数理解析研究所講究録
第 815 巻 1992 年 23-28

Topological symmetry of holomorphic function germs with isolated singularities

TAKASHI NISHIMURA 西村 尚史
Department of Mathematics, Faculty of Education
Yokohama National University
Yokohama 240, JAPAN

In this note, The author would like to propose the following problem (problem 1) which seems to be open apparently.

PROBLEM 1. *Let* $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ *be a holomorphic function germ having an isolated singular point at the origin. Let* $\bar f$ *be its complex conjugation. Then, is there a germ of homeomorphism of the source space* $h:(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ *such that* $\bar f = f\circ h$ *?*

Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a holomorphic function germ. We say f is *of real coefficient* if the identity germ $\bar f(z)=f(\bar z)$ holds.

PROBLEM 2. *Let* $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ *be a holomorphic function germ having an isolated singular point at the origin. Then, is there a germ of one parameter family* $F:(\mathbb{C}^n\times[0,1],0\times[0,1])\to(\mathbb{C},0)$ *such that the following 4 properties hold ?*

(1) F *depends on the parameter* $t\in[0,1]$ *continuously,*
(2) $F(\ ,t)$ *is holomorphic for any* t *of* $[0,1]$,
(3) $F(\ ,0)=f$ *and* $F(\ ,1)$ *is of real coefficient,*
(4) *there exists a germ of homeomorphism*

$$H:(\mathbb{C}^n\times[0,1],0\times[0,1])\to(\mathbb{C}^n\times[0,1],0\times[0,1])$$

of the form $H(z,t)=(H_1(z,t),t)$ *such that* $F\circ H(z,t)=f(z)$.

We see easily that the problem 1 is affirmative if the problem 2 is affirmative.

Trivially, in the case $n=1$ (one variable) the problem 2 is affirmative. The author learned from O.Saeki that the problem 2 has been solved affirmatively in the case $n=2$ (two variables) by S.M.Gusein-Zade ([**GZ**]). In §2, we will see that the problem 2 is affirmative in the case that the given function germ f has a non-degenerate Newton principal part in the sense of A.G.Kouchnirenko

([**Ko**]). Since having a non-degenerate Newton principal part in the sense of A.G.Kouchnirenko is a generic property, we can say that the problem 2 is affirmative for almost all function germs. On the other hand, there are attempts to find counterexamples of the problem 2 in three variables case ($n = 3$) (see [**S**]). However, the problem 2 seems to be still open in the case $n \geq 3$.

In §1, the author gives a similar problem as the problem 1 from a knot-theoretic view point, and also gives an alternative proof of the affirmative solution of the problem 1 in the case $n = 2$ from this view point. The problem 1 also seems to be still open in the case $n \geq 3$.

§1. Algebraic link

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ having an isolated singular point at the origin. We take a representative of f (denoted by f again). That is to say, f is a holomorphic function defined on some neighborhood U of the origin 0 in $\mathbb{C}^n$, that $f(0) = 0$, and that

$$\{z \in U \mid \frac{\partial f}{\partial z_1(z)} = \cdots = \frac{\partial f}{\partial z_n(z)} = 0\} = \{0\}.$$

Then, the hypersurface $f^{-1}(0)$ is equal to the origin in the case $n = 1$. For $n \geq 2$, there exists a sufficiently small positive number ε_0 such that for any ε $(0 < \varepsilon < \varepsilon_0)$ the hypersurface $f^{-1}(0)$ intersects transversally a small sphere εS^{2n-1} centered at the origin (ε is the radius of this sphere). Thus, the intersection $f^{-1}(0) \cap \varepsilon S^{2n-1}$ gives a smooth compact $(2n-3)$-dimensional manifold K_f (as a general reference on this subject, see [**M**]).

We are interested in the embedding of K_f in εS^{2n-1}, which we call *algebraic link*.

Remark 1.1: It is well-known that for any holomorphic function germ f : $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ having an isolated singular point at the origin, there exists a biholomorphic germ $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that the composition $f \circ h$ is a polynomial (c. f. [**W**]). This is the reason why we use the word "algebraic".

Remark 1.2: In the case $n = 2$, K_f may have several connected components (for instance, K_f has two connected components for $f = z_1^2 + z_2^2$). This is the reason why we use the word "link".

Remark 1.3: It is well-known that K_f is $(n-3)$-connected ([**M**]). Thus, K_f is connected in the case $n \geq 3$.

Remark 1.4: K_f is orientable.

Remark 1.5: It is well-known that the mapping $\phi_f : \varepsilon S^{2n-1} - K_f \to S^1$ given by $\phi_f(z) = \frac{f(z)}{||f(z)||}$ is a fibration, which we call Milnor's fibration (see [**M**]).

REMARK 1.6: It is also well-known that a fiber of the Milnor's fibration $\phi_f^{-1}(\theta)$ of the given function germ f is diffeomorphic to the intersection of the open ball $\varepsilon B^{2n} = \{z \in \mathbb{C}^n : ||z|| < \varepsilon\}$ and a smooth hypersurface $f^{-1}(t)$ for sufficiently small $t \neq 0$ (see [M]). Thus, we can see the topological structure of the given map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is determined by the Milnor's fibration of f.

