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Quantum group symmetry and lattice correlation functions

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Abstract. Quantum groups play a role of deformed symmetry of integrable models in lattice statistics or quantum field theory. We give a survey of how they arise in the context of the one dimensional XXZ spin chain. We then outline the recent works concerning the structure of the space of states of this model and its spin correlation functions (Joint work with Brian Davies, Omar Foda, Kei Miki, Tetsuji Miwa and Atsushi Nakayashiki).

§1. Deformed symmetry

In quantum mechanics one often encounters the situation that symmetry inherent in the system implies degeneracy in the energy spectrum. Suppose we have a Hamiltonian H and a set of infinitesimal generators J^α such that $[H, J^\alpha] = 0$. Clearly, if $|u\rangle$ is an eigenstate of H then $J^\alpha|u\rangle$, $J^\alpha J^\beta|u\rangle$, $\cdots$ are all eigenstates belonging to the same energy level as $|u\rangle$. Whence arises degeneracy.

As an illustration let us consider the simple Hamiltonian on the one dimensional chain

$$H_{XXX} = -\frac{1}{2}\sum_{k=1}^{N}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \sigma_k^z\sigma_{k+1}^z\right). \tag{1.1}$$

Here as usual the σ_k^α $(\alpha = x, y, z)$ stand for the Pauli matrices acting as σ^α on the k-th site of

$$V^{\otimes N} = \overbrace{V \otimes \cdots \otimes V}^{N}, \quad V = \mathbf{C}^2,$$

and as identity elsewhere. Setting

$$J^\alpha = \sum_{k=1}^{N}\sigma_k^\alpha = \sum_{k=1}^{N} 1 \otimes \cdots \otimes \sigma^\alpha \otimes \cdots \otimes 1$$

we find $[J^\alpha, H] = 0$ and

$$[J^z, J^\pm] = \pm 2J^\pm, \quad [J^+, J^-] = J^z. \qquad (2J^\pm = J^x \pm iJ^y)$$

This shows that H_{XXX} is symmetric under the Lie algebra $\mathfrak{sl}_2$. If

$$V^{\otimes N} = \oplus_i (\overbrace{W_i \oplus \cdots \oplus W_i}^{m_i})$$

is the decomposition into copies of irreducible representations W_i of $\mathfrak{sl}_2$, then each of the corresponding eigenvalues occurs with multiplicity $m_i \dim W_i$.

Next let us consider a more general case

$$H_{XXZ} = -\frac{1}{2}\sum_{k=1}^{N-1}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \Delta\sigma_k^z\sigma_{k+1}^z\right) + a\sigma_1^z + b\sigma_N^z \tag{1.2}$$

where $\Delta \in \mathbf{R}$ stands for an anisotropy parameter. Obviously we no longer have the Lie algebra symmetry. Nevertheless, if the boundary terms are so chosen that $a = -b = -(q - q^{-1})/4$ with

$$\Delta = \frac{q + q^{-1}}{2},$$

then the spectrum of (1.2) exhibits exactly the same multiplicity structure as in the case of $q = 1$ (1.1). To account for this fact one has to extend the notion of the symmetry and introduce the following 'deformed' operators:

$$\begin{aligned} J^+ &= \sum_{k=1}^{N} q^{\sigma^z} \otimes \cdots \otimes q^{\sigma^z} \otimes \sigma^+ \otimes 1 \otimes \cdots \otimes 1, \\ J^- &= \sum_{k=1}^{N} 1 \otimes \cdots \otimes 1 \otimes \sigma^- \otimes q^{-\sigma^z} \otimes \cdots \otimes q^{-\sigma^z}, \\ J^z &= \sum_{k=1}^{N} 1 \otimes \cdots \otimes 1 \otimes \sigma^z \otimes 1 \otimes \cdots \otimes 1. \end{aligned} \tag{1.3}$$

With the above choice of a, b one finds that $[J^\alpha, H_{XXZ}] = 0$; moreover the J^α obey the following commutation relations independently of the length of the chain:

$$[J^z, J^\pm] = \pm 2J^\pm, \quad [J^+, J^-] = \frac{q^{J^z} - q^{-J^z}}{q - q^{-1}}.$$

Since the RHS of the last equation is non-linear in the generators, the J^α no longer generate a Lie algebra. Instead they are regarded as defining an associative algebra, denoted $U_q(\mathfrak{sl}_2)$. More generally, with any simple Lie algebra $\mathfrak{g}$ or even a Kac-Moody Lie algebra one can associate a similar deformation $U_q(\mathfrak{g})$, commonly called quantum group or quantized enveloping algebra. The original Lie algebra $\mathfrak{g}$ is recovered in the limit $q \to 1$.

