbf{Q}^{\sim} : \mathcal{C} \to$ ($\mathbf{Q}-$vector spaces) be the constant functor defined by $\mathbf{Q}^{\sim}(\alpha) := \mathbf{Q}$ and $\mathbf{Q}^{\sim}(u) := 1_{\mathbf{Q}}$ for all $\alpha \in \mathcal{C}$ and $u \in \mathrm{mor}\,\mathcal{C}$. Since Ξ is connected as a graph of cones, we have ind lim $\mathbf{Q}^{\sim}_{\Xi} = \mathbf{Q}$.

Since each $D^0(\alpha)$ contains $\mathbf{Q}$, there exists a natural morphism of functors $\epsilon_\Xi : Q^{\sim}_{\Xi} \to D^0_{\Xi}$. By [I1, Lem.3.1], the $\mathbf{Q}$-linear map

$$\mathbf{Q} = \mathrm{ind\,lim}\, \mathbf{Q}^{\sim}_{\Xi} \to \mathrm{ind\ lim}\, D^0_{\Xi}$$

is injective. Hence we regard $\mathbf{Q}$ as a linear subspace of ind lim D^0_{Ξ}.

We recall some notations in [I2] with exchanging the roles of M and N.

We denote by $\mathbf{Q}(N)$ the quotient field of the group ring $\mathbf{Q}[N] = \bigoplus_{n\in N} \mathbf{Q}\mathrm{e}(n)$. For a nonsingular cone σ in $N_{\mathbf{R}}$, the elements $Q_0(\sigma)$ and $Q(\sigma)$ are defined by

$$Q_0(\sigma) = \prod_{y\in \mathrm{gen}\,\sigma} \frac{\mathrm{e}(y)}{1-\mathrm{e}(y)} \in \mathbf{Q}(N)$$

and

$$Q(\sigma) = \prod_{y\in \mathrm{gen}\,\sigma} \frac{1}{1-\mathrm{e}(y)} \in \mathbf{Q}(N)\ .$$

Let $\varepsilon : M \otimes \mathbf{C} \to M \otimes \mathbf{C}^*$ be the holomorphic map defined by $\varepsilon(m \otimes z) := m \otimes \exp(-z)$. For each $y \in N$, $\mathrm{e}(y)$ is a regular function on $M \otimes \mathbf{C}^*$, and the pull back $\varepsilon^*\mathrm{e}(y)$ is equal to $\exp(-y)$. For a nonsingular cone σ in $N_{\mathbf{R}}$,

$$x(\sigma)\varepsilon^*Q_0(\sigma) = \prod_{y\in \mathrm{gen}\,\sigma} \frac{y\exp(-y)}{1-\exp(-y)} = \prod_{y\in \mathrm{gen}\,\sigma} \frac{y}{\exp(y)-1}\ .$$

is an entire function on $M \otimes \mathbf{C}$. We denote by $[\varepsilon^*Q_0(\sigma)]_0$ the rational function $f_d/x(\sigma)$, where f_d is the homogeneous degree $d := \dim\sigma$ part of the Taylor expansion of $x(\sigma)\varepsilon^*Q_0(\sigma)$ at the origin.

For each α of the T-complex Ξ, we set

$$\omega(\alpha) := [\varepsilon(\alpha)^*Q_0(c(\alpha))]_0 \in D^0(\alpha)\ ,$$

where $\varepsilon(\alpha) = 1_{M(\alpha)} \otimes \exp(-*) : M(\alpha) \otimes \mathbf{C} \to M(\alpha) \otimes \mathbf{C}^*$. The class of $(\omega(\alpha))_{\alpha\in\Xi}$ in $\mathrm{ind}\lim D^0_\Xi$ is denoted by $\omega(\Xi)$.

The main result of [I1] is the following.

Theorem 2.1 *The class $\omega(\Xi) \in \mathrm{ind}\lim D^0_\Xi$ is in $\mathbf{Q}$, and this rational number is equal to the zeta zero value $Z(0)(C,\Gamma)$ of the cusp $V(C,\Gamma)$.*

The value $Z(0)(C,\Gamma)$ can be calculated as follows:

A morphsm of fuctors $\nu : D^0_\Xi \to \mathbf{Q}^\Xi_\Xi$ is said to be a *retraction* if the composition $\nu \cdot \epsilon_\Xi$ is the identity. It was shown that retractions always exist [I1, Lem.3.1]. Then $Z(0)(C,\Gamma)$ is equal to $\sum_{\alpha\in\Xi} \nu(\alpha)(\omega(\alpha))$.

