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SOME TOPICS RELATED WITH DISCRIMINANT POLYNOMIALS

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1. Introduction

The purpose of this note is to explain some results, conjectures and problems on discriminant polynomials of root systems.

Let Σ be a root system on a vector space V of dimension r. For simplicity, we always assume that Σ is irreducible in this note. Let W_Σ be its Weyl group. Then it is knwon by C. Chevalley that there are r number of algebraically independent homogeneous polynomials $x_1, x_2, \cdots, x_r$ on V such that $\mathbf{C}[V]^{W_\Sigma}$ is generated by $x_1, x_2, \cdots, x_r$. This implies that V_c/W_Σ is identified with an affine space S with the coordinate ring $\mathbf{C}[V]^{W_\Sigma}$, where V_c is the complexification of V.

Let D be a non-trivial anti-invariant of W_Σ. Then since its square D^2 is contained in $\mathbf{C}[V]^{W_\Sigma}$, there is a polynomial $F(x_1, x_2, \cdots, x_r)$ of $x_1, x_2, \cdots, x_r$ such that $D^2 = F(x_1, x_2, \cdots, x_r)$. In this note, we call F the *discriminant polynomial* (of Σ).

2. Invariant Differential Operators and b-Functions

We begin this note by explaining a relation between the b-function (or Bernstein-Sato polynomial) of F and that of a discriminant polynomial of a tangent space of a symmetric space.

Let $\underline{g}$ be a complex semisimple Lie algebra and let σ be its complex linear involution. Let $\underline{k}$ (resp. $\underline{p}$) be the $+1$ (resp. -1) eigenspace of σ of $\underline{g}$. We take an abelian subspace $\underline{a}$ of $\underline{p}$ consisting of semisimple elements. If Σ is equal to the root system of the symmetric pair $(\underline{g}, \underline{k})$, then $\underline{a}$ is identified with V_c. Let K be the connected closed subgroup of $Int\ \underline{g}$ with Lie algebra $\underline{k}$. Then, by an unpublished result of C. Chevalley, there are algebraically independent homogeneous polynomials $h_1(X), \cdots, h_r(X)$ on $\underline{p}$

such that $\mathbf{C}[\underline{p}]^K = \mathbf{C}[h_1, \cdots, h_r]$. As a result, the map φ of $\underline{p}$ to S defined by $\varphi(X) = (h_1(X), \cdots, h_r(X))$ is surjective and $\mathbf{C}[\underline{p}]^K \cong \mathbf{C}[x_1, \cdots, x_r]$ by φ. For any polynomial $f \in \mathbf{C}[\underline{p}]^K$, we denote by f^- the unique polynomial on S such that $f = f^- \circ \varphi$.

If we treat the algebra of K-invariant differential operators on $\underline{p}$ instead of $\mathbf{C}[\underline{p}]^K$, how do we formulate a claim analogous to the result of Chevalley mentioned above? To consider this question, we need some notation. Let $Diff(\underline{p})$ be the algebra of polynomial coefficient differential operators on $\underline{p}$ and let $Diff(\underline{p})^K$ be the subalgebra of $Diff(\underline{p})$ consisting of K-invariant differential operators. On the other hand, let D_S be the Weyl algebra on S, that is, $D_S = \mathbf{C}[x_1, \cdots, x_r, \partial/\partial x_1, \cdots, \partial/\partial x_r]$. For any $P \in Diff(\underline{p})^K$, there is a differential operator $\varphi_*(P)$ on S defined by $\varphi_*(P)f = (P(f \circ \varphi))^-$ ($\forall f \in C^\infty(S)$). Put $R_{\underline{p}} = \varphi_*(Diff(\underline{p})^K)$. Then a differential operator $Q \in D_S$ is *φ-liftable* if Q is contained in $R_{\underline{p}}$, that is, there is a differential operator $P \in Diff(\underline{p})^K$ such that $\varphi_*(P) = Q$. We note that φ is not injective. There is a constant coefficient K-invariant second order differential operator $\tilde{\Delta}$ on $\underline{p}$. By definition, $\tilde{\Delta}$ is unique up to a constant factor. Put $\Delta = \varphi_*(\tilde{\Delta})$.

