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On the Wronskian of the hypergeometric functions of type (n+1,m+1)

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Introduction

Many specialists believe that the rank of the hypergeometric system $E(n+1,m+1)$ is equal to $\binom{m-1}{n}$. Although this fact is fundamental in studing the hypergeometric functions, there is no rigourous proof of the fact in general case: $\lambda_j \notin \mathbf{Z}$ $(1\le j\le m)$, $\sum \lambda_j \notin \mathbf{Z}$. This is the brief resume of [K] in which we gave a proof of the fact based on the perfect pairing of certain twisted rational de Rham cohomology and twisted homology associated with the hypergeometric integral of type $(n+1,m+1)$.

§1. The hypergeometric function (HGF) of type (n+1,m+1) $(n<m)$

Let

$$W_{n+1,m+1} := \left\{ w = \begin{pmatrix} w_{00} & w_{01} & \cdots & w_{0m} \\ \vdots & & & \\ w_{n0} & w_{n1} & \cdots & w_{nm} \end{pmatrix} \in M(n+1,m+1,\mathbf{C}) \;\middle|\; \begin{array}{l} \text{rank } w=n+1 \\ \text{each} \\ \text{column}\neq 0 \end{array} \right\},$$

$$[t_0:t_1:\cdots:t_n] \in \mathbf{CP}^n,$$

$$\tau := \sum_{i=0}^{n} (-1)^i t_i dt_0 \wedge\cdots\wedge \widehat{dt_i} \wedge\cdots\wedge dt_n \;:\; \text{an } n\text{-form on } \mathbf{C}^{n+1}.$$

Let $\tilde{\lambda}=(\lambda_0,\lambda_1,\cdots,\lambda_m) \in (\mathbf{C}\setminus\mathbf{Z})^{m+1}$ be exponents with the condition

$$(1) \qquad \sum_{j=0}^{m} \lambda_j + n+1 = 0.$$

Then, for each $w \in W_{n+1,m+1}$, the n-form

$$\prod_{j=0}^{m}(w_{0j}t_0 + \cdots + w_{nj}t_n)^{\lambda_j}\cdot\tau$$

can be seen as an n-form on $\mathbf{C}P^n$ because of (1). Taking a suitable twisted cycle σ as a domain of integration, we define a function by the integral

$$\Phi(\tilde{\lambda},w) = \int_\sigma \prod_{j=0}^{m}\Big(\sum_{i=0}^{n} w_{ij}t_i\Big)^{\lambda_j}\cdot\tau,$$

which will be called the *HG integral* (or function) of type (n+1,m+1)

The groups $GL(n+1,\mathbf{C})$ and $H_{m+1} := \left\{\begin{pmatrix} h_0 & & & 0 \\ & h_1 & & \\ & & \ddots & \\ 0 & & & h_m\end{pmatrix}\right\}$ act on $W_{n+1,m+1}$ from left and right, respectively as $w \longrightarrow gwh$. $\Phi(\tilde{\lambda},w)$ is homogeneous under the above two kind of group actions:

$$(2)\quad \begin{cases} \Phi(\tilde{\lambda},gw) = \dfrac{1}{\det g}\,\Phi(\tilde{\lambda},w) & \text{for } g \in GL(n+1) \\ \Phi(\tilde{\lambda},wh) = \prod_{j=0}^{m} h_j^{\lambda_j}\cdot\Phi(\tilde{\lambda},w) & \text{for } h \in H_{m+1}. \end{cases}$$

Since

$$\frac{\partial^2\Phi}{\partial w_{ip}\partial w_{jq}} = \lambda_p\lambda_q\int_\sigma \frac{t_pt_q}{(\sum_i w_{ip}t_i)(\sum_j w_{jq}t_j)}\prod_{j=0}^{m}(\sum w_{ij}t_i)^{\lambda_i}\tau,$$

we have

$$\frac{\partial^2\Phi}{\partial w_{ip}\partial w_{jq}} = \frac{\partial^2\Phi}{\partial w_{iq}\partial w_{jp}} \qquad (0\le i,j\le n,\ 0\le p,q\le m).$$

