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Basic problems on singularities of isotropic mappings

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This is a preliminary report about the singularity theory of isotropic mappings.

We collect some remarks and problems toward the local classification of singularities of generic isotropic mappings.

A C^∞ mapping $f : N \to (M, \omega)$ from a C^∞ manifold N of dimension n to a C^∞ symplectic manifold (M, ω) of dimension $2n$ is called isotropic if $f^*\omega = 0$. In other word, an isotropic mapping is a parametrization of (maximal) "integral variety" of the differential equation $\omega = 0$ on M. (For the general theory of symplectic manifolds, see [W], for instance.)

The natural equivalence relation for the classification of isotropic mappings is defined as follows: Two isotropic mappings f and $g : N' \to (M', \omega')$ are called equivalent if there exist a diffeomorphism $\sigma : N \to N'$ and a symplectic diffeomorphism $\tau : M \to M'$, $(\tau^*\omega' = \omega)$, such that $\tau \circ f = g \circ \sigma$.

Similarly we define the symplectic equivalence of isotropic map-germs or jets.

In this report, all manifolds and mappings are assumed of class C^∞.

Though we do not mention here, some differential analytical objects appear in the study of isotropic mappimgs or "singular Lagrange varieties", [Z], [M], [I1], [I2], [I3].

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Isotropic deformations and unfoldings

Set $z_i = \xi_i + \sqrt{-1}x_i, 1 \le i \le n$ and set $\omega = \sum_{i=1}^n d\xi_i \wedge dx_i = d(\sum_{i=1}^n \xi_i dx_i)$. Then $(\mathbf{C}^n, \omega)$ is a symplectic (homogeneous) manifold of dimension $2n$, which is the local model of the symplectic geometry: By Darboux's theorem [AGV], any isotropic map-germ $N^n, x \to M^{2n}$ is equivalent to an isotropic map-germ $\mathbf{R}^n, 0 \to \mathbf{C}^n, 0$.

Denote by $I(n)$ the set of isotropic map-germs $\mathbf{R}^n, 0 \to \mathbf{C}^n, 0$.

Let $f_0 \in I(n)$ and $f_\lambda, \lambda \in \mathbf{R}^\ell, 0$, be an isotropic deformation of f. By definition, $F = (f_\lambda, \lambda) : \mathbf{R}^n \times \mathbf{R}^\ell, 0 \to \mathbf{C}^n \times \mathbf{R}^\ell, 0$ is a C^∞ map-germ and $f_\lambda^* \omega = 0$ for all $\lambda \in \mathbf{R}^\ell, 0$.

Let u denote the coordinate of $\mathbf{R}^n$. Since $f_\lambda^* \omega = d_u f_\lambda^*(\sum_{i=1}^n \xi_i dx_i) = 0$, there exists a family of (generating) functions e_λ uniquely up to the addition of a function of λ with

$$d_u e_\lambda = f_\lambda^*(\sum_{i=1}^n \xi_i dx_i),$$

where d_u means the exterior derivative with respect to u. Then

$$de_\lambda = \sum_{i=1}^n \xi_i \circ f_\lambda d(x_i \circ f_\lambda) + \sum_{j=1}^\ell \mu_{j\lambda} d\lambda_j,$$

for some function-germs $\mu_{j\lambda}(u)$. Set

$$\tilde{F} = (f_\lambda; \mu_\lambda, \lambda) : \mathbf{R}^n \times \mathbf{R}^\ell, 0 \to \mathbf{C}^n \times \mathbf{C}^\ell.$$

Then $\tilde{F}$ is isotropic and it is a lift of F with respect to the projection $\pi : \mathbf{C}^n \times \mathbf{C}^\ell \to \mathbf{C}^n \times \mathbf{R}^\ell, \pi(\xi, x; \mu, \lambda) = (\xi, x, \lambda)$. As easily verified, isotropic lifts of F are equivalent to each other. We call $\tilde{F}$ an isotropic unfolding of f. Then we have the following fundamental fact:

PROPOSITION 1. *Let $f : N^n, x \to M^{2n}, f(x)$ be an isotropic map-germ with the kernal rank $krf (= dim\ KerT_x f) = k$. Then f is equivalent to an isotropic unfolding of a $f_0 \in I(k)$ with $krf_0 = k$.*

PROOF: There exists symplectic coordinate $(p_1, \ldots, p_n; q_1, \ldots, q_n)$ of $M, f(x)$ such that $(q_{k+1}, \ldots, q_n) \circ f$ is a submersion. Then it sufficies to set $f_0 = (p_1, \ldots, p_k; q_1, \ldots, q_k) \circ f$.