Then we have the proposition below which gives a characterization of elements of $R_{\underline{p}}$.

Proposition 2.1. For any $P \in D_S$, the two conditions below are equivalent.

(1) P is φ-liftable.

(2) $ad(\Delta)^m P = 0$ for some $m \gg 0$.

Now let $R'_{\underline{p}}$ be the subalgebra of $R_{\underline{p}}$ generated by $x_1, \cdots, x_r$ and Δ. Then it seems true that $R'_{\underline{p}}$ coincides with $R_{\underline{p}}$. (I think that this kind of statements is regarded as an analogue of Chevalley's Theorem.)

Let $b_F(s)$ be the b-function of the discriminant polynomial $F(x)$. Then there is a differential operator $Q(x, \partial/\partial x)$ on S such that $QF(x)^{s+1} = b_F(s)F(x)^s$. The explicit form of $b_F(s)$ was conjectured in [YS] and later was proved by E.Opdam [Op]. The result is

$$b_F(s) = \prod_{i=1}^r \prod_{j=1}^{d_i - 1} (s + 1/2 + j/d_i).$$

We consider the pull-back of $F(x)$ to $\underline{p}$, that is, $F_{\underline{p}}(X) = F(\varphi(X))$ which is K-invariant and is called the *discriminant polynomial* of $\underline{p}$. It follows from the definition that the map φ is smooth outside the set $\{F_{\underline{p}} = 0\}$. Let $b_{\underline{p}}(s)$ be the b-function of $F_{\underline{p}}(X)$. Then it is an interseting problem to determine $b_{\underline{p}}(s)$. Still this problem being open, we obtain the proposition below which follows from that $R_{\underline{p}}$ is a subalgebra of D_S.

Proposition 2.2. $b_{\underline{p}}(s)$ is divisible by $b_F(s)$.

Now we restrict our attention to the case where Σ is of type A. Let m_α be the multiplicity of a root $\alpha \in \Sigma$. Since, in this case, all roots of Σ are W_Σ-conjugate, the integer $m = m_\alpha$ is independent of α.

Conjecture 2.3. If Σ is of type A, then $b_{\underline{p}}(s)$ is divisible by $b_F(s)b_F(s+(m-1)/2)$.

Example 2.4. (i) If Σ is of type A_1, then $F(x) = x_1$ and $F_{\underline{p}}(X)$ is a quadratic form of (dim $\underline{p}$)-variables. It is known that, in this case, $b_F(s) = s+1$ and $b_{\underline{p}}(s) = (s+1)(s+(m-1)/2)$, where m is the multiplicity of restricted roots, that is, $m = \dim \underline{p} - 1$.

(ii) We consider the case A_2. In this case, we may take as $F(x_1, x_2)$ the polynomial $x_1^3 + x_2^2$ and therefore its b-function is $b_F(s) = (s+1)(s+5/6)(s+7/6)$. On the other hand, there is a polynomial $Q(\mu)$ of μ whose coefficients are differential operators in D_S with the following conditions.

(1) $\quad Q(\mu)F(x)^{s+1} = b_F(s)b_F(s+(\mu-1)/2)(s+(\mu+2)/4)(s+(\mu+4)/4)F(x)^s.$

(2) Let $(\underline{g}, \underline{k})$ be a symmetric pair whose root system Σ is of type A_2. If m is the multiplicity of roots of Σ for the pair $(\underline{g}, \underline{k})$, then $Q(m) \in R_{\underline{p}}$.

Therefore Conjecture 2.3 seems true in this case.

I have to point out here the similarity of Proposition 2.2 and the argument due to T. Shintani (cf.[Sh]) on the determination of b-functions of relative invariants of prehomogeneous vector spaces obtained from a given prehomogeneous vector space by using *Castling transform.* In fact, in his talk [Gy], A. Gyoja said that the Chevalley's Theorem referred to in this section is regarded as a kind of a Castling transform. In particular, if I do not misunderstand, the polynomial $b_{\underline{p}}(s)/b_F(s)$ is an analogue of a *relative* b-function in his sense and seems to have a meaning.