Hence, passing to the Lie algebra version of (2), we have that

$\Phi(\tilde{\lambda},w)$, viewed as a function on $W_{n+1,m+1}$, satisfies the following system $E(n+1,m+1;\tilde{\lambda})$ of differential operators:

$$(*)\quad \begin{cases} \sum_{i=0}^{n} w_{ip}\dfrac{\partial\Phi}{\partial w_{ip}} = \lambda_p\Phi & (0\le p\le m) \quad (\ H\ \ \text{- homogeneity}\ ) \\ \sum_{p=0}^{m} w_{ip}\dfrac{\partial\Phi}{\partial w_{jp}} = -\delta_{ij}\Phi & (1\le i,j\le n) \quad (\ GL(n+1)\text{-homogeneity}\ ) \\ \begin{vmatrix} \partial_{ip} & \partial_{iq} \\ \partial_{jp} & \partial_{jq} \end{vmatrix}\Phi = 0 & (0\le i,j\le n,\ 0\le p,q\le m) \end{cases}$$

We set

$$W'_{n+1,m+1} = \left\{ w\in W_{n+1,m+1} \;\middle|\; \begin{array}{l} \text{those matrices of which every minor of} \\ \text{order } n+1 \text{ in } w \text{ is non-zero} \end{array} \right\}$$

which is an open dense subset of $W_{n+1,m+1}$. For each $w\in W'_{n+1,m+1}$ there exist $g\in GL(n+1)$ and $h\in H_{m+1}$ such that

$$g\cdot w\cdot h = \begin{pmatrix} 1 & & & 1 & \cdots & 1 & 1 \\ & 1 & & z_{1,n+1} & & z_{1,m-1} & -1 \\ & & \ddots & \vdots & & \vdots & \vdots \\ & & 1 & z_{n,n+1} & \cdots & z_{n,,m-1} & -1 \end{pmatrix}.$$

Using $u_i=t_i/t_0$ $(1\le i\le n)$ the non-homogeneous coordinates in $\mathbf{CP}^n$, we set

$$\begin{cases} f_i(u) = u_i & (1\le i\le n) \\ f_j(u) = 1 + z_{1j}u_1 + \cdots + z_{nj}u_n & (n+1\le j\le m-1) \\ f_m(u) = 1 - \sum_{i=1}^{n} u_i . \end{cases}$$

From now on we keep this notations. Then the integral $\Phi(\tilde{\lambda},w)$ takes the following form

$$\bar{\Psi}(\lambda,z) = \int_{\sigma} \prod_{j=0}^{m} f_j(u)^{\lambda_j} \cdot du_1 \wedge \cdots \wedge du_n$$

where, by our assumptions, $\lambda=(\lambda_1,\cdots,\lambda_m)$ satisfies the conditions

$$\begin{cases} \lambda_j \notin \mathbf{Z} & (1\le j\le m) \\ \sum_{j=1}^{m} \lambda_j \notin \mathbf{Z}. \end{cases}$$

If we take the integration over the twisted cycle $\Delta^n(\omega)$ associated with the n-simplex

$$\Delta^n = \{ u\in R^n \mid 0\le u_i \quad (1\le i\le n),\ \sum u_i \le 1 \}$$

$$\text{where } \omega = \mathrm{dlog}\{ \prod_{i=1}^{n} u_i^{\lambda_i} \ (1 - \sum u_i)^{\lambda_m} \},$$

then we obtain the following power series expansion of $\bar{\Psi}(\lambda,z)$:

$$\bar{\Psi}(\lambda,z) = \mathrm{const} \sum_{\nu} \frac{\prod_{i=1}^{n} (\alpha_i;\ \sum_j \nu_{ij}) \prod_j (\beta_j;\ \sum_i \nu_{ij})}{(\gamma;\ \sum_{i,j} \nu_{ij})\nu!} z^{\nu}$$

where

$$\nu = \begin{pmatrix} \nu_{1,n+1} & \cdots\cdots & \nu_{1,m-1} \\ \vdots & & \\ \nu_{n,n+1} & \cdots\cdots & \nu_{n,m-1} \end{pmatrix} \in M(n,m-n-1;\mathbf{Z}_{\ge 0})$$

and

$$\alpha_i = \lambda_i + 1, \quad \beta_j = -\lambda_j \quad \& \quad \gamma = -\sum_{i=1}^{n} \lambda_i - \lambda_m - n.$$

§2. Twisted de Rham theory of HG integrals

Since, in case of $z_{ij} \in R$, twisted cycles become *visible*, we assume, from now on, that

(H.2) $z_{ij} \in R$ for $1 \le i \le n$, $n+1 \le j \le m-1$.