Recall that in the Lie algebra case there is a 'universal' composition law of angular momentum. Whenever $\mathfrak{sl}_2$ acts on two independent spaces $|1\rangle$ and $|2\rangle$, the total actin on $|1\rangle \otimes |2\rangle$ is given by

$$\Delta J = J \otimes 1 + 1 \otimes J, \quad J \in \mathfrak{sl}_2.$$

Similarly we have a deformed version for $U_q(\mathfrak{sl}_2)$, called coproduct:

$$\begin{aligned} \Delta J^+ &= J^+ \otimes 1 + q^{J^z} \otimes J^+, \\ \Delta J^- &= J^+ \otimes q^{-J^z} + 1 \otimes J^-, \\ \Delta J^z &= J^z \otimes 1 + 1 \otimes J^z. \end{aligned} \tag{1.4}$$

The expression (1.3) is obtained by iterating this operation $N-1$ times. Notice that unlike the Lie algebra case the rule (1.4) is now sensitive to the order of the composition:

$$P \circ \Delta J^\alpha \neq \Delta J^\alpha \tag{1.5}$$

where $Pa \otimes b = b \otimes a$.

As it turns out, $U_q(\mathfrak{sl}_2)$ has the 'same' representation theory as in the Lie algebra case (if q is 'generic', meaning $q^N \neq 1$ for $N = 1, 2, 3, \cdots$). The details of the representations depend of course on q, but such essential features as the classification of irreducible representations, weight multiplicities, Clebsch-Gordan rule, etc. are all the same as in the undeformed case $q = 1$. This explains why H_{XXZ} has the same degeneracy structure as H_{XXX}.

The fact is, H_{XXZ} is a solvable model. Not only the multiplicities but all the eigenvalues can be described exactly. The symmetry under $U_q(\mathfrak{sl}_2)$ does not give information about the exact eigenvalues themselves. After all $\mathfrak{sl}_2$ is only 3-dimensional, while the dimension of the space $V^{\otimes N}$ is 2^N. Clearly $\mathfrak{sl}_2$ is too small; it is necessary to consider a much larger (deformed) symmetry.

§2. Abelian symmetry

A common feature of integrable models (whether classical or quantum) is the existence of an infinite number of conservation laws. In the context of (1.2) this means the existence of a family of operators $H_1 = H_{XXZ}$, H_2, H_3, $\cdots$ such that they all commute with each other: $[H_i, H_j] = 0$. If you will, this is an infinite abelian symmetry for (1.2). In fact such 'higher Hamiltonians' can be obtained by differentiating the transfer matrix of the 6 vertex model $T_{6V}(\theta)$

$$H_n = \left(\frac{d}{d\theta}\right)^n \log T_{6V}(\theta)\Big|_{\theta=0}.$$

The commutativity of the H_n is a consequence of that for the transfer matrix

$$[T_{6V}(\theta), T_{6V}(\theta')] = 0 \quad \forall \theta, \theta'. \tag{2.1}$$

Let us recall the setting of the 6 vertex model. It is defined on a two dimensional lattice, each edge having two possible states + or −, and the interaction is specified by giving a Boltzmann weight R_{ij}^{kl} to each configuration (i, j, k, l) round a vertex. Arranging the weights into a 4×4 matrix form we have

$$R(z) = \begin{pmatrix} 1 & & & \\ & \frac{1-z}{1-q^2z}q & \frac{1-q^2}{1-q^2z} & \\ & \frac{1-q^2}{1-q^2z}z & \frac{1-z}{1-q^2z}q & \\ & & & 1 \end{pmatrix} \tag{2.2}$$

where $z = e^{i\theta}$. This matrix is a typical solution of the Yang-Baxter equation (YBE)

$$R_{12}(z_1/z_2)R_{13}(z_1/z_3)R_{23}(z_2/z_3) = R_{23}(z_2/z_3)R_{13}(z_1/z_3)R_{12}(z_1/z_2). \tag{2.3}$$

When $q = 1$ it reduces to the first example of solutions due to Yang [1]. As is well known, YBE guarantees the commutativity (2.1) of the transfer matrix (under the periodic boundary condition).