I thank to M.Muro who is interested in the b-function of $F_{\underline{p}}$ and told me the literature [Sh].

3. A Classification of Weighted Homogeneous Polynomials with Some Additional Conditions : Three Variables Case

The subject of this section is a problem of finding certain weighted homogeneous polynomials which have some nice properties as discriminant polynomials have.

First we formulate the problem which we treat here. Let x, y, z be variables and let p, q, r be natural numbers such that $p < q < r$ and that p, q, r have no common factor. We consider three vector fields on (x, y, z)-space including the Euler operator with weight:

$$V_0 = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z},$$
$$V_1 = qy\frac{\partial}{\partial x} + \{rz + a_{22}(x,y)\}\frac{\partial}{\partial y} + a_{23}(x,y,z)\frac{\partial}{\partial z},$$
$$V_2 = rz\frac{\partial}{\partial x} + a_{32}(x,y,z)\frac{\partial}{\partial y} + a_{33}(x,y,z)\frac{\partial}{\partial z},$$

where $a_{ij}(x,y,z)$ are polynomials. In addition, we define a matrix M obtained from V_0, V_1, V_2 by

$$M = \begin{pmatrix} px & qy & rz \\ qy & rz + a_{22}(x,y) & a_{23}(x,y,z) \\ rz & a_{32}(x,y,z) & a_{33}(x,y,z) \end{pmatrix}.$$

Now we consider the conditions on V_0, V_1, V_2 below:

Condition 3.1.

(i) $\qquad [V_0, V_1] = (q-p)V_1, \qquad [V_0, V_2] = (r-p)V_2.$

(ii) There exist polynomials $f_j(x,y,z)$ $(j = 0,1,2)$ such that

$$[V_1, V_2] = f_0(x,y,z)V_0 + f_1(x,y,z)V_1 + f_2(x,y,z)V_2.$$

(iii) The polynomial $det(M)$ is not *trivial*. ($det(M)$ is trivial if it becomes z^3 by a weight preserving coordinate change.)

Condition 3.1 (i),(ii) claim that the $\mathbf{C}[x,y,z]$-module $L(det(M))$ spanned by V_0, V_1, V_2 becomes a Lie algebra. If V_0, V_1, V_2 satisfy Condition 3.1, it follows that $V_j det(M)/det(M)$ is a polynomial $(j = 0,1,2)$. Namely, V_0, V_1, V_2 and therefore all the vector fields of $L(det(M))$ are logarithmic along the set $\{(x,y,z); det(M) = 0\}$ in the sense of [Sa]. Conversely, it is possible to reconstruct the vector fields V_0, V_1, V_2 from the polynomial $det(M)$ of x, y, z.

If the root system Σ is of rank 3, the type of Σ is one of A_3, B_3, H_3. In this case, there exist vector fields V_0, V_1, V_2 satisfying Condition 3.1 such that $det(M)$ is its discriminant polynomial. In this sense, the polynomial $det(M)$ is regarded as an analogue of a discriminant polynomial. For this reason, it is natural to ask the following problem:

Problem 3.2. Find all the triples $\{V_0, V_1, V_2\}$ of vector fields satisfying Condition 3.1. Or equivalently, find all polynomials $F(x,y,z)$ of the form $F = det(M)$.

The following theorem answers to this problem.

Theorem 3.3. (i) If $(p,q,r) \neq (2,3,4), (1,2,3), (1,3,5)$, there is no triple $\{V_0, V_1, V_2\}$ of vector fields satisfying Condition 3.1.

(ii) If (p,q,r) is one of $(2,3,4),(1,2,3),(1,3,5)$, any polynomial $F(x,y,z)$ of the form $F = det(M)$ is reduced to one of the following polynomials up to a constant factor by a weight preserving coordinate change.

(ii.A) The case $(p,q,r) = (2,3,4)$. (This case corresponds to the root system of type A_3.)

(ii.A1) $16x^4z - 4x^3y^2 - 128x^2z^2 + 144xy^2z - 27y^4 + 256z^3$.