Set

$$H_j = \{u \in C^n \mid f_j(u) = 0\}, \quad X = C^n \setminus \bigcup_{j=1}^{m} H_j, \quad (H_j)_R = H_j \cap R$$

and suppose that

(H.3) m hyperplanes and the hyperplane at infinity are in general position in CP^n.

For simplicity of writing we set

$$U(u) = \prod_{j=1}^{m} f_j(u)^{\lambda_j},$$

$$\varphi\langle J\rangle = \varphi\langle j_1, \cdots, j_p\rangle := \frac{df_{j_1}}{f_{j_1}} \wedge \cdots \wedge \frac{df_{j_p}}{f_{j_p}}.$$

We know that

$$\#\{\text{bounded components of } R^n \setminus \bigcup_j H_{jR}\} = \binom{m-1}{n}.$$

Taking a bounded chamber Δ with the standerd orientation of R^n together with a branch of $U(u)$, the following integral

$$\int_\Delta U(u)\varphi\langle j_1, \cdots, j_n\rangle = \int_\Delta \prod f_j(u)^{\lambda_j} \cdot \frac{df_{j_1}}{f_{j_1}} \wedge \cdots \wedge \frac{df_{j_n}}{f_{j_n}}$$

is called a *hypergeometric integral* where $\mathrm{Re}\, \lambda_j > 0$. Classical examples suggest that $\binom{m-1}{n}$ hypergeometric integrals $\int_\Delta U(u)\varphi\langle 1, \cdots, n\rangle$ taken over $\binom{m-1}{n}$ bounded chambers may be all linearly independent solutions of the HG system $E(n+1,m+1;\lambda_0,\lambda_1-1,\cdots,\lambda_n-1,\lambda_n+1,\cdots,\lambda_m)$. To see this, it is necessary to study the twisted de Rham theory associated with the many-valued function $U(u)$. For $\varphi\in\Gamma(X,\varepsilon_X^p)$, we have $d(U\varphi)= U(d\varphi+\omega_\wedge\varphi)$ on the universal covering manifold $\tilde{X}$ where we set

$$\omega = \frac{dU}{U} = \sum_{j=1}^{m} \lambda_j \frac{df_j}{f_j}.$$

Hence instead of considering the exterior differentiation d on $\tilde{X}$, we are led to study the covariant differentiation

$$\nabla_\omega\varphi = d\varphi+\omega_\wedge\varphi.$$

Let $\mathcal{S}_\omega=\{h\in O_X \mid \nabla_\omega h=0\}$ which is a complex local system of rank one. It is known the following results:

1. The comparison theorem.

$$H^p(X,\mathcal{S}_\omega) \simeq H^p(X,\nabla_\omega) := \frac{\left\{\begin{array}{c}\nabla_\omega\text{-closed rational p-forms}\\ \text{which are holo. on X}\end{array}\right\}}{\nabla_\omega\left\{\begin{array}{c}\text{rational(p-1)-forms}\\ \text{which are holo. on X}\end{array}\right\}}$$

2. $$H^p(X,\nabla_\omega)=0 \qquad \text{for} \quad p\neq n,$$

$$H^n(X,\nabla_\omega) \simeq \frac{\{\{\varphi<j_1,\cdots,j_n>\mid 1\le j_1<\cdots<j_n\le m\}\}}{\omega_\wedge\{\{\varphi<j_1,\cdots,j_{n-1}>\mid 1\le j_1<\cdots<j_{n-1}\le m\}\}},$$

$$\simeq \{\{\varphi<j_1,\cdots,j_n>\mid 1\le j_1<\cdots<j_n\le m-1\}\}$$

and

$$\dim H^n(X,\nabla_\omega) = \binom{m-1}{n}.$$

3. Let $\mathcal{S}_\omega^\vee$ be the dual complex local system to $\mathcal{S}_\omega$; then

$$H_p(X,\mathcal{S}_\omega^\vee) = 0 \qquad \text{for} \quad p\neq n,$$

$$H_n(X,\mathcal{S}^\vee_\omega) = \sum \mathbf{C}\cdot\Delta_\nu(\omega) \quad \text{(direct sum)}$$

where summation runs over the twisted cycles associated with $\binom{m-1}{n}$ bounded chambers.