There is a well established procedure to find the spectrum of H_{XXZ} or of T_{6V}; one uses the Bethe Ansatz type method (as was first demonstrated in the classic paper of Yang-Yang [2] for H_{XXZ}), or one invokes a functional relation method [3]. Although they are good enough to give exact results, the aspect of the symmetry is somewhat hidden behind the scene in this approach. In the next section we wish to address this question, in search for the full symmetry of the problem.

§3. Quantum affine symmetry

We wish to contend that, to the integrablity of our Hamiltonian H_{XXZ}, the heart of the matter is its symmetry under the deformation of the *affine* Lie algebra $\widehat{\mathfrak{sl}}_2$.

Recall that $\widehat{\mathfrak{sl}}_2$ is a central extention of the loop algebra over 2×2 traceless matrices

$$\widehat{\mathfrak{sl}}_2 = \left\{ X(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & -a(\lambda) \end{pmatrix} \;\middle|\; a(\lambda), b(\lambda), c(\lambda) \in \mathbf{C}[\lambda, \lambda^{-1}] \right\} \oplus \mathbf{C}c \oplus \mathbf{C}d \tag{3.1}$$

with the bracket given by

$$[X(\lambda), Y(\lambda)] = [X(\lambda), Y(\lambda)]_{\mathrm{mat}} + \mathrm{Res}_{\lambda=0}\mathrm{tr}\Big(\frac{dX}{d\lambda}Y(\lambda)\Big)d\lambda\ c,$$
$$[c, \text{everything}] = 0, \quad [d, X(\lambda)] = \lambda\frac{d}{d\lambda}X(\lambda)$$

where $[\ ,\ ]_{\mathrm{mat}}$ denotes the commutator of matrices. This algebra has the following important classes of representations.

(i) Irreducible highest weight representations $V(\Lambda)$ of level 1. There are exactly two such, corresponding to the choice of the highest weight $\Lambda = \Lambda_0,\ \Lambda_1$. These representations are infinite dimensional.

(ii) Affinization of the 2-dimensional (spin 1/2) representation of $\mathfrak{sl}_2$. This space is $V_z = V \otimes \mathbf{C}[z, z^{-1}]$ spanned by the basis elements $v_\pm z^n$ ($n \in \mathbf{Z}$), on which $\widehat{\mathfrak{sl}}_2$ acts as

$$X(\lambda)(v_\pm z^n) = X(z)v_\pm z^n, \quad c = 0, \quad d = z\frac{d}{dz}.$$

We will use the same letters to denote the q-deformations of these representations.

The R matrix (2.2) arises naturally in the following way. Consider the tensor product $V_{z_1} \otimes V_{z_2}$ of the representations of type (ii). Is it the same as the tensor product in the opposite order $V_{z_2} \otimes V_{z_1}$? In the Lie algebra case the answer is trivially yes, since the transposition operator $P\ :\ V_{z_1} \otimes V_{z_2} \to V_{z_2} \otimes V_{z_1}$ gives an intertwiner (i.e. an operator that commutes with the Lie algebra generators). In the deformed case this is no longer so because of (1.5). If we demand that there exist an intertwiner

$$\check{R}(z_1, z_2)\ :\ V_{z_1} \otimes V_{z_2} \to V_{z_2} \otimes V_{z_1}$$

we get a linear equation for $\check{R}(z_1, z_2)$. Solving them we find that the solution is unique up to scalar multiple, and is given by $\check{R}(z_1, z_2) = PR(z_1/z_2)$ where $R(z)$ is the one (2.2).