(ii.A2) $2x^6 - 3x^4z + 18x^3y^2 - 18xy^2z + 27y^4 + z^3$.

(ii.B) The case $(p,q,r) = (1,2,3)$. (This case corresponds to the root system of type B_3.)

(ii.B1) $(x^6 - 30x^4y - 150x^3z + 225x^2y^2 + 2250xyz - 500y^3 + 5625z^2)z$.

(ii.B2) $(5x^6 + 6x^4y + 18x^3z - 3x^2y^2 + 18xyz - 4y^3 + 9z^2)z$.

(ii.B3) $(2x^6 - 30x^4y - 225x^3z + 150x^2y^2 + 1125xyz - 250y^3 + 5625z^2)z$.

(ii.B4) $(x^6 - 18x^4y - 108x^3z + 108x^2y^2 + 972xyz - 216y^3 + 2916z^2)z$.

(ii.B5) $790343001x^9 - 5991070554x^7y + 99323708638x^6z + 14600855556x^5y^2 - 3212905573500x^4yz - 16156757156904x^3z^2 + 18228136279584x^2y^2z + 170267363884296xyz^2 - 37837191974288y^3z + 476053650043848z^3$.

(ii.B6) $239625x^9 + 9591750x^7y - 16446850x^6z - 32413500x^5y^2 - 1023546300x^4yz + 3458880600x^3z^2 + 41506567200x^2y^2z + 508455448200xyz^2 - 112990099600y^3z + 996572678472z^3$.

(ii.B7) $13x^9 - 66x^7y - 714x^6z + 84x^5y^2 + 22932x^4yz + 222264x^3z^2 - 98784x^2y^2z - 518616xyz^2 + 115248y^3z + 3630312z^3$.

(ii.H) The case $(p,q,r) = (1,3,5)$. (This case corresponds to the *reflection group* of type H_3.)

(ii.H1) $-8x^9y^2 + 8x^7yz - 20x^6y^3 + 8x^5z^2 + 120x^4y^2z - 230x^3y^4 - 100x^2yz^2 + 450xy^3z - 135y^5 - 100z^3$.

(ii.H2) $-370014797021536x^{15} + 52259033400539715x^{12}y - 7543662620558607 0x^{10}z - 4178071306440x^9y^2 - 664088802409094940x^7yz + 1349632710555470280x^6y^3 + 1070387723782354680x^5z^2 - 2458979443167108840x^4y^2z - 1720082434973806980x^3y^4 + 895508991004499100x^2yz^2 + 4258642757221395720xy^3z - 1277592827166418716y^5 - 1472614785207398520z^3$.

(ii.H3) $-2943652093952x^{15} + 86180519706880x^{12}y - 3126428202240x^{10}z - 3553395309080x^9y^2 - 1917304399080x^7yz + 799477667460x^6y^3 + 71402468760x^5z^2 + 41222238120x^4y^2z - 12236330610x^3y^4 + 10705583700x^2yz^2 - 9287817210xy^3z + 2786345163y^5 - 405076140z^3$.

(ii.H4) $-195432883751468x^{15} - 4240356138903255x^{12}y - 633855510627010x^{10}z - 3923208421631520x^9y^2 + 3797498050261580x^7yz - 3969636123646760x^6y^3 + 810425383418840x^5z^2 + 1905527842803480x^4y^2z - 1112218128823340x^3y^4 -$

$828154338270700x^2yz^2 + 1603040457798360xy^3z - 480912137339508y^5 - 221396034150760z^3$.

(ii.H5) $1292566372387942 4x^{15} + 1072409508559238 40x^{12}y - 5033998385744832 0x^{10}z + 8134309555937136 0x^9y^2 - 1636327980840974 40x^7yz + 3754097667980118 0x^6y^3 + 4918169746397088 0x^5z^2 - 5848720934100714 0x^4y^2z + 1750422404969370x^3y^4 + 60543497116655100x^2yz^2 - 10979922358444230xy^3z + 3293976707533269y^5 - 14161021359488820z^3$.

