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On the volumes of integral convex polytopes satisfying certain conditions

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Let P be an integral convex polytope in $\mathbb{R}^n$, i.e., P is the convex hull of finite points of $\mathbb{Z}^n$. Assume that $\dim P = n$ and that $\mathrm{Int}(P) \cap \mathbb{Z}^n = \{0\}$. Then the dual polytope $P^* := \{ v \in \mathbb{R}^n \mid \langle v,u\rangle \geqq -1$ for all $u \in P \}$ is the convex hull of finite points of $\mathbb{Q}^n$. Let X_P be the union of $X_F := \mathrm{Spec}\mathbb{C}[(\mathbb{R}_{\geqq 0}F)^* \cap \mathbb{Z}^n]$ for all faces F of P, where $(\mathbb{R}_{\geqq 0}F)^* = \{ v \in \mathbb{R}^n \mid \langle v,u\rangle \geqq 0$ for all $u \in \mathbb{R}_{\geqq 0}F \}$. Then X_P is a compact toric variety whose anti-canonical divisor is ample and $(-K_{X_P})^n = n!\mathrm{vol}(P^*)$. On the other hand, Hensley[3] showed that the volume of P has a upper-bound $K(n)$. However, $K(n)$ is much greater than the volume of the following example P_m first given by Zaks, Perles and Wills[4] and no examples are known whose volumes are greater than $\mathrm{vol}(P_m)$.

Example. Assume that $n \geqq 3$. Let $y_1 = 2$, $y_2 = 3$ and $y_k = y_1y_2 \cdots y_{k-1} + 1$ for $k \geqq 3$. Then $y_1^{-1} + \cdots + y_{n-1}^{-1} + (y_n - 1)^{-1} = 1$. Let P_m be the convex hull of the $n + 1$ points ${}^t(y_1-1,-1, \ldots ,-1)$, ${}^t(-1,y_2-1,-1, \ldots .-1)$, $\ldots$, ${}^t(-1, \ldots, -1,y_{n-1}-1,-1)$, ${}^t(-1, \ldots ,-1,2(y_n-1)-1)$, ${}^t(-1,-1, \ldots ,-1)$.

Then $\mathrm{Int}(P_m) \cap \mathbb{Z}^n = \{0\}$, P_m^* is integral and $\mathrm{vol}(P_m) = \frac{1}{n!}y_1 \cdots y_{n-1}2(y_n-1) = \frac{2}{n!}(y_n-1)^2$.

In this note, we show that the above example has the maximal volume among n-dimensional integral convex polytopes satisfying certain additional conditions, if $n \geqq 3$.

Theorem. If $n \geqq 3$ and P is an n-dimensional integral simplex in $\mathbb{R}^n$ such that $\mathrm{Int}(P) \cap \mathbb{Z}^n = \{0\}$ and that P^* is also integral, then $\mathrm{vol}(P) \leqq \mathrm{vol}(P_m)$.

Remark. (1) If P is a simplex, then X_P is isomorphic to the quotient of $\mathbb{P}^n$ under the action of a finite subgroup of $(\mathbb{C}^\times)^n$. (2) X_P is $\mathbb{Q}$-factorial, i.e., X_P is a $\mathbb{Q}$-Fano variety, if and only if each (n-1)-dimensional face of P is a simplex. In particular, if P is a simplex, then X_P is a $\mathbb{Q}$-Fano variety.