4. A perfect pairing.

$$H_n(X,\mathcal{S}^\vee_\omega) \times H^n(X,\mathcal{S}_\omega) \xrightarrow{\text{a perfect pairig}} \mathbf{C}$$

$$\|$$

$$\sum_\nu \mathbf{C}\cdot\Delta_\nu(\omega) \times \{\{\ \varphi\langle j_1,\cdots,j_n\rangle \mid 1\le j_1< \cdots <j_n\le m-1\ \}\} \longrightarrow \mathbf{C}$$

$$(\ \sigma,\quad \varphi\ ) \longrightarrow \int_\sigma U\cdot\varphi$$

That the above pairing is perfect is equivalent to

$$\det\Big(\int_{\Delta_\nu(\omega)} U\cdot\varphi\langle J\rangle \Big) \neq 0.$$

§3. The Wronskian of the HG function of type (n+1,m+1).

3.1. To show the linear independence of the HG integrals

$$\int_{\Delta_\nu(\omega)} U\cdot\varphi\langle 1,\cdots,n\rangle \qquad \Big(\ 1 \le \nu \le \binom{m-1}{n}\ \Big),$$

we must make a proper choice of partial derivatives of the integrals and prove that the Wronskian is not zero. To give a better understanding of the paper, we begin by illustrating our idea by some important examples.

Example 1. $E(2,3+\ell)$ Appell's F_1 ($\ell=2$) Lauricella's F_D ($\ell\ge 3$)

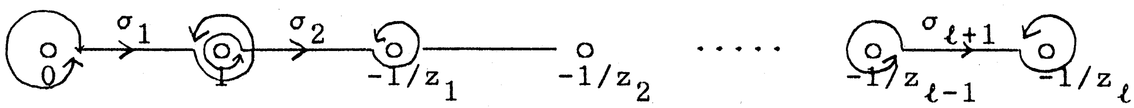
$$w = \begin{pmatrix} 1 & 0 & 1 & \cdots\cdots & 1 & 1 \\ 0 & 1 & z_1 & \cdots\cdots & z_\ell & -1 \end{pmatrix} \in M(2,3+\ell;R)$$

$$z=(z_1,\cdots,z_\ell)$$

Set

$$f_1=u, \quad f_2=1+z_1u, \quad \cdots, \quad f_{\ell+1}=1+z_\ell u, \quad f_{\ell+2}=1-u$$

$$U = \prod_{j=1}^{\ell+2} f_j^{\lambda_j}, \qquad \omega = d\log U, \qquad \varphi\langle i\rangle = df_i/f_i$$



$$\sigma_1, \cdots, \sigma_{\ell+1} \ : \text{ a basis of } H_1(X,\mathcal{S}_\omega^\vee)$$

Let $\sigma \in H_1(X,\mathcal{S}_\omega^\vee)$ and let

$$F(z) = \int_\sigma U\cdot\varphi\langle 1\rangle$$

$$= \int_\sigma u^{\lambda_1}(1+z_1u)^{\lambda_2} \cdots (1+z_\ell u)^{\lambda_{\ell+1}}(1-u)^{\lambda_{\ell+2}} \frac{du}{u}$$

be a HG integral; then we have

$$\frac{\partial F}{\partial z_1} = \lambda_2 \int_\sigma U\cdot\frac{du}{1+z_1u} = \frac{\lambda_2}{\lambda_1}\int_\sigma U\cdot\varphi\langle 2\rangle,$$

$$\frac{\partial F}{\partial z_2} = \frac{\lambda_3}{\lambda_2}\int_\sigma U\cdot\varphi\langle 3\rangle, \quad \cdots\cdots, \quad \frac{\partial F}{\partial z_\ell} = \frac{\lambda_{\ell+1}}{\lambda_\ell}\int_\sigma U\cdot\varphi\langle \ell+1\rangle.$$

Let

$$W = \begin{vmatrix} \int_{\sigma_1} U\cdot\varphi\langle 1\rangle & \cdots\cdots & \int_{\sigma_{\ell+1}} U\cdot\varphi\langle 1\rangle \\ \frac{\partial}{\partial z_1}\int_{\sigma_1} U\cdot\varphi\langle 1\rangle & \cdots\cdots & \frac{\partial}{\partial z_1}\int_{\sigma_{\ell+1}} U\cdot\varphi\langle 1\rangle \\ \vdots & & \vdots \end{vmatrix}$$

$$\left| \begin{matrix} \frac{\partial}{\partial z_\ell} \int_{\sigma_1} U\cdot\varphi\langle 1\rangle & \cdots\cdots & \frac{\partial}{\partial z_\ell} \int_{\sigma_{\ell+1}} U\cdot\varphi\langle 1\rangle \end{matrix} \right|$$

be the Wronskian of $\ell+1$ HG integrals $\int_\sigma U\cdot\varphi\langle 1\rangle$; then we have