(ii.H6) $-1867869826665 04x^{15} + 2486353531961860x^{12}y - 7162348657370280x^{10}z - 65602207020750310x^9y^2 - 100928478709658760x^7yz + 57027626933583559 5x^6y^3 - 216045842196795480x^5z^2 + 249187997641139190x^4y^2z - 1255852911490211520x^3y^4 - 3820523746342671 00x^2yz^2 + 2590390753955902080xy^3z - 777117226186770624y^5 - 630953822663324280z^3$.

(ii.H7) $-35621432x^{15} - 1893758097x^{12}y - 488175534x^{10}z - 7017940728x^9y^2 + 10940917428x^7yz - 19775803320x^6y^3 + 4789439928x^5z^2 + 23999272920x^4y^2z - 26525700180x^3y^4 - 15077834100x^2yz^2 + 48159052200xy^3z - 14447715660y^5 - 9451776600z^3$.

(ii.H8) $-331226567016381729996 8x^{15} + 200841939442465086259 20x^{12}y + 270237484774963928678 40x^{10}z - 171762826837922207649 720x^9y^2 - 922889076630730247835 720x^7yz + 271400302814021853751 3140x^6y^3 + 392136450941315730308 40x^5z^2 + 132791187293071671868 3080x^4y^2z - 912236473710813970745 6490x^3y^4 - 256831772005156780661 6700x^2yz^2 + 189657602904653098733 68110xy^3z - 568972808713959296201 0433y^5 - 468498359154678344764 3260z^3$.

Remark 3.4. (i)The polynomials in (ii.A1), (ii.B1), (ii.H1) are the discriminant polynomials of types A_3, B_3, H_3, respectively.

(ii) The polynomial in (ii.A2) is obtained by M.Sato.

(iii) Let $F(x,y,z)$ be one of the polynomials in Theorem 3.3. Then the curve $\{(y,z); F(0,y,z) = 0\}$ is regarded as the simple singularity of type E_6, E_7, E_8 if $F(x,y,z)$ is one of the polynomials in (ii.A), (ii.B), (ii.H), respectively. Is it possible to explain this observation?

Since it is known by P.Deligne, E.Brieskorn, K.Saito that if F is a discriminant polynomial, the complement of $F = 0$ in S is a $K(\pi, 1)$-space and that $\pi_1(\{F \neq 0\})$ is related with Artin braid groups (we used the notation in section 2), it is natural to ask the problem:

Problem 3.5. Let $F(x,y,z)$ be one of the polynomials in Theorem 3.3 and let T be the complement of $F(x,y,z) = 0$ in (x,y,z)-space.

(i) Is T a $K(\pi,1)$-space?

(ii) Compute the fundamental group of T.

Problem 3.5 (i) is a conjecture proposed in [Sa].

It is easy to generalize Problem 3.2 to n variables case which was originally formulated by Prof. M. Sato more than 15 years ago in connection with the study of prehomogeneous vector spaces. I formulate here the problem only in three variables case, because this is the unique case which I could succeed a classification of such vector fields by using Lap Top computer under the guidance of my colleague Prof. K.Okubo.

You can find topics related with the subject of this section in *RIMS Kokyuroku* **281** (1976), 40-105.

4. A Construction of Invariant Spherical Hyperfunctions

It is an important problem to construct *tempered* invariant spherical hyperfunctions on a semisimple symmetric space G/H because they contribute to the Plancherel formula for G/H. Last summer, S.Sano explained me an idea how to construct them in the case $SL(2,R)/SO(1,1)$. Computing those in this case, I was impressed by their interesting support property. In fact, their support is contained in the closure of a conjugacy class of a *Cartan subspace* as the case of characters of principal series representations of semisimple groups. The subject of this section is to explain a result on invariant spherical hyperfunctions which relates with the support property mentioned above. For the details, see [Se].