REMARK: To set up the general theory of isotropic unfoldings, it is better to regard $\mathbb{C}^n$ as $T^*\mathbb{R}^n$: Let B be a manifold of dimension n. Then there exists the unique one-form θ on the cotangent bundle T^*B, which is called the canonical one-form, such that, for any one-form α on B considered as a section $\alpha : B \to T^*B$ of the projection $\pi : T^*B \to B$, the induced one-form $\alpha^*\theta$ on B is equal to the one-form α. Set $\omega = d\theta$. Then (T^*B, ω) is a symplectic manifold of dimension $2n$.

Let N be n-manifold and $f : N \to T^*B$ an isotropic map-germ. We call $(F; i, j)$ an isotropic unfolding of f if $F : N' \to T^*B'$ is an isotropic map-germ such that $(\pi \circ F; i, j)$ is an unfolding of $\pi \circ f$ in the usual sense and $f = j^*(F \circ i)$ as one-form along $\pi \circ f$. Let $(\tilde{F}; \tilde{i}, \tilde{j}), \tilde{F} : \tilde{N} \to T^*\tilde{B}$ be another isotropic unfolding of f. Then $(\phi, \psi) : (\tilde{F}; \tilde{i}, \tilde{j}) \to (F; i, j)$ is called a morphism if $\phi : \tilde{N} \to N'$, $\psi : \tilde{B} \to B'$, (ϕ, ψ) is a morphism $(\pi \circ \tilde{F}; \tilde{i}, \tilde{j}) \to (\pi \circ F; i, j)$ in the usual sense, and $\tilde{F} = \psi^*(F \circ \phi)$ modulo closed one-form on $\tilde{B}$. Then the notion of versality of isotropic unfoldings is naturally defined. The characterization of versal isotropic unfoldings should be an important subject.

