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WKB analysis to global solvability and hypoellipticity

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1. Introduction

Let X be a compact manifold or an open domain in $\mathbf{R}^n$. We recall that a differential operator P is said to be *globally solvable* (resp. *hypoelliptic*) in X if and only if for every $f \in C_0^\infty(X)$ there exists $u \in D'(X)$ satisfying $Pu = f$. (resp. $u \in C^\infty(X)$ when $Pu \in C^\infty(X)$). P is said to be *locally solvable* (resp. *hypoelliptic*) at a point $p \in X$ if there exists a neighborhood U of p such that for all $f \in C_0^\infty(U)$, there exists $u \in D'(U)$ such that $Pu = f$ in U (resp. $p \notin singsupp(Pu)$ implies $p \notin singsupp(u)$). We note that local hypoellipticity at each point $p \in X$ implies the global hypoellipticity in X, and the global solvability implies the local solvability at each point $p \in X$.

We want to study the global regularity and global solvability of linear operators. In order to explain our problems, we recall that the problem is studied mainly from the viewpoints of

1. degenerate elliptic operators,
2. weakly hyperbolic operators,
3. constant coefficients operators.

Methods 1 and 2 are not adequate to study operators which are degenerate elliptic in some region and weakly hyperbolic in other region. Moreover at points where a symbol vanishes a small perturbation to equations may turns their type from degenerate elliptic to weakly hyperbolic one or vice versa. We want to analyze the regularity or the solvability of these mixed type operators. Method 3 is based on Fourier method and it is applicable to operators on the torus $\mathbf{T}^n = \mathbf{R}^n/2\pi\mathbf{Z}^n$. The advantage of this method is that we need not restrict the type of the equations. But it is not applicable for variable coefficients operators.

Our problems in this note are the followings

1. How can we describe the global solvability and the global regularity for linear operators which change its type everwhere in the domain ?
2. How can we describe the highly transcendental phenomena in the global solvability for operators with variable coefficients ?

We shall give answers to these problems by WKB analysis to global solvability. In principle, one can read informations for these phenomena from formal WKB solutions.

2. Second order equations

Let $X = \mathbf{T}^n = \mathbf{R}^n/2\pi\mathbf{Z}^n$ be an n-dimensional flat torus. For simplicity, we begin with operators of the form

$$(1) \qquad Pu = (D_x + ia(x)D_y)(D_x + ib(x)D_y)u + \gamma(x)D_y u$$

on $(x,y) \in \mathbf{T}^2$, where $a(x), b(x)$ and $\gamma(x)$ are 2π periodic complex-valued functions and $D_x = i^{-1}\partial/\partial x$, and so on.

Let us consider a WKB approximate solution ψ

$$(2) \quad \psi(x,\eta) = \exp\left(\int_0^x S(t,\eta)dt\right), \; S(x,\eta) = \sum_{j=-1}^{\infty} \eta^{-j} S_j(x) \;\; S_{-1}(x) = a(x) \text{ or } b(x).$$

to the partial Fourier transform of the equation $Pu = 0$ with respect to y

$$(3) \quad \hat{P}\hat{u} = -\left(\frac{\partial}{\partial x}\right)^2 \hat{u} + (a(x)+b(x))\,\eta\left(\frac{\partial}{\partial x}\right)\hat{u} + \left(b'(x)\eta + \gamma(x)\eta - a(x)b(x)\eta^2\right)\hat{u} = 0,$$

where η is a covariable of y. We easily see that S satisfies a Riccati equation

$$(4) \qquad R(S) \equiv -S^2 - S' + (a+b)\eta S + b'\eta + \gamma\eta - ab\eta^2 = 0.$$

We first consider the case where the following condition is satisfied

(A.1) There exists a smooth WKB formal solution (2) to (3), i.e, $S_j(x)$ are smoothly defined on $\mathbf{T}$ for all $j = 0, 1, 2, \ldots$.

The condition (A.1) is true if we assume

(A.1)$'$ Either $(a(x)-b(x))^{-\ell}\left(\frac{d}{dx}\right)^k \gamma(x)$ or $(a(x)-b(x))^{-\ell}\left(\frac{d}{dx}\right)^k (\gamma(x) - a'(x) + b'(x))$ is defined as a smooth function on $\mathbf{T}$ for $k = 0, 1, 2, \ldots$ and $\ell = 1, 2, \ldots$.