That the R matrix solves YBE can be seen by comparing the following maps:

$$\begin{aligned} V_{z_1} \otimes V_{z_2} \otimes V_{z_3} &\xrightarrow{\check{R}(z_1,z_2)\otimes id} V_{z_2} \otimes V_{z_1} \otimes V_{z_3} \\ &\xrightarrow{id\otimes\check{R}(z_1,z_3)} V_{z_2} \otimes V_{z_3} \otimes V_{z_1} \xrightarrow{\check{R}(z_2,z_3)\otimes id} V_{z_3} \otimes V_{z_2} \otimes V_{z_1} \end{aligned}$$

and

$$V_{z_1} \otimes V_{z_2} \otimes V_{z_3} \overset{id\otimes\check{R}(z_2,z_3)}{\longrightarrow} V_{z_1} \otimes V_{z_3} \otimes V_{z_2}$$
$$\overset{\check{R}(z_1,z_3)\otimes id}{\longrightarrow} V_{z_3} \otimes V_{z_1} \otimes V_{z_2} \overset{id\otimes\check{R}(z_1,z_2)}{\longrightarrow} V_{z_3} \otimes V_{z_2} \otimes V_{z_1}.$$

The composition gives left/right sides of (2.3) respectively. It can be shown that $V_{z_1}\otimes V_{z_2}$, $V_{z_1}\otimes V_{z_2}\otimes V_{z_3}$, $\cdots$ are all irreducible, so the above two maps $V_{z_1}\otimes V_{z_2}\otimes V_{z_3} \to V_{z_3} \otimes V_{z_2} \otimes V_{z_1}$ commuting with $U_q(\widehat{\mathfrak{sl}}_2)$ should be proportional to each other. It is easy to verify that the proportionality scalar is actually 1, proving YBE for $R(z)$.

Thus we have seen the following scheme:

$$U_q(\widehat{\mathfrak{sl}}_2) \;\Rightarrow\; \text{YBE} \;\Rightarrow\; \text{Commuting Transfer Matrix} = \text{Abelian Symmetry}.$$

It is then more or less clear that in some way the integrability of H_{XXZ} should be coded in $U_q(\widehat{\mathfrak{sl}}_2)$. In fact one can verify straightforwardly that $U_q(\widehat{\mathfrak{sl}}_2)$ provides the symmetry for H_{XXZ} in the following sense:

$$[U_q'(\widehat{\mathfrak{sl}}_2), H_{XXZ}] = 0, \qquad [d, H_n] \propto H_{n+1} \quad (n = 1, 2, \cdots). \tag{3.2}$$

Here $U_q'(\widehat{\mathfrak{sl}}_2)$ denotes the q-deformation of the subalgebra of $\widehat{\mathfrak{sl}}_2$ obtained by dropping d from (3.1). The commutation relations (3.2) hold as operators on the infinite tensor product $V^{\otimes\infty} = \cdots\otimes V\otimes V\otimes V\otimes\cdots$, on which $U_q'(\widehat{\mathfrak{sl}}_2)$ acts by iterating the coproduct (1.4) infinite times.

Unfortunately there seems to be no way of making (3.2) exactly true for a finite chain; the commutativity holds only in the infinite lattice limit. Then both H_{XXZ} and $U_q'(\widehat{\mathfrak{sl}}_2)$ are defined only formally, and the main issue is how to extract finite and exact information from here.

§4. Space of states

To gain insight into the difficulty of working directly on the infinite lattice, let us first look at the limiting case $q \to 0^-$, of equivalently $\Delta \to -\infty$. After an appropriate rescaling and addition of a constant term our Hamiltonian becomes in this limit

$$\frac{1}{2}\sum(\sigma_k^z\sigma_{k+1}^z + 1), \tag{4.1}$$

which is already diagonal in the natural basis $v_\pm$ of $V = \mathbf{C}^2$. Let us write the pure tensor $\cdots \otimes v_{\epsilon_k} \otimes v_{\epsilon_{k+1}} \otimes \cdots$ simply as $\cdots \varepsilon_k \varepsilon_{k+1} \cdots$. A moment's thought shows the following :

(1) The eigenvalues of (4.1) are all non-negative. There are two ground states having the energy 0, namely

$$\cdots + - + - + - \cdots$$
$$\cdots - + - + - + \cdots$$

(2) An eigenstate has finite energy (=eigenvalue) if and only if it has the form

$$\cdots \pm \mp \pm \mp \cdots (\text{disturbance}) \cdots \pm \mp \pm \mp \cdots. \tag{4.2}$$

Thus there are altogether 4 possible boundary conditions.