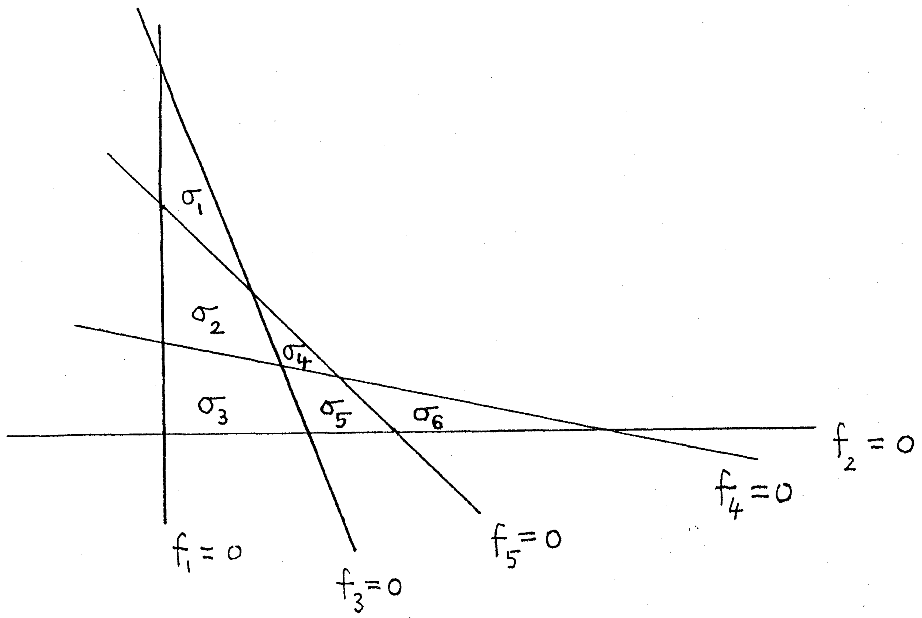
$$W = \frac{\lambda_2 \cdots \lambda_{\ell+1}}{z_1 \cdots z_\ell} \det\left(\int_{\sigma_\nu} U\cdot\varphi\langle j\rangle \right) \neq 0$$

because of the perfect pairing.

Example 2. $E(3,6)$.

$$w = \begin{pmatrix} 1 & & & 1 & 1 & 1 \\ & 1 & & z_{11} & z_{12} & -1 \\ & & 1 & z_{21} & z_{22} & -1 \end{pmatrix} \in M(3,6;\mathbf{R}),$$

$$z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}.$$



Set

$f_1=u_1$, $f_2=u_2$, $f_3=1+z_{11}u_1+z_{21}u_2$, $f_4=1+z_{12}+z_{22}u_2$,

$f_5=1-u_1-u_2$,

$$U = \prod_{j=1}^{5} f_j^{\lambda_j}, \qquad \omega = d\log U$$

$$\varphi\langle ij\rangle = \frac{df_i}{f_i} \wedge \frac{df_j}{f_j}.$$

Let $\sigma \in H_2(X, \mathcal{S}_\omega^\vee)$ and let

$$F(z) = \int_\sigma U\cdot\varphi\langle 12\rangle$$

$$= \int_\sigma u_1^{\lambda_1} u_2^{\lambda_2} (1+z_{11}u_1+z_{21}u_2)^{\lambda_3} (1+z_{12}u_1+z_{22}u_2)^{\lambda_4} \cdot$$

$$\cdot (1-u_1-u_2)^{\lambda_5} \frac{du_1}{u_1} \wedge \frac{du_2}{u_2}$$

be a HG integral. We have

$$\frac{\partial F}{\partial z_{11}} = \int_\sigma \lambda_3 u_1 \frac{U}{1+z_{11}u_1+z_{21}u_2} \frac{du_1}{u_1} \wedge \frac{du_2}{u_2}$$