Remark. (A.1)$'$ is fullfilled if the function $a - b \not\equiv 0$ is analytic and if, either γ or $\gamma - a' + b'$ is flat on the set $\{x : a(x) = b(x)\}$. In case $a - b$ is not necessarily analytic then (A.1)$'$ is satisfied if either γ or $\gamma - a' + b'$ vanishes in some neighborhood of the set $\{x : a(x) = b(x)\}$. We note that if $\gamma \equiv 0$ or $\gamma = a' - b'$ then (A.1)$'$ holds without any restriction on a and b. The condition is, so to speak, a Levi condition.

By the condition (A.1) we can construct a formal solution $S(x,\eta)$ to (4). We consider three cases:

(a) $Re\, a(x) \not\equiv 0$ and $Re\, b(x) \not\equiv 0$.

(b) Either $Re\, a(x) \equiv 0$ or $Re\, b(x) \equiv 0$ holds. And there exists $j \geq 0$ such that

$$\int_0^{2\pi} Re S_j(x)dx \neq 0.$$

(c) Either $Re\, a(x) \equiv 0$ or $Re\, b(x) \equiv 0$ holds. And the condition

$$\int_0^{2\pi} ReS_j(x)dx = 0$$

holds for all $j = 1, 2, \ldots$.

Theorem 1. *Suppose that* (A.1) *is satisfied. Moreover, assume that* $Rea(x)$ *and* $Reb(x)$ *do not change their sign. Then if either* (a) *or (b) is satisfied,* P *is globally hypoelliptic on* $\mathbf{T}^2$ *and Fredholm solvable. In the case* (a), *the results are also true for the perturbed operator* $P + \delta(x)$ *with sufficiently small supremum norm of the zeroth order term* $\delta(x)$.

Remarks. (i) Theorem 1 can be viewed as a result for degenerate elliptic operators. We recall that D.Fujiwara and H.Omori [4] established global hypoellipticity for $D_x^2 + \varphi(x)D_y^2$, where φ is $C^\infty(\mathbf{R})$ 2π periodic real-valued nonnegative function, identically equal to 0 and 1 on some subintervals of $[0, 2\pi]$. The interesting point of this example is that the operator degenerates identically on some domain of $\mathbf{T}^2$. Concerning this we note that among the operators satisfying the condition (a) there are operators which may be identically degenerate elliptic in some subdomain and weakly hyperbolic and of mixed type in other regions. This shows the quite different feature of global regularity.

(ii) In 1974, S.Greenfield and N.Wallach [8] showed that a scalar constant coefficients differential operator $P(D)$ on the n-dimensional torus is globally hypoelliptic if and only if its full symbol satisfies a Siegel type conditon. An interesting example is a globally hypoelliptic hyperbolic operator on $\mathbf{T}^2$, $D_x + cD_y$, where $c \in \mathbf{R} \setminus 0$ is an irrational non Liouville number. Later on P.Popivanov established similar results, using Siegel type estimates, for linear systems of constant differential operators on $\mathbf{T}^2$ (cf. [13]) and for some linear operators with polynomial coefficents in the Schwartz class $\mathcal{S}(\mathbf{R}^n)$ (cf. [14]). J. Hounie [9] gives a necessary and sufficient condition for global solvability for first order systems $\partial_t u + b(t)A$, where A is an essentially self-adjoint operator on a compact manifold. The second author has also obtained results for global hypoellipticity on tori for operators not satisfying Siegel type conditions [17], and for the Mathieu operator [18].

In order to study the case (c) we begin with a rather special case. We set

$$\tau_a = \int_0^{2\pi} Im\, a(x)dx,\ \tau_b = \int_0^{2\pi} Im\, b(x)dx. \tag{5}$$

Then we have

Theorem 2. *Suppose that* (A.1) *and* (c) *are satisfied. Moreover, suppose that* $Rea(x)$ *and* $Reb(x)$ *do not change their sign. Assume that* $Rea(x) \equiv 0$ *implies*

$$\tau_a/(2\pi) \in \mathbf{Q}\ (\textit{respectively } Reb(x) \equiv 0 \textit{ implies } \tau_b/(2\pi) \in \mathbf{Q}). \tag{6}$$

Then if

$$\int_0^{2\pi} ImS_0(x) \neq \eta\tau_a\ (\textit{respectively, } \neq \eta\tau_b)\ \textit{ for all } \eta \in \mathbf{Z} \tag{7}$$

the operator P is globally hypoelliptic. Suppose further that (7) *is not true. Then P is globally hypoelliptic if and only if*

$$(*) \qquad \text{there exists } j \geq 1 \text{ such that } \int_0^{2\pi} Im S_j(x)dx \neq 0.$$