Assume that $n \geqq 3$. Let P be an n-dimensional integral simplex, i.e., P is the convex hull of $n+1$ points u_1, u_2, ... and u_{n+1} of $\mathbb{Z}^n$ such that $u_1 - u_{n+1}$, $u_2 - u_{n+1}$, ... and $u_n - u_{n+1}$ are linearly independent. Assume that $\mathrm{Int}(P) \cap \mathbb{Z}^n = \{0\}$. Then the dual polytope P^* is the convex hull of the $n+1$ points $v_1, v_2, \ldots, v_{n+1}$ of $\mathbb{Q}^n$ defined by $\langle v_j, u_k \rangle = -1$ if $j \neq k$. Also assume that P^* is integral, i.e., $v_j \in \mathbb{Z}^n$. Hence $a_j := \langle v_j, u_j \rangle + 1$ are positive integers. Let $u_0 = a_1^{-1}u_1 + a_2^{-1}u_2 + \dots + a_{n+1}^{-1}u_{n+1}$. Then $\langle v_j, u_0 \rangle = -a_1^{-1} - \dots - a_{j-1}^{-1} + a_j^{-1}(a_j - 1) - a_{j+1}^{-1} - \dots - a_{n+1}^{-1} = 1 - \sum_{k=1}^{n+1} a_k^{-1}$ for all j. Since $0 \in P^* = \overline{v_1 v_2 \dots v_{n+1}}$, we have:

Proposition 1. $u_0 = 0$ and $a_1^{-1} + a_2^{-1} + \dots + a_{n+1}^{-1} = 1$.

Proposition 2. $\mathrm{vol}(P) \leqq \frac{1}{n!} a_1 a_2 \dots a_n$.

Proof. Let L be the sublattice of $\mathbb{Z}^n$ generated by $u_1 - u_{n+1}$, $u_2 - u_{n+1}$, ... and $u_n - u_{n+1}$. Then $n!\mathrm{vol}(P) = [\mathbb{Z}^n:L]$. On the other hand, $\langle v_j, a_k^{-1}(u_k - u_{n+1}) \rangle = \delta_{jk}$ for $1 \leqq j, k \leqq n$. Hence the $\mathbb{Z}$-module M generated by $a_1^{-1}(u_1 - u_{n+1})$, $a_2^{-1}(u_2 - u_{n+1})$, ... and $a_n^{-1}(u_n - u_{n+1})$ contains $\mathbb{Z}^n$. Therefore, $[\mathbb{Z}^n:L] \leqq [M:L] = a_1 a_2 \dots a_n$. q.e.d.

Hence the theorem follows from the following proposition.

Proposition 3. Let a_1, a_2, ... and a_{n+1} be positive integers such that $a_1 \leqq a_2 \leqq \dots \leqq a_{n+1}$ and that $a_1^{-1} + a_2^{-1} + \dots + a_{n+1}^{-1} = 1$. Then $a_1 a_2 \dots a_n \leqq y_1 y_2 \dots y_{n-1} 2(y_n - 1)$.

For the proof we need the following two lemmas.

Lemma 4. Let a_1, a_2, ... and a_{n+1} be those as in Proposition 3. Then there exist positive integers b_{n-1} and b_n satisfying the following three conditions.

(C1) $a_{n-1}, b_{n-1} \leqq b_n$.

(C2) $b_{n-1}^{-1} + b_n^{-1} \leqq a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} < b_{n-1}^{-1} + (b_n-1)^{-1}$.

(C3) $2b_{n-1}b_n \geqq a_{n-1}a_n$.

Proof. Since $n \geqq 3$ and $a_{n-2} \leqq a_{n-1}$, we have $a_{n-1} \geqq 3$. There exists a integer q greater than 1 such that $q^{-1} \leqq a_n^{-1} + a_{n+1}^{-1} < (q-1)^{-1}$. Then $2q \geqq a_n$, because $a_n \leqq a_{n+1}$. Hence if $q \geqq a_{n-1}$, then $b_{n-1} = a_{n-1}$ and $b_n = q$ satisfy C1, C2 and C3. Now assume that $q < a_{n-1}$. Then $a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} > 2a_{n-1}^{-1}$.

(I) First, we consider the case that a_{n-1} is even, i.e., $a_{n-1} = 2r$ for a positive integer r. Then $r^{-1} < a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} < 1$. Hence there exists a positive integer b_n such that $b_{n-1} = r$ and b_n satisfy C2. Since $r^{-1} + a_n^{-1} = 2a_{n-1}^{-1} + a_n^{-1} \geqq a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1}$, we have $b_n \geqq a_n$. Hence b_{n-1} and b_n satisfy C1 and C3.

