Topological symmetry of holomorphic function germs with isolated singularities

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In this note, The author would like to propose the following problem (problem 1) which seems to be open apparently.

**Problem 1.** Let $f : (\mathbb{C}^{n}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ having an isolated singular point at the origin. Let $\bar{f}$ be its complex conjugation. Then, is there a germ of homeomorphism of the source space $h : (\mathbb{C}^{n}, 0) \rightarrow (\mathbb{C}^{n}, 0)$ such that $\bar{f} = f \circ h$?

Let $f : (\mathbb{C}^{n}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ. We say $f$ is of real coefficient if the identity germ $\bar{f}(z) = f(\bar{z})$ holds.

**Problem 2.** Let $f : (\mathbb{C}^{n}, 0) \rightarrow (\mathbb{C}, 0)$ be a holomorphic function germ having an isolated singular point at the origin. Then, is there a germ of one parameter family $F : (\mathbb{C}^{n} \times [0,1], 0 \times [0,1]) \rightarrow (\mathbb{C}, 0)$ such that the following 4 properties hold?

1. $F$ depends on the parameter $t \in [0,1]$ continuously,
2. $F(\cdot, t)$ is holomorphic for any $t$ of $[0,1]$,
3. $F(\cdot, 0) = f$ and $F(\cdot, 1)$ is of real coefficient,
4. there exists a germ of homeomorphism

$$H : (\mathbb{C}^{n} \times [0,1], 0 \times [0,1]) \rightarrow (\mathbb{C}^{n} \times [0,1], 0 \times [0,1])$$

of the form $H(z, t) = (H_1(z, t), t)$ such that $F \circ H(z, t) = f(z)$.

We see easily that the problem 1 is affirmative if the problem 2 is affirmative.

Trivially, in the case $n = 1$ (one variable) the problem 2 is affirmative. The author learned from O. Saeki that the problem 2 has been solved affirmatively in the case $n = 2$ (two variables) by S.M. Gusein-Zade ([GZ]). In \S 2, we will see that the problem 2 is affirmative in the case that the given function germ $f$ has a non-degenerate Newton principal part in the sense of A.G. Kouchnirenko.
([Ko]). Since having a non-degenerate Newton principal part in the sense of A.G.Kouchnirenko is a generic property, we can say that the problem 2 is affirmative for almost all function germs. On the other hand, there are attempts to find counterexamples of the problem 2 in three variables case \((n = 3)\) (see [S]). However, the problem 2 seems to be still open in the case \(n \geq 3\).

In §1, the author gives a similar problem as the problem 1 from a knot-theoretic view point, and also gives an alternative proof of the affirmative solution of the problem 1 in the case \(n = 2\) from this view point. The problem 1 also seems to be still open in the case \(n \geq 3\).

§1. Algebraic link

Let \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) be a holomorphic function germ having an isolated singular point at the origin. We take a representative of \(f\) (denoted by \(f\) again). That is to say, \(f\) is a holomorphic function defined on some neighborhood \(U\) of the origin 0 in \(\mathbb{C}^n\), that \(f(0) = 0\), and that

\[
\{z \in U \mid \frac{\partial f}{\partial z_1(z)} = \cdots = \frac{\partial f}{\partial z_n(z)} = 0\} = \{0\}.
\]

Then, the hypersurface \(f^{-1}(0)\) is equal to the origin in the case \(n = 1\). For \(n \geq 2\), there exists a sufficiently small positive number \(\epsilon_0\) such that for any \(0 < \epsilon < \epsilon_0\) the hypersurface \(f^{-1}(0)\) intersects transversally a small sphere \(\epsilon S^{2n-1}\) centered at the origin (\(\epsilon\) is the radius of this sphere). Thus, the intersection \(f^{-1}(0) \cap \epsilon S^{2n-1}\) gives a smooth compact \((2n - 3)\)-dimensional manifold \(K_f\) (as a general reference on this subject, see [M]).

We are interested in the embedding of \(K_f\) in \(\epsilon S^{2n-1}\), which we call algebraic link.

REMARK 1.1: It is well-known that for any holomorphic function germ \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)\) having an isolated singular point at the origin, there exists a biholomorphic germ \(h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)\) such that the composition \(f \circ h\) is a polynomial (c. f. [W]). This is the reason why we use the word "algebraic".

REMARK 1.2: In the case \(n = 2\), \(K_f\) may have several connected components (for instance, \(K_f\) has two connected components for \(f = z_1^2 + z_2^2\)). This is the reason why we use the word "link".

REMARK 1.3: It is well-known that \(K_f\) is \((n - 3)\)-connected ([M]). Thus, \(K_f\) is connected in the case \(n \geq 3\).

REMARK 1.4: \(K_f\) is orientable.