In order to study the case where the finiteness condition (6) is not true we define $\tilde{S}(x,\eta)$ by any realization of the formal sum $S(x,\eta)$

$$\tilde{S}(x,\eta) = \sum_{j=-1}^{\infty} \eta^{-j}\chi_j(\eta)S_j(x),$$

where $\chi_j(\eta)$ are suitably chosen cutoff functions whose supports tend to infinity as $j \to \infty$. Then we have

Theorem 3. *Assume that* (A.1) *and* (c) *are satisfied. Moreover, suppose that* $Rea(x)$ *and* $Reb(x)$ *do not change their sign. Then P is globally hypoelliptic on* $\mathbf{T}^2$ *if and only if there exists* $N \geq 0$ *such that the Siegel conditions*

$$(8) \qquad \liminf_{\eta\to\infty,\eta\in\mathbf{Z}} |\eta|^N \left|\exp\left(\int_0^{2\pi} G(x,\eta)dx\right) - 1\right| > 0,$$

are satisfied for $G(x,\eta) = \tilde{S}(x,\eta)$ *and* $G(x,\eta) = \eta(a(x)+b(x)) - \tilde{S}(x,\eta)$. *The conditions* (8) *are also sufficient for the Fredholm solvability of P. We note that the conditions* (8)*are independent of the realizations* $\tilde{S}(x,\eta)$ *of the formal sum* $S(x,\eta)$.

Theorems 2 and 3 are more clearly stated for the special operator (1) such that $\gamma \equiv 0$ or $\gamma = a' - b'$. We denote the operator by P_0. (cf. [6]).

Theorem 4. *Suppose that* $Re\, a(x)$ *and* $Re\, b(x)$ *do not change their sign. Then if* $Re\, a(x) \equiv 0$ (*resp.* $Re\, b(x) \equiv 0$) *the equation* $P_0 u = f$ *has a solution* $u \in \mathcal{D}'$ *for every* $f \in C^\infty(\mathbf{T}^2)$ *such that* $\int_0^{2\pi}\int_0^{2\pi} f(x,y)dxdy = 0$ *if and only if*

$$(9) \qquad \frac{\tau_a}{2\pi}\ (\text{resp. } \frac{\tau_b}{2\pi}) \text{ is an irrational non Liouville number.}$$

Moreover (9) *is necessary and sufficient for the global hypoellipticity of* P_0.

3. Equations of arbitrary order

We shall study the following equation

$$(10) \qquad P = p_m + \sum_{\alpha+j\leq m-1} a_\alpha(x)\partial_x^j D_y^\alpha, \; p_m = \prod_{j=1}^{m}(\partial_x - b_j(x)D_y).$$

We have the following

Theorem 5. *Assume that WKB solutions corresponding to* $b_j(x)(1 \leq j \leq m)$ *are smoothly defined on* $\mathbf{T}$. *Moreover, suppose that* $Reb_j(x)$ $(j = 1,\ldots,m)$ *do not change their* *sign and that if $m \geq 3$ there exists a point $x_0 \in \mathbf{T}$ such that $b_j(x_0) \neq b_i(x_0)$ for $i \neq j$. Then the operator P is globally hypoelliptic on $\mathbf{T}^2$ if and only if there exists $N \geq 0$ such that the Siegel conditions,*

$$\liminf_{\eta \to \infty, \eta \in \mathbf{Z}} |\eta|^N \left| \exp\left(\int_0^{2\pi} G(x, \eta) dx \right) - 1 \right| > 0 \tag{11}$$

are satisfied for $G(x, \eta) = \tilde{S}_j(x, \eta)(1 \leq j \leq m)$, where $\tilde{S}_j(x, \eta)$ is any realization of the formal WKB solution $S_j(x, \eta)$ corresponding to $b_j(x)$ as in Theorem 3.

4. Systems

We are going to study operators such that Rea or Reb changes its sign. Let us consider an $m \times m$ system of equations with smooth periodic coefficients on $\mathbf{T}^2$

$$Pu \equiv D_x u - \left(\sum_{r=0}^{\infty} A_r(x) D_y^{1-r} \right) u = f(x, y) \tag{12}$$

where $u = {}^t(u_1, \ldots, u_m)$ and $f(x, y)$ is an m vector smooth function and where $A_r(x)$ are smooth $m \times m$ matrix-valued functions and the negative powers of D_y denote pseudodifferential operators with the symbol η^{1-r} with modifications near $\eta = 0$. We assume

(A.2) The eigenvalues of $A_0(x)$ are distinct and 2π periodic functions on $0 \leq x \leq 2\pi$.