(II) Next, we consider the case that a_{n-1} is odd, i.e., $a_{n-1} = 2r + 1$ for a positive integer r. Then $q \leqq 2r$.

(i) Assume that $q < 2r$. Then $1 > a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} \geqq (2r+1)^{-1} + q^{-1} \geqq (2r+1)^{-1} + (2r-1)^{-1} > r^{-1}$. Hence there exists a positive integer b_n such that $b_{n-1} = r$ and b_n satisfy C2. Since $(r^{-1} + (a_n + 1)^{-1}) - (a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1}) = ((2r+1)^{-1} - a_{n+1}^{-1}) + (r^{-1}(2r+1)^{-1} - a_n^{-1}(a_n + 1)^{-1}) > 0$, we have $b_n \geqq a_n +$

2. Hence b_{n-1} and b_n satisfy C3, because $2rb_n - a_{n-1}a_n \geqq 2r(a_n + 2) - (2r+1)a_n = 4r - a_n \geqq 4r - 2q > 0$.
(ii) Assume that $q = 2r$. Suppose that $r = 1$. Then $q^{-1} \leqq 1 - a_{n-2}^{-1} - a_{n-1}^{-1} \leqq 1 - 2(2r+1)^{-1} = 3^{-1}$. Hence $r = \frac{q}{2} > 1$. (ii-i) Assume that $r^{-1} < a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1}$. Then there exists a positive integer b_n such that $b_{n-1} = r$ and b_n satisfy C2. Since $a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} < (2r+1)^{-1} + (2r-1)^{-1} = 4r(4r^2-1)^{-1} = r^{-1} + r^{-1}(4r^2-1)^{-1}$, we have $b_n > r(4r^2-1) > 2(2r+1)$. Hence $2rb_n > (2r+1)4r \geqq a_{n-1}a_n$. Therefore, b_{n-1} and b_n satisfy C1 and C3. (ii-ii) Assume that $a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} \leqq r^{-1}$. Since $(r + 1)^{-1} < (2r + 1)^{-1} + (2r)^{-1} = a_{n-1}^{-1} + q^{-1}$, there exists a positive integer b_n such that $b_{n-1} = r + 1$ and b_n satisfy C2. Then $b_n \geqq r(r+1)$, because $r^{-1} = (r+1)^{-1} + r^{-1}(r+1)^{-1}$. Hence b_{n-1} and b_n satisfy C1. Moreover, $2b_{n-1}b_n \geqq 2r(r+1)^2 > (2r+1)4r \geqq a_{n-1}a_n$, if $r \geqq 3$. Finally, we consider the case that $r = 2$. Then $a_n \leqq 2q = 8$. If $a_n \leqq 7$, then $2b_{n-1}b_n \geqq 2\cdot3\cdot6 > 35 \geqq a_{n-1}a_n$. If $a_n = 8$, then $a_{n-1}^{-1} + a_n^{-1} + a_{n+1}^{-1} \leqq 5^{-1} + 8^{-1} + 8^{-1} = 9/20$. Hence $b_n \geqq (9/20 - 1/3)^{-1} > 8$. Therefore, $2b_{n-1}b_n \geqq 2\cdot3\cdot9 > a_{n-1}a_n$. q.e.d.

Lemma 5. Let a_1, a_2, ... and a_n be positive integers such that $a_1 \leqq a_2 \leqq \dots \leqq a_n$. If $a_1^{-1} + a_2^{-1} + \dots + a_n^{-1} <$ (resp. $\leqq$) $1 \leqq$ (resp. $<$) $a_1^{-1} + \dots + a_{n-1}^{-1} + (a_n - 1)^{-1}$, then $a_1a_2 \dots a_n \leqq y_1y_2 \dots y_n$ (resp. $y_1 \dots y_{n-1}(y_n - 1)$).

Proof. In the case that $n = 3$, it is easy to verify the lemma. So assume that $n \geqq 4$.