Now, we consider the nonsingular fan $\tilde{\Delta} \cup \{0\}$ of $M_{\mathbf{R}}$. For $\rho \in \tilde{\Delta}$ and an integer $n \geq 0$, we denote by $\mathrm{Index}(\rho, n)$ the set of maps $\mathbf{f} : \mathrm{gen}\,\rho \to \mathbf{Z}_+ := \{c \in \mathbf{Z}\ ;\ c > 0\}$ with $\sum_{a\in \mathrm{gen}\,\rho} \mathbf{f}(a) = n$. We use mainly $\mathrm{Index}(\rho, r)$ and denote it simply by $\mathrm{Index}(\rho)$. An element $\mathbf{f}$ of $\mathrm{Index}(\rho, n)$ is said to be an index of norm n on ρ.

Let σ be a nonsingular cone of maximal dimension in $M_{\mathbf{R}}$. Then $\sigma^\vee$ is a nonsingular cone of dimension r in $N_{\mathbf{R}}$. The bijection $x(\sigma,\ ) : \mathrm{gen}\,\sigma \to \mathrm{gen}\,\sigma^\vee$ is defined so that $\langle a, x(\sigma, b)\rangle$ is 1 if $a = b$ and is zero otherwise for $a, b \in \mathrm{gen}\,\sigma$. We set $x^*(\sigma) := \prod_{a\in \mathrm{gen}\,\sigma} x(\sigma, a) = x(\sigma^\vee)$.

For $\mathbf{f} \in \mathrm{Index}(\rho, n)$ and $\sigma \in \tilde{\Delta}(\rho\prec)(r)$, we set

$$I(\sigma, \mathbf{f}) := \frac{\prod_{a\in \mathrm{gen}\,\rho} x(\sigma, a)^{\mathbf{f}(a)}}{x^*(\sigma)}$$

and we define

$$I(\tilde{\Delta}, \mathbf{f}) := \sum_{\sigma\in\tilde{\Delta}(\rho\prec)(r)} I(\sigma, \mathbf{f})\ .$$

Then $I(\tilde{\Delta}, \mathbf{f})$ is an integer if $n = r$ (cf. [I2, Thm.3.2]).

For each integer $n \geq 0$, we define $b_n := B_n/n!$, where B_n's are the Bernoulli numbers defined by $1/(1 - \exp(-z)) = \sum_{n=0}^{\infty}(B_n/n!)z^{n-1}$. For an index $\mathbf{f}$ on a cone ρ, we set $b_{\mathbf{f}} := \prod_{a\in\mathrm{gen}\,\rho} b_{\mathbf{f}(a)} \in \mathbf{Q}$.

Let $(\tilde{V}, X)$ be the toroidal desingularization of the cusp singularity $V(C^*, \Gamma^*)$ associated to the fan $\tilde{\Delta}\cup\{\mathbf{0}\}$. Then there exists a natural one-to-one correspondence between $\tilde{\Delta}(1)/\Gamma^*$ and the set of irreducible components of X. We denote $D(\gamma)$ the prime divisor corresponding to $\gamma \in \tilde{\Delta}(1)$. If we assume that the fan $\tilde{\Delta} \cup \{\mathbf{0}\}$ is sufficiently fine, then these prime divisors are nonsingular and X has only normal crossings. Then by expanding the formula for χ_∞ in the introduction, we get an equality

$$\chi_\infty(C^*, \Gamma^*) = \sum_{\rho\in\tilde{\Delta}/\Gamma^*} \sum_{\mathbf{f}\in\mathrm{Index}(\rho)} b_{\mathbf{f}} \prod_{a\in\mathrm{gen}\,\rho} D(\gamma(a))^{\mathbf{f}(a)}$$

where $\gamma(a) := \mathbf{R}_0 a \in \tilde{\Delta}(1)$ and the products of divisors mean the intersection numbers.

The following theorem is a consequence of Sczech's equality [S2]

$$I(\tilde{\Delta}, \mathbf{f}) = \prod_{a\in\mathrm{gen}\,\rho} D(\gamma(a))^{\mathbf{f}(a)}$$

which is written in our notation in [I2, Thm.3.2].

Theorem 2.2 *The rational number*

$$\sum_{\rho\in\tilde{\Delta}/\Gamma^*} \sum_{\mathbf{f}\in\mathrm{Index}(\rho)} b_{\mathbf{f}}\, I(\tilde{\Delta}, \mathbf{f})$$

is equal to the arithmetic genus defect $\chi_\infty(C^*, \Gamma^*)$ *of the cusp* $V(C^*, \Gamma^*)$.

Note that we need not assume that X has only simple normal crossings by [I2, Thm.4.9].

By Theorems 2.1 and 2.2, the both invariants $Z(0)(C,\Gamma)$ and χ_∞ are described by the elements of the homogeneous quotient of the polynomial ring $S^*(N_{\mathbf{Q}})$. This fact makes it possible to compare these two invariants.

For the proof of Theorem, we need some systematic culculation on the T-complexes. For the detail, see [I5]. For the historical meaning of this equality, see also [SO].

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