By a partial Fourier transform with respect to y it follows from (12)

$$\hat{P}\hat{u} \equiv D_x \hat{u} - \left(\sum_{r=0}^{\infty} A_r(x) \eta^{1-r} \right) \hat{u} = \hat{f}(x, \eta). \tag{13}$$

By a standard formal reduction procedure and (A.2) we have that (13) with $\hat{f} = 0$ has a formal solution of the form

$$\hat{u}(x, \eta) \sim P_0(x) \left(I + P_1(x) \eta^{-1} + \cdots \right) e^{\eta \Lambda(x, \eta)},$$

where $P_0(x)$ is invertible and the dots denotes negative powers of η, and where Λ is a diagonal matrix given by

$$\Lambda(x, \eta) = \text{ diag } \left(\int_0^x \lambda_1(t, \eta) dt, \ldots, \int_0^x \lambda_m(t, \eta) dt \right).$$

Here $\lambda_j(x, \eta)$ is a formal power series of η^{-1} with coefficients 2π periodic in x, $\lambda_j(t, \eta) \sim \lambda_j^0(t) + \eta^{-1} \lambda_j^1(t) + \cdots$. We take any realizations $\tilde{\lambda}_j(t, \eta)$ of formal power series $\lambda_j(t, \eta)$ and we define $\tilde{\Lambda}(x, \eta)$ by replacing λ_j by $\tilde{\lambda}_j$. For the sake of simplicity, we denote these realizations by the same letters. The following argument is independent of the choice of realizations of formal solutions. Then we have

Theorem 6. *Suppose that* (A.2) *is satisfied. Then P is globally hypoelliptic if and only if the following conditions are satisfied.*

(I)*Re* $\lambda_j(x)$ $(1 \le j \le m)$ *do not change their sign on the interval* $0 \le x \le 2\pi$.
(II)*There exists* $N > 0$ *such that for all* $j, 1 \le j \le m$

$$(14) \qquad \liminf_{\eta\to\infty, \eta\in \mathbf{Z}} \left|1 - \exp\left(\int_0^{2\pi} \tilde{\Lambda}(x,\eta)\eta dx\right)\right| |\eta|^N > 0.$$

Remark. We note that the condition (14) is independent of the realization $\tilde{\Lambda}(x,\eta)$ of $\Lambda(x,\eta)$. Clearly, there are arbitrariness of the choices of the diagonal matrix Λ. One may take another solution to (13), $\hat{u} \sim \hat{Z}e^{\eta\tilde{\Sigma}(x,\eta)}$, where

$$\tilde{\Sigma}(x,\eta) = \operatorname{diag}\left(\int_0^x \tilde{\mu}_j(t,\eta)dt\right)_{j\to 1,\ldots m} \text{ and } \hat{Z} = Q_0(x)\left(I + Q_1(x)\eta^{-1} + \cdots\right), \ \det Q_0(x) \neq 0.$$

We can prove that under (A.2) (II) holds for $\tilde{\Lambda}(x,\eta)$ if and only if (II) holds for $\tilde{\Sigma}(x,\eta)$. This implies that (II) is invariant under such transformations of unknown functions. We also note that $\lambda_j^0(x)$ are formally invariant. (cf. [16]).

Proposition 7. *Let* (A.1)′ *be true and assume that either* $Re\, a(x)$ *or* $Re\, b(x)$ *changes its sign at some point* $p \in [0, 2\pi]$. *Then the operator* P *is not globally hypoelliptic at* p *(hence it is not locally hypoelliptic) and its adjoint* P^* *is not locally solvable at* p *(hence* P^* *is not globally solvable as well). In particular if we consider* P_0 *with say* $c(x) = a'(x) - b'(x)$ *and require in addition that* $a(x)$ *has a primitive* $A(x)$ *such that* $ReA(p) = 0$, $ReA(x) \ge 0$, $x \in [0, 2\pi]$ *and*

$$\frac{A(2\pi) - A(0)}{2\pi i} \in \mathbf{Z}$$

then the kernel of $P_0 u = 0$ *is infinite dimensional and it contains an infinite subset* $L \subset \mathcal{D}'(\mathbf{T}^2)$ *such that every finite number of elements of* L *are linearly independent,* $L \bigcap C^\infty(\mathbf{T}^2) = \emptyset$ *and the singular support of every element of* L *contains the point* p. *If* $Re\, b(x)$ *satisfies the same conditions as* $Rea(x)$ *the conlusions above hold for the adjoint operator* P_0^*.