Sublemma. Let ε be a positive real number not greater than $1/2$, let a, b, c and d be positive integers with $c \leqq d$. Assume that $a^{-1} + b^{-1} <$ (resp. $\leqq$) $\varepsilon \leqq (a - 1)^{-1}$ and that $c^{-1} + d^{-1} <$ (resp. $\leqq$) $\varepsilon \leqq$ (resp. $<$) $c^{-1} + (d - 1)^{-1}$. Then $ab \geqq cd$.

Proof. Since $a^{-1} + b^{-1} \leqq (a - 1)^{-1}$, we have $b \geqq a(a - 1)$. Clearly, $a \leqq c$. If $a = c$, then $b \geqq d$ and hence $ab \geqq cd$. So assume that $c - a \geqq 1$. Since $a^{-1} + b^{-1} < c^{-1} + (d - 1)^{-1}$, we have $b > ac(d - 1)(a(c+d-1) - c(d-1))^{-1}$. Hence $ab - cd > c(d((c-a)(d-a) - c) + a(d-a))(a(c+d-1) - c(d-1))^{-1}$. Here we note that $a(c + d - 1) - c(d - 1) > 0$, because $a^{-1} < c^{-1} + (d - 1)^{-1}$. Hence if $E := (c - a)(d - a) - c > 0$, then $ab > cd$. Since $c - a \geqq 1$, if $d \geqq a + c$, then $ab > cd$. So we assume that $d < a + c$. Since $(a - 1)^{-1} \geqq c^{-1} + d^{-1} > c^{-1} + (2c)^{-1}$, we have $c - a > \frac{c}{3} - 1$.

(I) First, we consider the case that $c - a \geqq 3$. Then $(c - a)^2 \geqq 3(c - a) > c - 3$. Hence $E > (c - a)(d - a) - (c - a)^2 - 3 = (c - a)(d - c) - 3 \geqq (c - a)(d - c - 1)$. Therefore, if $d > c$, then $ab > cd$. (i) Assume that $c = d$ is even, i.e., $c = d = 2r$ for a positive integer r. Since $r^{-1} = c^{-1} + d^{-1} \leqq \varepsilon \leqq c^{-1} + (d - 1)^{-1} < (r - 1)^{-1}$, $a = r$ or $r + 1$. When

$a = r + 1$, $ab \geqq a^2(a-1) = r(r+1)^2 \geqq (2r)^2 = cd$. When $a = r$, $b^{-1} \leqq \varepsilon - r^{-1} < (2r)^{-1} + (2r-1)^{-1} - r^{-1} = (2r)^{-1}(2r-1)^{-1}$. Hence $ab \geqq 2r^2(2r-1) > (2r)^2 = cd$, because $r = \frac{c}{2} \geqq \frac{a+3}{2} > 2$. (ii) Assume that $c = d$ is odd, i.e., $c = d = 2r + 1$ for a positive integer r. Since $(r+1)^{-1} < c^{-1} + d^{-1} \leqq \varepsilon \leqq c^{-1} + (d-1)^{-1} < r^{-1}$, $a = r + 1$. Hence $b^{-1} \leqq \varepsilon - a^{-1} \leqq c^{-1} + (d-1)^{-1} - a^{-1} = (3r+1)(2r(2r+1)(r+1))^{-1}$. Therefore, $ab \geqq 2r(r+1)^2(2r+1)(3r+1)^{-1} > (2r+1)^2 = cd$, because $r \geqq 3$.

(II) Next, we consider the case that $c - a = 2$. Since $a \geqq 3$ and $\frac{c}{3} - 1 < c - a = 2$, we have $5 \leqq c \leqq 8$. (i) When $c = 8$ (resp. 7), $d < a + c = 14$ (resp. 12) and $b \geqq a(a-1) = 30$ (resp. 20). Hence $ab \geqq 6 \cdot 30 > 8 \cdot 14 > cd$ (resp. $ab \geqq 5 \cdot 20 > 7 \cdot 12 > cd$). (ii) When $c = 6$, $ab \geqq a^2(a-1) = 48$. If $d < 8$, then $cd < ab$. If $d \geqq 8$, then $E \geqq 2(8-4) - 6 > 0$ and hence $ab > cd$. (iii) When $c = 5$, $a = 3$. If $d \geqq 6$, then $E \geqq 2(6-3) - 5 > 0$. If $d = 5$, then $b^{-1} \leqq c^{-1} + (d-1)^{-1} - a^{-1} = 7/60 < 1/8$ and hence $ab \geqq 3 \cdot 9 > 5 \cdot 5 = cd$.