In order to have a finite theory, it is necessary to restrict the considerations to the subspace spanned by the states (4.2), throwing away the rest of vectors from $V^{\otimes\infty}$. We now ask the following question in the general case $q \neq 0$: does the space of states, i.e.

$$\mathcal{F} = \text{ the space of finite excitations over the ground states}$$

allow for an action of $U_q(\widehat{\mathfrak{sl}}_2)$, and if so what is it as a representation space?

Concerning this question we have a crucial piece of information at hand. Consider the half infinite sequences of $\pm$

$$p = (\cdots p_3, p_2, p_1), \quad p_j \in \{+,-\}, \quad p_j = (-)^j \quad \text{for } j \gg 0.$$

Such sequences are called Λ_0-paths. The fact is, there is a one-to-one correspondence

$$\text{the set of } \Lambda_0\text{-paths} \quad \longleftrightarrow \quad \text{basis of } V(\Lambda_0).$$

A similar result holds for $V(\Lambda_1)$ by taking the other boundary condition for the paths. This type of result originates in the corner transfer matrix method [4,5], and was established quite generally in the framework of the crystal base theory of $U_q(\mathfrak{g})$ [6].

This suggests that at least near $q = 0^-$ we can take $V(\Lambda_{0,1})$ as a substitute for the half-infinite product $\cdots \otimes V \otimes V \otimes V$. We are thus led to the following BASIC HYPOTHESIS:

$$\mathcal{F} \simeq \oplus_{i,j=0,1} V(\Lambda_i) \otimes V(\Lambda_j)^* \tag{4.3}$$

where $V(\Lambda_j)^*$ denotes the dual representation (a lowest weight representation of level -1).

As is known from the previous works the XXZ model has three distinct regimes

$$\begin{aligned} &(a) \quad \Delta > 1 \quad (q > 1) \\ &(b) \quad -1 < \Delta < 1 \quad (|q| = 1) \\ &(c) \quad \Delta < -1 \quad (q < -1) \end{aligned}$$

The regime (a) is ferroelectric but rather trivial, the second regime (b) is critical, and the last regime (c) is anti-ferroelectric. The picture (4.3) holds in the limit $q \to 0^-$, and we expect it to be correct throughout regime (c). From now on we restrcit our attention exclusively to this case. Turning around the logic leading to (4.3), we now take it as the definition of $\mathcal{F}$ and try to rebuild the theory by interpreting such notions as ground states, Hamiltonian, local operators etc. purely in terms of representation theory.

To be precise the symbol $\otimes$ in (4.3) is understood as a completion of the algebraic tensor product with respect to the q-adic topology (formal power series in q). It is necessary also to complete the spaces $V(\Lambda_i)$. We will not go into these technical points; see [7].

§5. Local structures

The space $\mathcal{F}$ manifestly admits an action of $U_q(\widehat{\mathfrak{sl}}_2)$. However the naïve picture of the infinite tensor product is lost here. There is one more problem to be settled. In getting to the picture (4.3) we had the following process in mind: pick a particular site, split the lattice into the left and the right halves, and replace them by $V(\Lambda_i)$ and $V(\Lambda_j)^*$ respectively depending on which boundary conditions we take, i.e.

$$\begin{gathered}\Big(\cdots\otimes V\otimes V\Big)\otimes\Big(V\otimes V\otimes\cdots\Big)\\ \Downarrow\\ V(\Lambda_i)\otimes V(\Lambda_j)^*.\end{gathered}$$

This process depends on the place of splitting. We could even replace $V^{\otimes\infty} = \cdots\otimes V\otimes V\otimes V\otimes\cdots$ by the tensor product of three parts $V(\Lambda_{i'})$, $V^{\otimes n}$ and $V(\Lambda_{j'})^*$, leaving a finite number of V's in between. All these should give one and the same space, otherwise the picture would be meaningless.

The key to connect these various pictures is the vertex opertors. By definition they are the operators

$$\Phi(z)\ :\ V(\Lambda_i)\longrightarrow V(\Lambda_{1-i})\otimes V_z \tag{5.1}$$

that commute with the action of $U_q(\widehat{\mathfrak{sl}}_2)$. To be precise they are infinite sums of the form

$$\Phi(z)=\sum_{\pm,n\in\mathbf{Z}}\Phi_{\pm\,n}\otimes v_{\pm}z^{-n},$$

where each $\Phi_{\pm\,n}$ maps a weight space $V(\Lambda_i)_\nu$ to another weight space $V(\Lambda_{1-i})_{\nu'}$ with $\nu'=\nu\mp(\Lambda_1-\Lambda_0)+n\delta$, shifting the weight by the null root δ.