REMARK 1.5: It is well-known that the mapping \(\phi_f : \epsilon S^{2n-1} \to K_f \to S^1\) given by \(\phi_f(z) = \frac{f(z)}{||f(z)||}\) is a fibration, which we call Milnor's fibration (see [M]).
REMARK 1.6: It is also well-known that a fiber of the Milnor’s fibration $\phi_{f}^{-1}(\theta)$ of the given function germ $f$ is diffeomorphic to the intersection of the open ball $\varepsilon B^{2n} = \{z \in \mathbb{C}^{n} : ||z|| < \varepsilon\}$ and a smooth hypersurface $f^{-1}(t)$ for sufficiently small $t \neq 0$ (see [M]). Thus, we can see the topological structure of the given map germ $f : (\mathbb{C}^{n}, 0) \to (\mathbb{C}, 0)$ is determined by the Milnor’s fibration of $f$.

DEFINITION 1: Let $\varepsilon S^{2n-1}$ be the set $\{z \in \mathbb{C}^{n} : ||z|| = \varepsilon\}$. We fix one orientation of $\varepsilon S^{2n-1}$. Let $L$ be an oriented submanifold of $\varepsilon S^{2n-1}$.

(1) We say $(\varepsilon S^{2n-1}, L)$ is invertible if there exists an orientation preserving homeomorphism $h : \varepsilon S^{2n-1} \to \varepsilon S^{2n-1}$ such that the following two properties hold:

(1.1) $h(L) = L$

(1.2) the restriction $h|_{L} : L \to L$ is orientation reversing.

(2) We say $(\varepsilon S^{2n-1}, L)$ is strongly invertible if there exists a one parameter family $H : \varepsilon S^{2n-1} \times [0,1] \to \varepsilon S^{2n-1}$ with the following 5 properties:

(2.1) $H$ depends on the parameter $t \in [0,1]$ continuously,

(2.2) $H(, t)$ is a homeomorphism for any $t$ of $[0,1]$

(2.3) $H(, 0)$ is the identity mapping

(2.4) $H(, 1) = h$ maps $L$ to itself homeomorphically

(2.5) the restriction $h|_{L} : L \to L$ is orientation reversing.

Of course, the strong invertibleness is a stronger notion than the invertibleness. The following is a similar problem as our problem 1.

PROBLEM 3. Let $f : (\mathbb{C}^{n}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ having an isolated singular point at the origin. Then, is $(\varepsilon S^{2n-1}, K_{f})$ strongly invertible?

The author learned the following fact from M. Yamamoto ([Y]). This proposition 1 gives a direct proof for the affirmative solution of problem 1 in the case $n = 2$.

PROPOSITION 1 (M. YAMAMOTO). In the case $n = 2$, every algebraic link $(\varepsilon S^{3}, K_{f})$ is strongly invertible.

PROOF OF PROPOSITION 1: First, we need one definition.

DEFINITION 2: Let $(S^{3}, K)$ be a classical knot. Take $l$ tubular neighborhoods $V_{1}, \ldots, V_{l}$ of $K$ in $S^{3}$ such that $K \subset V_{1} \subset V_{2} \subset \cdots \subset V_{l}$ and two boundaries of $V_{i}$ and $V_{i+1}$ are disjoint for each $i$ $(1 \leq i \leq l-1)$. Let $K_{i}(\subset \partial V_{i})$ be a
(p, q)-cabling of K, where p and q are relatively prime. Let L be the union of $K_1, K_2, \ldots, K_t$. We say L a (lp, lq) cable link of K.

In the case $n = 2$, every algebraic link $(eS^3, K_f)$ can be constructed in the following way (c. f. [P]).

Let $(S^3, T_0)$ be a trivial knot. Let $T_r = K_1 \cup \cdots \cup K_{\alpha}$, where $K_i$ be a connected component of $T_r$. Let $L_i$ be a $(s, t)$ cable link of $K_i$. We set

$$T_{r+1} = K_1 \cup \cdots \cup K_i \cup \cdots \cup K_{\alpha} \cup L_i \quad \text{or} \quad K_1 \cup \cdots \cup K_{i-1} \cup K_{i+1} \cup \cdots \cup K_{\alpha} \cup L_i.$$

Then, since every torus knot is strongly invertible, by this construction, every $(S^3, T_r)$ is also strongly invertible for any $r \in \mathbb{N}$.

Thus, every algebraic link in the case $n = 2$ is strongly invertible. ■

**Proof that Proposition 1 implies the affirmative solution of the problem 1 in the case $n = 2$:** By Proposition 1, there exists a homeomorphism $h_1 : (eS^3, K_f) \to (eS^3, K_f)$ such that the mapping $\phi_{f h_1} : eS^3 - K_f \to S^1$ given by $\phi_{f h_1}(z) = \frac{f(h_1(z))}{||f(h_1(z))||}$ is a fibration. Since for classical fibered link $(S^3, L)$ the oriented fibration structure of it is unique up to isotopy (c. f. [R]), we see there exists a homeomorphism $h_2 : (eS^3, K_f) \to (eS^3, K_f)$ such that

$$\frac{f(z)}{||f(z)||} = \frac{\bar{f}(h_2(z))}{||\bar{f}(h_2(z))||},$$

for any $z$ of $eS^3 - K_f$.