(III) Finally, we consider the case that $c - a = 1$. Since $a \geqq 3$ and $\frac{c}{3} - 1 < 1$, we have $4 \leqq c \leqq 5$. (i) When $c = 5$, $b \geqq a(a-1) = 12$ and $d < a + c = 9$. Hence $ab \geqq 4 \cdot 12 > 5 \cdot 9 > cd$. (ii) When $c = 4$, $b \geqq 6$ and $4 \leqq d < 7$. If $d = 4$, then $cd = 4 \cdot 4 < 3 \cdot 6 \leqq ab$. If $d = 5$, then $b^{-1} < c^{-1} + (d-1)^{-1} - a^{-1} = 1/6$ and hence $ab \geqq 3 \cdot 7 > 4 \cdot 5 = cd$. If $d = 6$, then $b^{-1} < 7/60 < 1/8$ and hence $ab \geqq 3 \cdot 9 > 4 \cdot 6 = cd$. q.e.d.

Proof of Lemma 5 continued. Suppose that $a_1^{-1} + a_2^{-1} + \dots + a_n^{-1} <$ (resp. $\leqq$) $1 \leqq$ (resp. $<$) $a_1^{-1} + \dots + a_{n-1}^{-1} + (a_n - 1)^{-1}$ and that $a_1 a_2 \dots a_n > y_1 y_2 \dots y_n$ (resp. $y_1 \dots y_{n-1}(y_n - 1)$). Let $\varepsilon = 1 - a_1^{-1} - \dots - a_{n-2}^{-1}$. Then $a_{n-1}^{-1} + a_n^{-1} <$ (resp. $\leqq$) $\varepsilon \leqq$ (resp. $<$) $a_{n-1}^{-1} + (a_n - 1)^{-1}$ and $\varepsilon \leqq \frac{1}{2}$, because $n \geqq 4$. Hence we may assume that $a_1^{-1} + a_2^{-1} + \dots + a_{n-1}^{-1} < 1 \leqq a_1^{-1} + \dots + a_{n-2}^{-1} + (a_{n-1} - 1)^{-1}$, by using the above sublemma repeatedly. On the other hand, $(a_n - 1)^{-1} \geqq$ (resp. $>$) $1 - a_1^{-1} - a_2^{-1} - \dots - a_{n-1}^{-1} \geqq 1 - y_1^{-1} - y_2^{-1} - \dots - y_{n-1}^{-1} = (y_n - 1)^{-1}$, by [1,2]. Hence $a_n \leqq y_n$ (resp. $\leqq y_n - 1$). Therefore, $a_1 a_2 \dots a_{n-1} > y_1 y_2 \dots y_{n-1}$. By the induction for n, we have $a_1 a_2 a_3 > y_1 y_2 y_3$, a contradiction.

q.e.d.

Proof of Proposition 3. By Lemma 4, there exist positive integers b_{n-1} and b_n such that $b_n = \max\{ b_j \mid 1 \leqq j \leqq n \}$, that $b_1^{-1} + b_2^{-1} + \dots + b_n^{-1} \leqq 1 < b_1^{-1} + \dots + b_{n-1}^{-1} + (b_n - 1)^{-1}$ and that $2b_{n-1}b_n \geqq a_{n-1}a_n$, where $b_j = a_j$ for $j = 1$ through $n-2$. Hence $a_1 a_2 \dots a_n \leqq 2b_1 b_2 \dots b_n \leqq 2y_1 \dots y_{n-1}(y_n - 1)$, by Lemma 5.

q.e.d.

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