The theory of such (q-deformed) vertex operators has been developed by [8], see also [9]. It turns out that for each $i=0,1$ there exist unique such operators up to scalar multiple. Moreover if we complete the spaces in the q-adic sense, then they are isomorphisms. (This latter property is a special feature of the spin $1/2$ representation, reflecting its 'perfectness' in the sense of [6].) In this sense we may freely identify the space $V(\Lambda_i)$ with $V(\Lambda_{1-i})\otimes V$ via the vertex operators.

It is now possible to define the shift operator T by one lattice unit

$$\begin{gathered}\Big(\cdots\otimes V\otimes V\Big)\otimes\Big(V\otimes V\otimes V\otimes\cdots\Big)\\ \downarrow\\ \Big(\cdots\otimes V\Big)\otimes V\otimes\Big(V\otimes V\otimes V\otimes\cdots\Big)\\ \downarrow\\ \Big(\cdots\otimes V\Big)\otimes\Big(V\otimes V\otimes V\otimes V\otimes\cdots\Big)\end{gathered} \tag{5.2}$$

to be the composition of

$$\begin{aligned} V(\Lambda_i)\otimes V(\Lambda_j)^* &\xrightarrow{\Phi\otimes id} V(\Lambda_{1-i})\otimes V\otimes V(\Lambda_j)^* \\ &\xrightarrow{id\otimes\Phi^*} V(\Lambda_{1-i})\otimes V(\Lambda_{1-j})^* \end{aligned}$$

where $\Phi = \Phi(1)$ and $\Phi^* \ : \ V\otimes V(\Lambda_j)^* \to V(\Lambda_{1-j})^*$ is a similar vertex operator. Iterating the vertex operators one also obtains the identification

$$\begin{aligned} V(\Lambda_i)\otimes V(\Lambda_j)^* &\longrightarrow V(\Lambda_{1-i})\otimes V\otimes V(\Lambda_j)^* \\ &\longrightarrow V(\Lambda_i)\otimes V\otimes V\otimes V(\Lambda_j)^* \\ &\longrightarrow \cdots \\ &\longrightarrow V(\Lambda_{i+n})\otimes V^{\otimes n}\otimes V(\Lambda_j)^* \end{aligned} \tag{5.3}$$

for any n (where we put $\Lambda_{i+2}=\Lambda_i$).

This allows one to define the local operators such as σ_n^α acting on the n-th site of the lattice in the naïve picture. It is the composition of

$$\begin{array}{ccc} V(\Lambda_i)\otimes V(\Lambda_j)^* & \longrightarrow & V(\Lambda_{i+n})\otimes V^{\otimes n}\otimes V(\Lambda_j)^* \\ & & \downarrow id\otimes(\sigma_n^\alpha)\otimes id \\ V(\Lambda_i)\otimes V(\Lambda_j)^* & \longleftarrow & V(\Lambda_{i+n})\otimes V^{\otimes n}\otimes V(\Lambda_j)^* . \end{array}$$

Here the middle arrow is defined by $\sigma^\alpha\otimes\cdots\otimes\mathrm{id} \ : \ V\otimes\cdots\otimes V\to V\otimes\cdots\otimes V$. (We are now numbering the lattice sites in the opposite order, so that $n\to\infty$ corresponds to the left end and $n\to-\infty$ to the right end.)

What are the Hamiltonian and its eigenstates? If we introduce the spectral parameter z in the definition (5.2) and replace Φ by $\Phi(z)$ and similarly for Φ^* the resulting operator $T(z)$ is shown to correspond to the transfer matrix of the six vertex model. (It is well known that $T(z=1)$ gives the translation.) By differentiation the Hamiltonian is given in terms of the translation and the grading operator d by

$$H_{XXZ} = \frac{1-q^2}{2q}\left(T^2dT^{-2}-d\right).$$

Here T^2 is used rather than T, as the former acts on the space $V(\Lambda_i)\otimes V(\Lambda_j)^*$ without changing boundary conditions.