Isotropic map-germs of kernel rank one

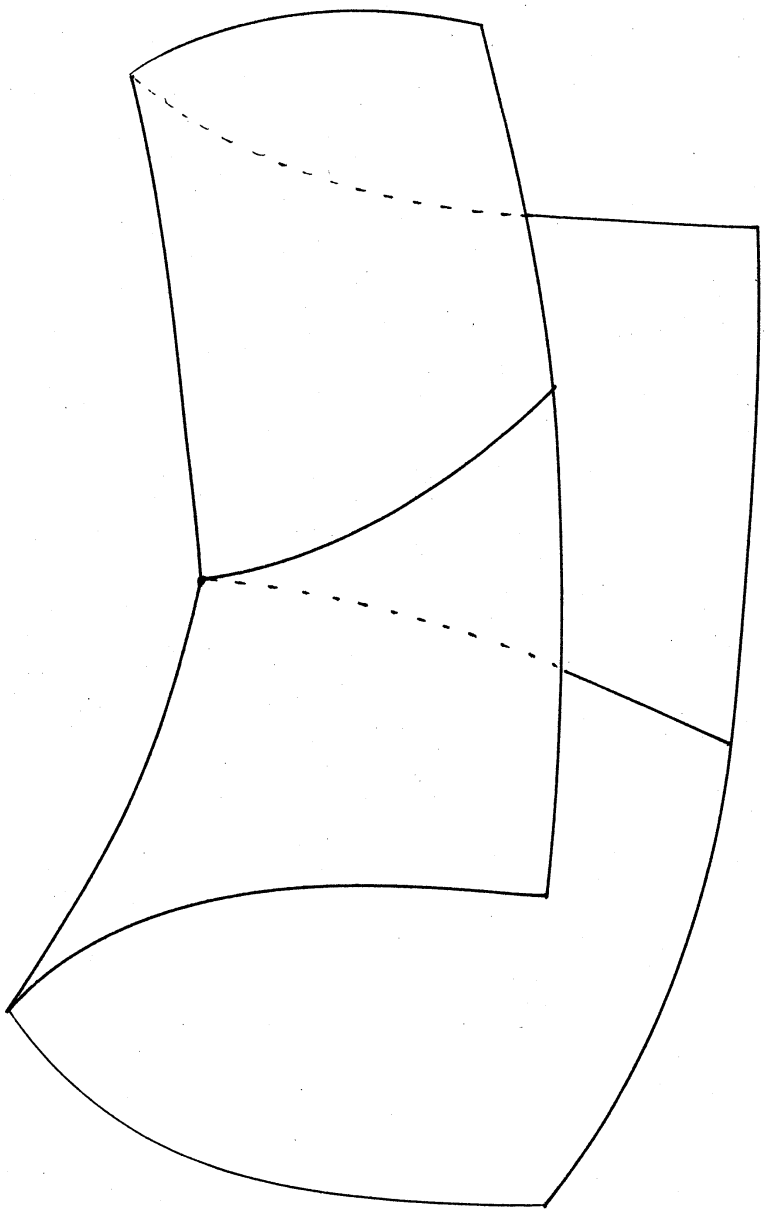
By Proposition 1, any isotropic map-germ of kernel rank one is equivalent to an isotropic unfolding of a map-germ $f : \mathbb{R}, 0 \to \mathbb{C}, 0$. Remark that f is automatically isotropic, and any deformation $(f_\lambda, \lambda) : \mathbb{R} \times \mathbb{R}^{n-1}, 0 \to \mathbb{C} \times \mathbb{R}^{n-1}, 0$ is also isotropic. Simply write $f_\lambda = (\xi, x)$. Then $\tilde{F} = (f_\lambda; \mu, \lambda)$, where

$$\begin{aligned} \mu_j &= \frac{\partial}{\partial \lambda_j}\left(\int_0^u \xi \frac{\partial x}{\partial u} du\right) - \xi \frac{\partial x}{\partial \lambda_j} \\ &= \int_0^u \left(\frac{\partial \xi}{\partial \lambda_j}\frac{\partial x}{\partial u} - \frac{\partial \xi}{\partial u}\frac{\partial x}{\partial \lambda_j}\right) du, \qquad 1 \le j \le n-1. \end{aligned}$$

In fact the local classification of generic isotropic mappings of kernel rank one is given in [I2], [Z]. (See also [G2]).

EXAMPLE: Let $f = (u^2, 0)$. Consider the one-parameter deformation $F = (f_\lambda, \lambda) = (u^2, u\lambda, \lambda)$ of f. Then $\tilde{F} = (u^2, u\lambda, -(2/3)u^3, \lambda) : \mathbb{R}^2, 0 \to \mathbb{C}^2, 0$, which is called the open Whitney umbrella [A1],[G1],[I2].

Figure: The open Whitney umbrella.



Isotropic map-germs of kernel rank two

Now we turn to the ploblem of the classification of isotropic map-germs of kernel rank 2. By Proposition 1, the first stage of attaking the problem is devided into the following two steps: Describe elements of $I(2)$ and then study the isotropic deformations of them.

Let $f : \mathbb{R}^2, 0 \to \mathbb{C}^2, 0$. Set $f = (\xi_1 \circ f, x_1 \circ f, \xi_2 \circ f, x_2 \circ f) = (P_1, Q_1, P_2, Q_2)$. Then

$$f^*\omega = (J(P_1, Q_1) + J(P_2, Q_2))du \wedge dv,$$

where (u, v) is the cordinate of $\mathbb{R}^2$, and $J(\, , \,)$ means the Jacobian. Therefore f is isotropic if and only if

$$J(P_1, Q_1) + J(P_2, Q_2) = 0.$$

This is a non-linear first order partial differential equation.

REMARK: An isotropic map-germ $f : \mathbb{R}^2, 0 \to \mathbb{C}^2$ is regarded as an infinitesimal Jacobian preserving deformation of $g = (Q_1, Q_2)$. Similarly an infinitesimal isotropic deformation of f is an isotropic map-germ $\xi : \mathbb{R}^2, 0 \to T^*\mathbb{C}^2 = \mathbb{H}^2$.