DEFINITION 1: Let εS^{2n-1} be the set $\{z \in \mathbb{C}^n \mid ||z|| = \varepsilon\}$. We fix one orientation of εS^{2n-1}. Let L be an oriented submanifold of εS^{2n-1}.
(1) We say $(\varepsilon S^{2n-1}, L)$ is *invertible* if there exists an orientation preserving homeomorphism $h : \varepsilon S^{2n-1} \to \varepsilon S^{2n-1}$ such that the following two properties hold:

(1.1) $h(L) = L$

(1.2) the restriction $h|_L : L \to L$ is orientation reversing.

(2) We say $(\varepsilon S^{2n-1}, L)$ is *strongly invertible* if there exists a one parameter family $H : \varepsilon S^{2n-1} \times [0,1] \to \varepsilon S^{2n-1}$ with the following 5 properties:

(2.1) H depends on the parameter $t \in [0,1]$ continuously,

(2.2) $H(\ ,t)$ is a homeomorphism for any t of $[0,1]$

(2.3) $H(\ ,0)$ is the identity mapping

(2.4) $H(\ ,1) = h$ maps L to itself homeomorphically

(2.5) the restriction $h|_L : L \to L$ is orientation reversing.

Of course, the strong invertibleness is a stronger notion than the invertibleness. The following is a similar problem as our problem 1.

PROBLEM 3. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ having an isolated singular point at the origin. Then, is $(\varepsilon S^{2n-1}, K_f)$ strongly invertible ?*

The author learned the following fact from M. Yamamoto ([**Y**]). This proposition 1 gives a direct proof for the affirmative solution of problem 1 in the case $n = 2$.

PROPOSITION 1 (M. YAMAMOTO). *In the case $n = 2$, every algebraic link $(\varepsilon S^3, K_f)$ is strongly invertible.*

PROOF OF PROPOSITION 1: First, we need one definition.

DEFINITION 2: Let (S^3, K) be a classical knot. Take l tubular neighborhoods $V_1, \ldots, V_l$ of K in S^3 such that $K \subset V_1 \subset V_2 \subset \cdots \subset V_l$ and two boundaries of V_i and V_{i+1} are disjoint for each i $(1 \leq i \leq l-1)$. Let $K_i (\subset \partial V_i)$ be a

(p,q)-cabling of K, where p and q are relatively prime. Let L be the union of $K_1, K_2, \ldots, K_l$. We say L a (lp, lq) *cable link of* K.

In the case $n = 2$, every algebraic link $(\varepsilon S^3, K_f)$ can be constructed in the following way (c. f. [**P**]).

Let (S^3, T_0) be a trivial knot. Let $T_r = K_1 \cup \cdots \cup K_\alpha$, where K_i be a connected component of T_r. Let L_i be a (s,t) cable link of K_i. We set

$$\begin{aligned} T_{r+1} = & K_1 \cup \cdots \cup K_i \cup \cdots \cup K_\alpha \cup L_i \qquad \text{or} \\ & K_1 \cup \cdots \cup K_{i-1} \cup K_{i+1} \cup \cdots \cup K_\alpha \cup L_i. \end{aligned}$$

Then, since every torus knot is strongly invertible, by this construction, every (S^3, T_r) is also strongly invertible for any $r \subset \mathbb{N}$.

Thus, every algebraic link in the case $n = 2$ is strongly invertible. ∎

PROOF THAT PROPOSITION 1 IMPLIES THE AFFIRMATIVE SOLUTION OF THE PROBLEM 1 IN THE CASE $n = 2$: By proposition 1, there exists a homeomorphism $h_1 : (\varepsilon S^3, K_f) \to (\varepsilon S^3, K_f)$ such that the mapping $\phi_{\bar{f}h_1} : \varepsilon S^3 - K_f \to S^1$ given by $\phi_{\bar{f}h_1}(z) = \frac{\bar{f}(h_1(z))}{\|\bar{f}(h_1(z))\|}$ is a fibration. Since for classical fibered link (S^3, L) the oriented fibration structure of it is unique up to isotopy (c. f. [**R**]), we see there exists a homeomorphism $h_2 : (\varepsilon S^3, K_f) \to (\varepsilon S^3, K_f)$ such that

$$\frac{f(z)}{\|f(z)\|} = \frac{\bar{f}(h_2(z))}{\|\bar{f}(h_2(z))\|}$$

for any z of $\varepsilon S^3 - K_f$.