$$= \frac{\lambda_3}{z_{11}} \int_\sigma U\cdot \frac{d(1+z_{11}u_1+z_{21}u_2)}{1+z_{11}u_1+z_{21}u_2} \wedge \frac{du_2}{u_2} \quad = -\frac{\lambda_3}{z_{11}} \int_\sigma U\cdot\varphi\langle 23\rangle,$$

$$\frac{\partial F}{\partial z_{12}} = -\frac{\lambda_4}{z_{12}} \int_\sigma U\cdot\varphi\langle 24\rangle, \qquad \frac{\partial F}{\partial z_{21}} = \frac{\lambda_3}{z_{21}} \int_\sigma U\cdot\varphi\langle 13\rangle,$$

$$\frac{\partial F}{\partial z_{22}} = \frac{\lambda_4}{z_{22}} \int_\sigma U\cdot\varphi\langle 14\rangle, \qquad \frac{\partial^2 F}{\partial z_{11}\partial z_{22}} = \frac{\lambda_3\lambda_4}{\det z} \int_\sigma U\cdot\varphi\langle 34\rangle.$$

Let

$$W = \begin{vmatrix} \int_{\sigma_1} U\cdot\varphi\langle 12\rangle & \cdots\cdots & \int_{\sigma_6} U\cdot\varphi\langle 12\rangle \\ \frac{\partial}{\partial z_{11}} \int_{\sigma_1} U\cdot\varphi\langle 12\rangle & \cdots\cdots & \frac{\partial}{\partial z_{11}} \int_{\sigma_6} U\cdot\varphi\langle 12\rangle \end{vmatrix}$$

$$\begin{vmatrix} \vdots & & \vdots \\ \frac{\partial}{\partial z_{22}} \int_{\sigma_1} U\cdot\varphi\langle 12\rangle & \cdots\cdots & \frac{\partial}{\partial z_{22}} \int_{\sigma_6} U\cdot\varphi\langle 12\rangle \\ \frac{\partial^2}{\partial z_{11}\partial z_{22}} \int_{\sigma_1} U\cdot\varphi\langle 12\rangle & \cdots\cdots & \frac{\partial^2}{\partial z_{11}\partial z_{22}} \int_{\sigma_6} U\cdot\varphi\langle 12\rangle \end{vmatrix}$$

be the Wronskian of 6 HG integrals $\int_{\sigma_\nu} U\cdot\varphi\langle 12\rangle$; then we have

$$W = \left(-\frac{\lambda_3}{z_{11}}\right)\left(-\frac{\lambda_4}{z_{12}}\right)\left(\frac{\lambda_3}{z_{21}}\right)\left(\frac{\lambda_4}{z_{22}}\right)\left(\frac{\lambda_3\lambda_4}{\det z}\right) \times \det\left(\int_{\sigma_\nu} U\cdot\varphi\langle ij\rangle\right).$$

On the other hand, $\{\ \varphi\langle ij\rangle \mid 1\le i<j\le 4\ \}$ forms a basis of $H^2(X,\nabla_\omega)$ and hence the perfect pairing yields

$$W = \frac{(\lambda_3\lambda_4)^3}{\prod_{i,j=1}^{2} z_{ij}\cdot\det z} \det\left(\int_{\sigma_\nu} U\cdot\varphi\langle ij\rangle\right) \neq 0.$$

3.2. *General case.*

$$w = \begin{pmatrix} 1 & & & 1 & \cdots\cdots & 1 & 1 \\ & 1 & & z_{1,n+1} & \cdots\cdots & z_{1,m-1} & -1 \\ & & \ddots & \vdots & & \vdots & \vdots \\ & & 1 & z_{n,n+1} & \cdots\cdots & z_{n,m-1} & -1 \end{pmatrix} \in M(n+1,m+1;R).$$

We set

$$z = \begin{pmatrix} z_{1,n+1} & \cdots\cdots & z_{1,m-1} \\ \vdots & & \vdots \\ z_{n,n+1} & \cdots\cdots & z_{n,m-1} \end{pmatrix} \in M(n,m-n-1;R)$$