This time, let $\underline{g}_0$ be a real semisimple Lie algebra and let σ be its involution. Then we have a symmetric pair $(\underline{g}_0,\underline{h}_0)$ and a direct sum decomposition $\underline{g}_0 = \underline{h}_0 + \underline{q}_0$. For simplicity, we assume that $(\underline{g}_0,\underline{h}_0)$ is irreducible in the sequel. From the definition, $\underline{h}_0$ acts on $\underline{q}_0$ via the adjoint action. We also assume that the complexifications of $\underline{g}_0$, $\underline{h}_0$, $\underline{q}_0$, are $\underline{g}$, $\underline{k}$, $\underline{p}$ of section 2, respectively. (I am sorry that the notation are confusing.) In the sequel, we use the notation of section 2 without any comment. Then, from the definition, $Diff(\underline{p})$ is regarded as an algebra of differential operators on $\underline{q}_0$. Let $Diff_{const}(\underline{p})^K$ be the subalgebra of $Diff(\underline{p})^K$ consisting of constant coefficient differential operators. From the definition, $P_j = ad(\widetilde{\Delta})^{d_j} h_j$ $(j = 1,2,\cdots,r)$ are contained in $Diff_{const}(\underline{p})^K$. We now recall the following lemma due to Harish-Chandra which supports the claim after Proposition 2.1.

Lemma 4.1. (cf.[HC]) The differential operators $P_1, P_2, \cdots, P_r$ are algebraically independent and generate $Diff_{const}(\underline{p})^K$.

For any $\lambda = (\lambda_1, \cdots, \lambda_r) \in \mathbf{C}^r$, we define a system of differential equations M_λ on $\underline{q}_0$ by

$$(P_j - \lambda_j)u = 0 \qquad (j = 1, \cdots, r)$$

$$\tau(Y)u = 0 \qquad (\forall Y \in \underline{h}_0),$$

where, for any $Y \in \underline{h}_0$, $\tau(Y)$ is the vector field on $\underline{q}_0$ defined by

$$(\tau(Y)f)(X) = \tfrac{d}{dt} f(X + t[X, Y]) \,|_{t=0} \; (\forall f \in \mathbf{C}^{\infty}(\underline{q}_0)).$$

Solutions to the system M_λ are called *invariant spherical hyperfunctions* on $\underline{q}_0$.

There is a deep relation between the system M_λ with the discriminant polynomial $F_{\underline{p}}$. To explain this, we introduce *logarithmic* vector fields along the set $\{F_{\underline{p}} = 0\}$. (For a general theory of logarithmic vector fields, see [Sa]). We put $\widetilde{L}_j = [\widetilde{\Delta}, h_j] - \widetilde{\Delta} h_j$ $(j = 1, 2, \cdots, r)$. Then each $\widetilde{L}_j$ is a vector field on $\underline{q}_0$ which is logarithmic along the set $\{F_{\underline{p}} = 0\}$. Namely, there exist polynomials $c_j(X) \in \mathbf{C}[\underline{p}]^K$ $(j = 1, 2, \cdots, r)$ such that $L_j F_{\underline{p}} = c_j(X) F_{\underline{p}}$. Accordingly we see that $L_j = \varphi_*(\widetilde{L}_j)$ $(j = 1, 2, \cdots, r)$ are vector fields logarithmic along the set $\{F = 0\}$. Conversely, the differential operator Δ is obtained from L_j $(j = 1, \cdots, r)$ by the lemma below.

Lemma 4.2. There is a vector field L_0 on S such that

$$\Delta = \tfrac{1}{2} \textstyle\sum_{j=1}^{r} \frac{\partial}{\partial x_j} L_j + L_0.$$

In the sequel, we assume the condition below on the symmetric pair $(\underline{g}_0, \underline{h}_0)$ unless otherwise stated.

Condition 4.3. There is a normal real form $\underline{g}_1$ of $\underline{g}$ such that $\underline{k} \cap \underline{g}_1$ is its maximal compact subalgebra.

In this case, Lemma 4.2 is refined as follows.

Lemma 4.2'. $\quad \Delta = \frac{1}{2} \sum_{j=1}^{r} \frac{\partial}{\partial x_j} L_j.$

As a direct consequence of Lemma 4.2', we have the following.

Proposition 4.4. $\quad \widetilde{\Delta} \, |F_{\underline{p}}|^s = s^2 q_0 \, |F_{\underline{p}}|^{s-1}$, where $q_0 = \widetilde{\Delta} F_{\underline{p}} \in \mathbf{C}[\underline{p}]^K$.