Example. If $a(x) = \sin(2x)$ one checks easily that all conditions in the last part of Proposition 7 are fulfilled for the operator P_0 and the point $p = 0$.

5. Examples

We are going to give various examples which illustrate our results.

Example 1. ([Greenfield and Wallach; 8]). Let us consider the operator

$$P = (\partial_{x_1} - \tau\partial_{x_2})(\partial_{x_1} - \mu\partial_{x_2}), \ \text{ on } \mathbf{T}^2, \ \partial_{x_j} = \frac{\partial}{\partial x_j}, \quad \tau > 0, \mu > 0.$$

Then $\tau_a = 2\pi\tau, \tau_b = 2\pi\mu$. Hence by Theorem 4 we have

$Lu = f$ is solvable for any f such that $\int_0^{2\pi}\int_0^{2\pi} f(x,y)dxdy = 0$ if and only if τ and μ are irrational non Liouville numbers. (Siegel condition).

Example 2.

$$\hat{P} = \partial_x^3 - \eta^3 a(x)^3, \quad a \neq 0.$$

Smooth WKB solutions are constructed because $a \neq 0$. Let $S = \sum_{j=-1}^{\infty} \eta^{-j} S_j(x)$ be any WKB solution. Then we have $S_{-1} = \omega a(x)$, $\omega^3 = 1$. We easily see that $S_j = \omega^{-j} \times$ (functions depending only on a). The above theorems imply that the global hypoellipticity is equivalent to the following conditions:

1) $Re\omega a(x)$ do not change their sign on $\mathbf{T}$ for all ω such that $\omega^3 = 1$.

2) Siegel conditions for ω such that $Re\omega a(x) \equiv 0$ are satisfied.

The first condition is equivalent to say that $a(x)$ for $0 \leq x \leq 2\pi$ lies in one of the six sectors bounded by three lines $Imz = 0$, $Im\omega z = 0$ and $Im\omega^2 z = 0$. The case (a) is equivalent to that $a(x)(0 \leq x \leq 2\pi)$ is not entirely contained in the boundary.

We consider the case when the diophantine condition appears, that is $Re\omega a(x) \equiv 0$ for some $\omega, \omega^3 = 1$. For the sake of simplicity, let us assume that this holds for the WKB solution corresponding to $\omega = 1$. Because $a \neq 0$, $b(x) = Ima(x)$ does not vanish on $\mathbf{T}$. Hence $Ima(x)$ does not change its sign. Therefore we have, for $\omega \neq 1$

$$Re \int_0^{2\pi} \omega i b(x) dx = -Im\omega \int_0^{2\pi} b(x) dx = -(Im\omega) \int_0^{2\pi} b(x) dx \neq 0.$$

Therefore the diophantine condition is necessary only for the WKB solution corresponding to $\omega = 1$.

Example 3. We next consider a fourth order equation of mixed type

$$\hat{P} = \partial_x^4 - \eta^4 a(x)^4, \quad a \neq 0.$$

By the same calculations as above we see that WKB solutions $S = \sum_{j=-1}^{\infty} \eta^{-j} S_j(x)$ are of the form, $S_{-1} = a(x)\omega$ with $\omega^4 = 1$, and $S_n = \omega^{-n} \times$ (functions depending only on a). Then $Rea\omega$ does not change its sign for all ω if and only if Rea and Ima do not change their sign. It follows that if $Rea \neq 0$ and $Ima \neq 0$ then the operator is elliptic. If $Rea = 0$ or $Ima = 0$ then the operator is of mixed type. In this case, we can show that a Siegel condition is necessary and sufficient for the global hypoellipticity.

6. Operators not satisfying (A.1)

The following example shows that we cannot drop (A.1) in general.

$$Q = (D_x + ia(x)D_y)^2 - b(x)D_y \tag{15}$$

where $a(x)$ is a 2π periodic real-valued real- analytic function having zero of exact order $2k \neq 0$ at $x = 0$ and $b(x) = -(cx^{2\mu-1} + id)^2, 2 \leq 2\mu \leq k$. Q is not C^∞ globally hypoelliptic on $\mathbf{T}^2$. We note that (A.1) for Q reads $b = 0$. Hence Q does not satisfy (A.1). (cf. [5]).