The strict motivation of the study is the following:

CONJECTURE 2. *(Givental' [G1]) Any isotropic mapping $f : N^2 \to M^4$ is approximated by an isotropic f' such that, for any $x \in N$, the germ f'_x is an immersion or equivalent to the open Whitney unbrella.*

Consider the more weak conjecture.

CONJECTURE 3. *Any isotropic mapping $f : N^2 \to M^4$ is approximated by an isotropic f' such that, for any $x \in N$, $krf'_x \leq 1$.*

Then we remark that Conjecture 3 implies Conjecture 2, by the result of [I2],[Z]. Furthermore, Zakalykin [Z] anounces that, if, at each point, f composed with a Lagrangian fibration is finite, then Conjecture 3 (therefore Conjecture 2) is true. But I think further study on isotropic map-germs of kernel rank two is needed to solve Conjecture 2 completely.

Isotropic jets

The notion of jet is essential for the usual singularity theory. Here we give some foundation for the counterpart of the singularity theory of isotropic mappings.

Set $J^r = \{j^r f(0) \mid f : \mathbb{R}^2, 0 \to \mathbb{C}^2, 0\}$, and $J^r_I = \{z \in J^r \mid z = j^r f(0)$, for some $f \in I(2)\}$, $r = 1, 2, \ldots, \infty$. The first fundamental problem of the study is the following:

PROBLEM 4. *Describe the set* J^r_I.

Now we introduce an auxiliary notion:

DEFINITION 5: A map-germ $f : \mathbb{R}^2, 0 \to \mathbb{C}^2, 0$ is called ℓ-isotropic, ($\ell = 1, 2, \ldots, \infty$), if $f^*\omega \in m_2^\ell \Omega$, that is, $j^{\ell-1}(f^*\omega)(0) = 0$, where Ω denotes E_2-module of 2-form germs on $\mathbb{R}^2, 0$. A jet $z \in J^r$ is called ℓ-isotropic if $z = j^r f(0)$ for some ℓ-isotropic f.

Now set $J^r_{\ell-I} = \{z \in J^r \mid z$ is ℓ-isotropic$\}$. Then we have a sequence of sets:

$$J^r \supset J^r_{1-I} \supset J^r_{2-I} \supset \cdots \supset J^r_{r-I} \supset J^r_{r+1-I} \supset \cdots \supset J^r_{\infty-I} \supset J^r_I.$$

Set $f = f_1 + f_2 + \cdots$, formally, where $f_i = (P_{1i}, Q_{1i}, P_{2i}, Q_{2i})$ is homogeneous of degree i, $i = 1, 2, \ldots$. Then $J(P_1, Q_1) + J(P_2, Q_2) = h_0 + h_2 + \cdots$, with

$$h_k = \sum_{i+j=k+2} J(P_{1i}, Q_{1j}) + J(P_{2i}, Q_{2j}),$$

$k = 0, 1, 2, \ldots$. Hence we have

LEMMA 6. *f is ℓ-isotropic if and only if $h_k = 0$ for $k \leq \ell$.*

Then it is easy to see the following lemmas:

LEMMA 7. *$J^r_{\ell-I}$ is algebraic (resp. semi-algebraic) if $\ell \leq r$ (resp. $r < \ell < \infty$).*

LEMMA 8. *$J^1_{1-I} = J^1_I$, which is identified with the set of linear isotropic mappings $\mathbb{R}^2 \to \mathbb{C}^2$. Moreover $J^1_I \subset Hom_{\mathbb{R}}(\mathbb{R}^2, \mathbb{C}^2) \cong \mathbb{R}^8$ is a quadratic hypersurface with $Sing J^1_I = \{0\}$.*

For $s \leq r$, we denote by $\pi^r_s : J^r \to J^s$ the canonical projection. Then we have

PROPOSITION 9. $J^r_{r-I} - (\pi^r_1)^{-1}(0) \subset J^r_I$.