Remark 4.5. We return to the general case, forgetting Condition 4.3. Then the statement below seems to be true:

There is a polynomial $q_0(X) \in \mathbf{C}[\underline{p}]^K$ and a constant α such that

$$\widetilde{\Delta}\,|F_{\underline{p}}|^s = s(s+\alpha)q_0\,|F_{\underline{p}}|^{s-1}.$$

As a consequence, $s+\alpha$ has to be a factor of the b-function of $F_{\underline{p}}$.

We put $\underline{q}_0' = \{X \in \underline{q}_0; F_{\underline{p}}(X) \neq 0\}$. By definition, $\underline{q}_0'$ has finitely many connected components. For any connected component Ω of $\underline{q}_0'$, we define a function $|F_{\underline{p}}|_\Omega^s$ on $\underline{q}_0$ ($s \in \mathbf{C}$) by $|F_{\underline{p}}|_\Omega^s(X) = |F_{\underline{p}}(X)|^s$ if $X \in \Omega$ and $|F_{\underline{p}}|_\Omega^s(X) = 0$ otherwise. Needless to say, $|F_{\underline{p}}|_\Omega^s$ is a continuous function on $\underline{q}_0$ if Re $s > 0$ and is extended to a $D'(\underline{q}_0)$-valued meromorphic function of s on the whole s-space, where $D'(\underline{q}_0)$ is the space of distributions on $\underline{q}_0$. Moreover, it is clear that $Y_\Omega = |F_{\underline{p}}|_\Omega^s\,|_{s=0}$ is the characteristic function of Ω. As a corollary to Proposition 4.4, we have the following.

Proposition 4.6. $\widetilde{\Delta} Y_\Omega = \left(s^2 q_0\,|F_{\underline{p}}|_\Omega^{s-1}\right)_{s=0}$.

For simplicity, we put $Z_\Omega = \left(s^2 q_0\,|F_{\underline{p}}|_\Omega^{s-1}\right)_{s=0}$. In spite that it is not clear whether $\left(s^2\,|F_{\underline{p}}|_\Omega^{s-1}\right)_{s=0}$ is holomorphic near $s = 0$ or not, Z_Ω is well-defined because of Proposition 4.6. From the definition, $Supp(Z_\Omega)$ is contained in the set $\{X \in \underline{q}_0; F_{\underline{p}}(X) = 0, (dF_{\underline{p}})_X = 0\}$. Then we obtain the theorem below which is related with the support property mentioned at the first part of this section. For its proof, we need Lemma 4.1 and Proposition 4.6.

Theorem 4.7. We assume that Condition 4.3 holds for the symmetric pair $(\underline{g}_0, \underline{h}_0)$. If there are connected components $\Omega_1, \cdots, \Omega_k$ of $\underline{q}_0'$ and constants $c_1, \cdots, c_k$ such that

$$\textstyle\sum_{j=1}^k c_j Z_{\Omega_j} = 0,$$

we have the following.

(i) $\eta = \sum_{j=1}^k c_j Y_{\Omega_j}$ is a solution to the system M_λ with $\lambda = (0, \cdots, 0)$.

(ii) Let $\lambda = (\lambda_1, \cdots, \lambda_r)$ be arbitrary. If $f(X)$ is an analytic solution to M_λ, then $f(X)\eta(X)$ is a hyperfunction solution to M_λ.

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==================== 余談 ====================

母校を訪れて

＜学術を究むるところ大寒に入る＞

昭和五十二年　新田次郎

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「新田氏の俳句のこと」　　遠藤一郎

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この句はまず「学術を究むるところ」と大学を定義しておられる。これは重い定義である。それに続く「大寒」は、「学術を究むる」に対応し、大学像にぴったりの季語であろう。大学は春風たいとうであってはならず、興奮状態の夏、沈滞凋落ムードの秋であってもならない。寒稽古や寒行に象徴されるきびしい修練の季節、大寒こそふさわしい。

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「電気通信大学の夜間・社会人の教育」より抜粋