Thus, we may conclude there exists a germ of homeomorphism $h : (C^2, 0) \to (C^2, 0)$ such that $f = \bar{f} \circ h$. ■

§2 Function germs having non-degenerate Newton principal parts

Let $f : (C^n, 0) \to (C, 0)$ be a holomorphic function germ. We write $f(z) = \sum a_{\nu} z^\nu$, where $\nu = (\nu_1, \ldots, \nu_n)$ goes through multi-integers $\mathbb{N}^n$ and $z^\nu = z_1^{\nu_1} z_2^{\nu_2} \cdots z_n^{\nu_n}$ as usual. Let $\Gamma(f)$ be the convex hull of $\cup_{\nu} (\nu + (\mathbb{R}_+)^n)$, where the union is taken for all $\nu$ such that $a_{\nu} \neq 0$. Let $\Gamma(f)$ be the union of compact boundaries of $\Gamma(f)$. We say $f$ has a non-degenerate Newton principal part if $f_{\Delta}(z) = \sum_{\nu \in \Delta} a_{\nu} z^\nu$ is non-singular on $(C^*)^n = (C - \{0\})^n$ for any $\Delta$ of $\Gamma(f)$. $f$ is said to be convenient if the intersection of $\Gamma(f)$ with each coordinate axis is non-empty. These definitions are due to A. G. Kouchnirenko ([Ko], see also [O]).

The problem 2 is affirmative for a holomorphic function germ which has a non-degenerate Newton principal part (proposition 2).
PROPOSITION 2. Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a holomorphic function germ with isolated singular point at the origin. Suppose \( f \) has a non-degenerate Newton principal part. Then there exists a germ of one parameter family \( F : (\mathbb{C}^n \times [0, 1], 0 \times [0, 1]) \to (\mathbb{C}, 0) \) such that the following 4 properties hold:

1. \( F \) depends on the parameter \( t \in [0, 1] \) continuously,
2. \( F(t) \) is holomorphic for any \( t \) of \([0,1]\),
3. \( F(0) = f \) and \( F(1) \) is of real coefficient,
4. there exists a germ of homeomorphism

\[
H : (\mathbb{C}^n \times [0, 1], 0 \times [0, 1]) \to (\mathbb{C}^n \times [0, 1], 0 \times [0, 1])
\]

of the form \( H(z, t) = (H_1(z, t), t) \) such that \( F \circ H(z, t) = f(z) \).

PROOF OF PROPOSITION 2: By the geometric characterization of finite determinacy ([W]), we see

LEMA 1. Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a function germ with isolated singularities which has a non-degenerate Newton principal part. Then, there exists a biholomorphic map germ \( h : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0) \) such that the composition \( f \circ h \) is convenient and non-degenerate.

We write \( f \circ h = \sum b_\lambda z^\lambda \). Let \( V_{fh} \) be the set of coefficients of all polynomials having terms only on \( f(f \circ h) \). Namely,

\[
V_{fh} = \{ \sum c_\lambda z^\lambda \mid c_\lambda = 0 \text{ if and only if } b_\lambda = 0 \text{ or } \lambda \notin \Gamma(f \circ h) \}.
\]

We also set

\[
U_{fh} = \{ \sum c_\lambda z^\lambda \in V_{fh} \mid \text{it has a non-degenerate Newton principal part} \}.
\]

Then,

LEMA 2 ([O]). \( U_{fh} \) is a non-empty Zariski open subset of \( V_{fh} \).

Thus, we can choose a germ of one parameter family \( F : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) such that

1. \( F \) depends on the parameter \( t \in [0, 1] \) analytically,
2. \( F(t) \) is convenient and has a non-degenerate Newton principal part for any \( t \) of \([0,1]\),
3. \( F(0) = f \circ h \) and \( F(1) \) is of real coefficient.

This germ of one parameter family \( F \) is the desired one because

LEMA 3 (COMBINING [O] AND [K]). Let \( F : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a germ of one parameter family such that
(1) $F$ depends on the parameter $t \in [0, 1]$ analytically,
(2) $F(t, t)$ is convenient and has a non-degenerate Newton principal part for any $t$ of $[0, 1]$.

Then, there exists a germ of homeomorphism

$$H : (\mathbb{C}^n \times [0, 1], 0 \times [0, 1]) \rightarrow (\mathbb{C}^n \times [0, 1], 0 \times [0, 1])$$

of the form $H(z, t) = (H_1(z, t), t)$ such that $F \circ H(z, t) = f(z)$.

References