PROOF: Consider the natural action of Diff($\mathbb{R}^2,0$)$\times$Symp($\mathbb{C}^2,0$) on J^r_I. Let $\pi^r_1(z) \in \Sigma^0$. Then the jet z is equivalent to $j^r(P_1,u,P_2,v)(0)$ for polynomials P_1, P_2 of degree $\leq r$, such that (P_1,u,P_2,v) is r-isotropic. Then the polynomial form of degree $\leq r-1$, $dP_1du + dP_2dv \in m^r\Omega$. Therefore $dP_1du + dP_2dv = 0$ as form. Thus $z \in J^r_I$.

Let $\pi^r_1(z) \in \Sigma^1$. Then z is equivalent to $j^r(P_1,u,P_2,Q_2)(0)$ for polynomials P_1,P_2,Q_2 of degree $\leq r$ such that (P_1,u,P_2,Q_2) is r-isotropic. Then $dP_1du + dP_2dQ_2 \in m^r\Omega$. Therefore

$$\frac{\partial P_1}{\partial v} = \frac{\partial P_2}{\partial u}\frac{\partial Q_2}{\partial v} - \frac{\partial P_2}{\partial v}\frac{\partial Q_2}{\partial u} + \rho, \qquad \rho \in m^r.$$

Set $\tilde{P}_1 = P_1 - \int_0^v \rho dv$ and $f' = (\tilde{P}_1, u, P_2, Q_2)$. Then $j^r f'(0) = z$ and f' is isotropic. Hence z is isotropic; $z \in J^r_I$.

Q.E.D.

Set $J^r_{\ell-I,0} = J^r_{\ell-I} \cap (\pi^r_1)^{-1}(0)$. Then we have

LEMMA 10. *$J^r_{r+1-I,0}$ is an algebraic set in J^r.*

LEMMA 11. $J^2_{3-I,0} = J^2_{I,0} (= J^2_I \cap (\pi^r_1)^{-1}(0))$.

REMARK: $J^2_{I,0}$ is identified with the set of homogeneous isotropic polynomial mappings $\mathbb{R}^2 \to \mathbb{C}^2$.

COROLLARY 12. $J^2_{3-I} = J^2_I$.

For the classification of isotropic 2-jets, we see

PROPOSITION 13. *Any jet $z \in J^2_{I,0}$ is equivalent to the jet $J^2(P_1,Q_1,P_2,Q_2)(0)$ of one of followings:*

$$(0, uv, 0, (1/2)(u^2+v^2)), \quad (0, uv, 0, (1/2)(u^2-v^2)), \quad (0, (1/2)(u^2+v^2), 0, 0),$$

$$(0, uv, 0, 0), \qquad (0, (1/2)u^2, 0, 0), \qquad (0,0,0,0).$$

PROOF: P_1, Q_1, P_2, Q_2 are necesarily linearly dependent over $\mathbb{R}$. Therefore the image is contained in a Lagrange plane. Then, by the classification of quadratic mappings $\mathbb{R}^2, 0 \to \mathbb{R}^2$, we have Proposition 13. (See [Gi]).

REMARK: For the isotropic 3-jets, the classification needs more initimate study. In fact, for instance, the image of the isotropic map-germ $f = (u^3, v^3, -3u^2v, uv^2)$ is not contained in any proper submanifold.

Here we refer the following fact, which is easy to see:

PROPOSITION 14. *Let $f \in I(2)$. If the image of f is contained in a proper submanifold, then it is contained in a Lagrange submanifold.*

Now in general it seems natural to expect

OPTIMISTIC CONJECTURE 15. *For any $r < \infty$, there exists $\ell = \ell(r) < \infty$, such that $J^r_{\ell - I} = J^r_I$. Further, $J^\infty_{\infty - I} = J^\infty_I$.*

Based on the arguments of Zakalykin [Z], we have

THEOREM 16. *Let $z = j^r f(0) \in J^r_{r+k-1-I}$ with $r \geq k$. Assume $f_1 = \cdots f_{k-1} = 0$, and $f_k : \mathbb{R}^2, 0 \to \mathbb{C}^2, 0$ is finite as map-germ. Then $z \in J^r_I$.*