Thus, we may conclude there exists a germ of homeomorphism $h : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $f = \bar{f} \circ h$. ∎

§2 FUNCTION GERMS HAVING NON-DEGENERATE NEWTON PRINCIPAL PARTS

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ. We write $f(z) = \sum a_\nu z^\nu$, where $\nu = (\nu_1, \ldots, \nu_n)$ goes through multi-integers $\mathbb{N}^n$ and $z^\nu = z_1^{\nu_1} z_2^{\nu_2} \ldots z_n^{\nu_n}$ as usual. Let $\Gamma_+(f)$ be the convex hull of $\cup_\nu (\nu + (\mathbb{R}_+)^n)$, where the union is taken for all ν such that $a_\nu \neq 0$. Let $\Gamma(f)$ be the union of compact boundaries of $\Gamma_+(f)$. We say f has a *non-degenerate Newton principal part* if $f_\Delta(z) = \sum_{\nu \in \Delta} a_\nu z^\nu$ is non-singular on $(\mathbb{C}^*)^n = (\mathbb{C} - \{0\})^n$ for any Δ of $\Gamma(f)$. f is said to be *convenient* if the intersection of $\Gamma(f)$ with each coordinate axis is non-empty. These definitions are due to A. G. Kouchnirenko ([**Ko**], see also [**O**]).

The problem 2 is affirmative for a holomorphic function germ which has a non-degenerate Newton principal part (proposition 2).

PROPOSITION 2. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ with isolated singular point at the origin. Suppose f has a non-degenerate Newton principal part. Then there exists a germ of one parameter family $F : (\mathbb{C}^n \times [0,1], 0 \times [0,1]) \to (\mathbb{C}, 0)$ such that the following 4 properties hold:*

(1) F depends on the parameter $t \in [0,1]$ continuously,
(2) $F(\ ,t)$ is holomorphic for any t of $[0,1]$,
(3) $F(\ ,0) = f$ and $F(\ ,1)$ is of real coefficient,
(4) there exists a germ of homeomorphism

$$H : (\mathbb{C}^n \times [0,1], 0 \times [0,1]) \to (\mathbb{C}^n \times [0,1], 0 \times [0,1])$$

of the form $H(z,t) = (H_1(z,t), t)$ such that $F \circ H(z,t) = f(z)$.

PROOF OF PROPOSITION 2: By the geometric characterization of finite determinacy ([**W**]), we see

LEMMA 1. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a function germ with isolated singularities which has a non-degenerate Newton principal part. Then, there exists a biholomorphic map germ $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that the composition $f \circ h$ is convenient and non-degenerate.*

We write $f \circ h = \sum b_\lambda z^\lambda$. Let V_{fh} be the set of coefficients of all polynomials having terms only on $\Gamma(f \circ h)$. Namely,

$$V_{fh} = \{\sum c_\lambda z^\lambda \mid c_\lambda = 0 \text{ if and only if } b_\lambda = 0 \text{ or } \lambda \notin \Gamma(f \circ h)\}.$$

We also set

$$U_{fh} = \{\sum c_\lambda z^\lambda \in V_{fh} \mid \text{it has a non-degenerate Newton principal part}\}.$$

Then,

LEMMA 2 ([**O**]). *U_{fh} is a non-empty Zariski open subset of V_{fh}.*

Thus, we can choose a germ of one parameter family $F : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that

(1) F depends on the parameter $t \in [0,1]$ analytically,
(2) $F(\ ,t)$ is convenient and has a non-degenerate Newton principal part for any t of $[0,1]$,
(3) $F(\ ,0) = f \circ h$ and $F(\ ,1)$ is of real coefficient.

This germ of one parameter family F is the desired one because

LEMMA 3 (COMBINING [**O**] AND [**K**]). *Let $F : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of one parameter family such that*

(1) F *depends on the parameter* $t \in [0,1]$ *analytically,*
(2) $F(\ ,t)$ *is convenient and has a non-degenerate Newton principal part for any* t *of* $[0,1]$.

Then, there exists a germ of homeomorphism

$$H : (\mathbb{C}^n \times [0,1], 0 \times [0,1]) \to (\mathbb{C}^n \times [0,1], 0 \times [0,1])$$

of the form $H(z,t) = (H_1(z,t), t)$ *such that* $F \circ H(z,t) = f(z)$.

∎

References

[**GZ**] S. M. Gusein-Zade, *Dynkin diagrams for singularities of functions of two variables*, Functional Anal. Appl. **8** (1974), 295-300.

[**Ki**] H. King, *Topological type in families of germs*, Inventiones math. **62** (1980), 1-13.

[**Ko**] A. G. Kouchnirenko, *Polyhèdres de Newton et nombres de Milnor*, Inventiones math. **32** (1976), 1-31.

[**M**] J. Milnor, "Singular points of complex hypersurfaces," Ann. of Math. Studies 61, Princeton Univ. Press, 1968.

[**O**] M. Oka, *On the bifurcation of the multiplicity and topology of the Newton boundary*, J. Math. Soc. Japan **31** (1979), 435-450.

[**P**] F. Pham, "Cours de 3ème cycle. Dept. Math. de la Faculté des Sciences de Paris," (à paraître au Centre de Math. de l'Ecole Polytechnique).

[**R**] D. Rolfsen, "Knots and links," Publish or Perish, 1976.

[**S**] O. Saeki, *Theory of super-isolated singularities and its applications*, R. I. M. S. Koukyuroku **764** (1991).

[**Y**] M. Yamamoto, private communication.

[**W**] C. T. C. Wall, *Finite determinacy of smooth map-germs*, Bull. London Math. Soc. **13** (1981), 481-539.