We expect the ground states to be the singlet (i.e. that they belong to the trivial representation) under $U_q(\widehat{\mathfrak{sl}}_2)$. The natural candidate for them are the canonical elements

$$|\mathrm{vac}\rangle_i = \sum u_j\otimes u_j^* \quad \in V(\Lambda_i)\otimes V(\Lambda_i)^* \tag{5.4}$$

where $u_j \in V(\Lambda_i)$ and $u_j^* \in V(\Lambda_i)^*$ are dual bases. It can be shown that

$$T|\mathrm{vac}\rangle_i = const.|\mathrm{vac}\rangle_{i+1}, \quad H_{XXZ}|\mathrm{vac}\rangle_i = 0.$$

One can also introduce a creation operator of quasi-particles

$$\varphi_\pm^*(z) \; : \; V(\Lambda_i) \otimes V(\Lambda_j)^* \longrightarrow V(\Lambda_{1-i}) \otimes V(\Lambda_j)^*$$

such that

$$T\varphi_\pm^*(z)T^{-1} = \tau(z)^{-1}\varphi_\pm^*(z), \quad [H_{XXZ}, \varphi_\pm^*(z)] = \epsilon(z)\varphi_\pm^*(z)$$

with some scalar $\tau(z)$, $\epsilon(z)$. The construction is based on vertex operators of different type, and we leave the details to [7]. We only mention that the formulas for the momentum $\log \tau(z)^{-1}$ and the energy $\epsilon(z)$ can be derived using the q-KZ equation for vertex operators [8], and that they agree with those obtained by the Bethe Ansatz if we identify $z = e^{i\theta}$ with the quasi-momentum of the Bethe vectors.

§6. Correlation functions

Finally let us come to the correlation functions of local operators.

It is convenient to regard the space $V(\Lambda_i) \otimes V(\Lambda_i)^*$ with $\mathrm{End}(V(\Lambda_i))$, the space of all linear maps from $V(\Lambda_i)$ to itself. In this language a natural inner product covariant under $U_q(\widehat{\mathfrak{sl}}_2)$ is

$$\langle f|g\rangle = \frac{\mathrm{tr}_{V(\Lambda_i)}\left(q^{-2\rho}fg\right)}{\mathrm{tr}_{V(\Lambda_i)}\left(q^{-2\rho}\right)} \quad f, g \in \mathrm{End}(V(\Lambda_i)).$$

Here $\rho = \Lambda_0 + \Lambda_1$. Combining this with the the definition of ground states (5.4) and the local operators (5.3), we find that the correlation functions can be expressed as a trace (over a highest weight representation $V(\Lambda_i)$, not over $V(\Lambda_i) \otimes V(\Lambda_j)^*$) of products of vertex operators.

Fortunately, for level 1, the representations $V(\Lambda_i)$ [10] and the vertex operators [11] can be bozonized. Hence the trace can be evaluated explicitly. We quote below from [11] an explicit formula of the expectation value of the local operator $L = E_{\epsilon_n\epsilon'_n} \otimes \cdots \otimes E_{\epsilon_1\epsilon'_1}$ (E_{ij} signifies the matrix unit) in the sector $V(\Lambda_i)$.

Introduce the following notations

$$A = \{a_1, \cdots, a_s\} = \{j \mid \varepsilon'_j = -1\}, \quad B = \{b_1, \cdots, b_t\} = \{j \mid \varepsilon_j = +1\},$$
$$(s + t = n, \quad a_i < a_j, \quad b_i < b_j \quad \text{for} \quad i < j),$$
$$h(z) = (q^2z; x)_\infty(xz; x)_\infty(q^2z^{-1}; x)_\infty(xz^{-1}; x)_\infty.$$