We denote by H^j the set of homogeneous polynomials of u, v of degree j. Associated to the initial part f_k, we define $\Phi_{kj} : (H^j)^4 \to H^{k+j-2}$ by

$$\Phi_{kj} = J(P_{1k}, \ \ ) + J(Q_{1k}, \ \ ) + J(P_{2k}, \ \ ) + J(Q_{2k}, \ \ ).$$

Then the key to prove Theorem 16 is the following fact proved by Zakalykin [ ]:

LEMMA 17. *If f_k is finite, then Φ_{kj} is surjective for $j \geqq k$.*

Furthermore we need

LEMMA 18. *If f_k is finite, then similarly defined $\Phi_{k\infty} : (m^\infty)^4 \to m^\infty$ is surjective.*

PROOF OF THEOREM 16: For the given leading terms $f_k, \ldots, f_r$, we determine f_{r+1} by the condition $h_{r+k-1} = 0$, using Lemma 17, $j = r + 1$. Determine f_{r+2} by $h_{r+k} = 0$, and so on. Then we have $z = j^r f'(0)$ with ∞-isotropic f'. By Lemma 18, $z = j^r f''(0)$ with isotropic f''.

Q.E.D.

EXAMPLE: Let $f_2 = (0, uv, 0, (1/2)(u^2 + v^2))$, (cf. Proposition 13).

Define $\Phi : E \times E \to E$ by $\Phi(A, B) = J(uv, A) + J((1/2)(u^2 + v^2), B)$. Then $\Phi : m^j \times m^j \to m^j, j = 1, 2, \ldots$, and $\Phi : m^\infty \times m^\infty \to m^\infty$ are all surjective.

In fact, to solve $\Phi(A, B) = u(B_v - A_u) + v(A_v - B_u) = C$, set $C = -uD - vK$; if $C \in m^j$ then $D, K \in m^{j-1}$. Then it sufficies to solve

$$A_u - B_v = D, \qquad -A_v + B_u = E.$$

Fixing $A = \int_0^u (D + B_v) du$, we need to solve the wave equation

$$\frac{\partial^2 B}{\partial u^2} - \frac{\partial^2 B}{\partial v^2} = vD + uE.$$

Since $(\partial^2/\partial u^2) - (\partial^2/\partial v^2) : m^{j+2} \to m^j$ is surjective, $j = 0, 1, \ldots, \infty$, we have the result.

REMARK: Morimoto and Homma informed to me that the above situation is closely related to the notions of prolongation, involutivity and Spencer cohomology, [Go]. I am very interested in this aspect, and I think further intimate investigations are needed to progress this point of view and to construct new general theory.

Analytic approximation, stability and determinacy

Let denote by $C_I^\infty(N,M)$ the space of proper isotropic mappings $N \to M$ endowed with Whitney C^∞ topology, and by $C_I^{an}(N,M)$ the subspace of isotropic mappings f such that, for any $x \in N$, f_x is equivalent to an analytic isotropic map-germ.

CONJECTURE 19. *$C_I^{an}(N,M) \subset C_I^\infty(N,M)$ is dense.*

If Conjecture 19 is affermative, then the stability of isotropic mappings becomes rather easy to characterize: An isotropic map-germ $f : N,x \to M$ is symplectically stable if and only if f is infinitesimally symplectically stable and f is equivalent to an analytic map-germ: f is called infinitesimally symplectically stable if

$$VI(f) = tf(V_N) + wf(VH_M),$$

where $VI(f)$ (resp. V_M, VH_M) is the set of isotropic one-forms $N,x \to T^*M$ along f (resp. the set of vector fields over N,x, the set of Hamiltonian vector fields over $M, f(x)$; a Hamiltonian vector field is naturally considered as one-form on T^*M.

A natural candidate for the characterization of finite determinacy of isotropic map-germ is the condition that, for some $k < \infty$,

$$m_N^k V(f) \cap VI(f) \subset tf(m_N V_N) + wf(m_M VH_M).$$

In any case, the fundamental question would be the following:

QUESTION 20. *For any $\xi \in VI(f)$, is there an isotropic deformation f_λ such that $\xi = (\partial f_\lambda / \partial t)|_{t=0}$?*

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