In case we do not assume (A.1) WKB solutions to (10) have in general essential singularities at points x such that $b_j(x) = b_k(x)$ for some $j \neq k$. We shall discuss a bit on this case. We assume

(A.3) The set $\Gamma = \{x \in \mathbf{T}; b_j(x) = b_k(x) \text{ for some } j \neq k\}$ is a finite set and $b_j(x) - b_k(x)$ for $j \neq k$ have finite order zeros on Γ.

We note that (A.3) is weaker than (A.1) in some cases. Moreover, if $b_j(x)$ are analytic then (A.3) is automatically satisfied. Then we have

Theorem 8. *Suppose that* (A.3)*is satisfied. Then the operator P is globally hypoelliptic in $\mathbf{T}^2 \setminus \{(x,y); x \in \Gamma\}$ if and only if* (8) *is satisfied for any $G = \tilde{S}_j$ where $\tilde{S}_j$ denotes the realization of a singular WKB solution S_j corresponding to b_j in a class of C^∞ functions. We note that the conditions* (8)*are independent of such realizations.*

We remark that the realizations $\tilde{S}_j$ of S_j exists by a generalized Borel-Ritt theorem. In the preceeding theorem the operators which do not satisfy (A.1) generally have so-called turning points on the set of x such that (A.1) is not satisfied. In order to study the regularity on such points we consider apriori estimate of solutions, which is also important when considering (quasi-linear) perturbations of equations.

Let $k \geq 0$ and $\ell \in \mathbf{R}$. The Sobolev norm of a distribution $u(x,y)$ with exponents k for x, ℓ for y is denoted by $||u||_{k,\ell}$. We recall that our solutions are always smooth with respect to x. Then we have

Theorem 9. *Suppose that all WKB solutions for* (10)*are constructed so that the coefficients of η^{-j}, $S_j(x)$ is smooth in x for $j \leq \mu, \mu \geq 1$. Assume further that, for any WKB solution $S = \sum_{j=-1}^{\infty} \eta^{-j} S_j(x)$ we have, for some $\kappa < \mu$*

$$\int_0^{2\pi} Re S_\kappa(x)dx \neq 0, \tag{16}$$

where S_κ is the coefficient of $\eta^{-\kappa}$ of S. Then we have, for any $\ell \in \mathbf{R}$

$$||u||_{0,\ell} \leq c_0 ||Pu||_{0,\ell} + c_1 ||u||_{0,\kappa-\mu+\ell}, \quad u \in D' \cap C_x^\infty \tag{17}$$

for some c_0 and c_1 independent of u.

By (17) we can estimate $||u||_{0,\ell}$ for $\ell \in \mathbf{R}$ because $\kappa - \mu < 0$. By differentiating the equation and by induction on k we can prove

$$||u||_{k,\ell} \leq \sum_{m+n \leq k+\ell} ||Pu||_{m,n} + c_1 \sum_{m+n<k+\ell, m<k} ||u||_{m,n+\kappa-\mu} \tag{18}$$

for any integer k. (18) implies the global hypoellipticity of P.

7. Global solvability for operators which change the sign of $Rea(x)$ and $Reb(x)$.

We first note that if $Rea(x)$ or $Reb(x)$ changes its sign, then we cannot expect the global regularity or global solvability because of Theorem 6. Our object here is to find a class of functions such that $P_0u = f$ is globally solvable on $\mathbf{T}^2$. Assume

$$\int_0^{2\pi} Rea(x)dx \int_0^{2\pi} Reb(x)dx > 0,$$

i.e, they have the same sign. We may assume $\int_0^{2\pi} Rea(x)dx > 0$ and $\int_0^{2\pi} Reb(x)dx > 0$. We set

$$\kappa = \int_0^{2\pi} ((Rea)_-(x) + (Reb)_-(x))\, dx$$

where $(Rea)_-$(resp. $(Reb)_-$) denotes the negative part of Rea. (resp. Reb). Then we have

$P_0u = f$ has a unique solution for any smooth f such that $\int_0^{2\pi}\int_0^{2\pi} f(x,y)dxdy = 0$ if and only if

(**) $$\sup_{\eta} \sup_{x\in[0,2\pi]} |\hat{f}(x,\eta)|e^{2|\eta|\kappa} < \infty.$$

The condition (**) is sufficient for the local solvability at the point where Rea or Reb changes its sign. The important point is that (**) implies the local solvability at points which are far away from points where Rea and Reb change their sign.

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