We prepare the integration variables ξ_a $(a \in A)$, ζ_b $(b \in B)$ and set $\eta_j = \xi_{a_j}$ $(1 \le j \le s)$, $= \zeta_{b_{n+1-j}}$ $(s < j \le n)$, $\bar{\eta} = \prod_j \eta_j$ and $\bar{z} = \prod_j z_j$. Then we have

$$
\begin{aligned}
&P^{\epsilon_n, \cdots, \epsilon_1}_{\epsilon'_n, \cdots, \epsilon'_1}(z_n, \cdots, z_1 \,|\, x, y \,|\, i) \\
&= (-1)^t q^{\sum_{a \in A} a + \sum_{b \in B} b - n(n+1)/2} \prod_{a \in A} \oint_{C_a} \frac{d\xi_a}{2\pi i(\xi_a - z_a)} \prod_{b \in B} \oint_{C_b} \frac{d\zeta_b}{2\pi i(\zeta_b - z_b)} \\
&\times \prod_{a \in A} \prod_{a < j \le n} \frac{z_j - q^2 \xi_a}{z_j - \xi_a} \prod_{b \in B} \prod_{b < j \le n} \frac{\zeta_b - q^2 z_j}{\zeta_b - z_j} \prod_{j < k} \frac{\eta_k - \eta_j}{\eta_k - q^2 \eta_j} \\
&\times \frac{h(1)^n \prod_{j<k} h(z_j/z_k) h(\eta_j/\eta_k)}{\prod_{j,k} h(\eta_j/z_k)} \frac{\sum_{m \in \mathbf{Z} + i/2} (\bar{z}/\bar{\eta})^{2m} y^{2m} x^{m^2 - i/4}}{(x;x)_\infty \mathrm{tr}_{V(\Lambda_i)}(x^{-d} y^{\alpha})}.
\end{aligned}
\tag{6.1}
$$

Note that the last factor of the above equation can be rewritten into

$$
\left(\frac{\bar{z}}{\bar{\eta}}\right)^i \frac{(-(y\bar{z}/\bar{\eta})^2 x^{1+i}; x^2)_\infty (-(\bar{\eta}/y\bar{z})^2 x^{1-i}; x^2)_\infty}{(-y^2 x^{1+i}; x^2)_\infty (-y^{-2} x^{1-i}; x^2)_\infty}.
$$

The contours of integration should be chosen as follows. Both C_a and C_b are anti-clockwise, and the z_i $(1 \le i \le n)$ lie inside of C_a and outside of C_b. Other relevant poles with small multipliers q, x (e.g., $q^2 z_i$) are inside, and those with large multipliers are outside of the contours C_a and C_b. Eventually we specialize $z_n = \cdots = z_1 = 1$, $x = q^4$ and $y = q^{-1}$ to get the correlation $\langle L \rangle$.

Having an exact (if unwieldy) formula we must check it against known results. In fact there are only two of them. In a remarkable paper [12] Baxter derived the following formula for the one-point function (the spontaneous staggered polarization)

$$
\langle \sigma_1^z \rangle = \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n}}{1 + q^{2n}} \right)^2.
$$

It can be shown after some work that Baxter's formula can be recovered from (6.1) by taking $n = 1$. There is one more and much easier quantity, the nearest neighbor σ^z correlation $\langle \sigma_{k+1}^z \sigma_k^z \rangle$ which can be obtained by differentiating the ground state energy with respect to the parameter Δ. Again the formula (6.1) is shown to reproduce the correct result.

§7. Summary

We have seen that the symmetry under the *affine* quantum group $U_q(\widehat{\mathfrak{sl}}_2)$ is crucial, both to build the model from the R matrix, and to solve the model too. We postulated that for the XXZmodel in the anti-ferroelectric regime $\Delta < -1$

$$\text{the space of states } = \sum_{i,j=0,1} V(\Lambda_i) \otimes V(\Lambda_j)^*.$$

The local operators are defined using the different realizations of this space

$$V(\Lambda_i) \otimes V(\Lambda_j)^* \xrightarrow{\simeq} V(\Lambda_{i+n}) \otimes V^{\otimes n} \otimes V(\Lambda_j)^*$$

given by the vertex operators. The correlation functions of the local operators are given as a trace of products of vertex operators, and for level one the bozonization method makes it possible to derive an integral formula for the correlators.

The above scheme can be generalized to other Lie algebras and/or higher level representations. The RSOS models related to the coset pair $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g})$ can also be formulated in the same spirit. To derive the correlation functions, however, one needs to develop the bozonization method for higher level representations. More interesting and difficult are the problems of taking the continuum limit (to see whether the correlation functions can be described e.g. by differential equations), or going to other classes of models such as the eight vertex model or the chiral Potts model. Since they are beyond the scope of the present method, we mention them here as future problems.

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