& for simplicity,

$$z\begin{pmatrix} i_1 i_2 & \cdots & i_p \\ k_1 k_2 & \cdots & k_p \end{pmatrix} = \det\begin{pmatrix} z_{i_1k_1} & \cdots\cdots & z_{i_1k_p} \\ \vdots & & \vdots \\ z_{i_pk_1} & \cdots\cdots & z_{i_pk_p} \end{pmatrix},$$

$$\varphi\langle j_1,\cdots,j_n\rangle = \frac{df_{j_1}}{f_{j_1}} \wedge\cdots\wedge \frac{df_{j_n}}{f_{j_n}},$$

$$F(\lambda,z) = \int_\sigma U\cdot\varphi\langle 12\cdots n\rangle \qquad \text{where} \quad \sigma\in H_n(X;\mathscr{S}_\omega^\vee).$$

Set $I=\{i_1,\cdots,i_p\}$, $K=\{k_1,\cdots,k_p\}$, $H=\{n+1,\cdots,m\}\setminus K$

$$L=\{1,\cdots,n\}\setminus I \underset{\text{set}}{=} \{\ell_1,\cdots,\ell_{n-p}\}$$

where we suppose

$$1\le i_1< \cdots <i_p\le n, \quad n+1\le k_1< \cdots <k_p\le m-1, \quad 1\le \ell_1< \cdots <\ell_{n-p}\le n.$$

Then we get

$$U = \prod_{i=1}^{n} u_i^{\lambda_i}\cdot \prod_{j=n+1}^{m-1}(u_1z_{1j}+ \cdots + u_nz_{nj})^{\lambda_j}\cdot(1-\sum u_i)^{\lambda_m}$$

$$\frac{\partial^p F}{\partial z_{i_1k_1}\cdots\partial z_{i_pk_p}} = \lambda_{k_1}\cdots\lambda_{k_p}\int_\sigma u_{i_1}\cdots u_{i_p}\cdot\frac{1}{\prod_{k\in K} f_k}\, U\cdot\varphi\langle 1\cdots n\rangle$$

$$= \prod_{k\in K}\lambda_k\int_\sigma U\cdot\frac{1}{u_{\ell_1}\cdots u_{\ell_{n-p}}\prod_{k\in K} f_k}\, du_1\wedge\cdots\wedge du_n.$$

Since

$$du_{\ell_1}\wedge\cdots\wedge du_{\ell_{n-p}}\wedge df_{k_1}\wedge\cdots\wedge df_{k_p}$$

$$= du_{\ell_1}\wedge\cdots\wedge du_{\ell_{n-p}}\wedge z\begin{pmatrix} i_1 & \cdots & i_p \\ k_1 & \cdots & k_p\end{pmatrix} du_{i_1}\wedge\cdots\wedge du_{i_p}$$

$$= z\binom{I}{K} \mathrm{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ & L & & I \end{pmatrix} du_1 \wedge \cdots \wedge du_n$$

$$\therefore \quad \frac{\partial^p F}{\partial z_{i_1k_1} \cdots \partial z_{i_pk_p}} = \frac{\mathrm{sgn}\begin{pmatrix} 12\cdots n \\ L \quad I \end{pmatrix}}{z\binom{I}{K}} \left(\prod_k \lambda_k \right) \int_\sigma U\cdot\varphi\langle LK\rangle .$$

On the other hand, since

$$1 \le \ell_1 < \cdots < \ell_{n-p} \le n, \quad n+1 \le k_1 < \cdots < k_p \le m-1,$$

$\{\varphi\langle LK\rangle\}$ is a subset of the basis $\{\varphi\langle j_1\cdots j_n\rangle \mid 1 \le j_1 < \cdots < j_n \le m-1\}$.

Since

$$\#\{\varphi\langle LK\rangle\} = \sum_{p=0}^{n} \binom{n}{n-p}\binom{m-n-1}{p} = \binom{m-1}{n} = \dim H^n(X,\nabla_\omega),$$

we see that $\{\varphi\langle LK\rangle\}$ coincides with the basis $\{\varphi\langle j_1\cdots j_n\rangle\}$. Put $N=\binom{m-1}{n}$ and let $\sigma_1,\cdots,\sigma_N$ be the twisted n-cycles associated with the N bounded chambers, which form a basis of $H_n(X,\mathcal{S}_\omega^\vee)$. Set

$$F_\nu(\lambda,x) = \int_{\sigma_\nu} U\cdot\varphi\langle 1\cdots n\rangle;$$

then

$$W = \det\left(\frac{\partial^p F_\nu}{\partial z_{i_1k_1} \cdots\cdots \partial z_{i_pk_p}} \right)_{\substack{\nu=1,\cdots,N \\ I,K}}$$

$$= \left(\prod_{I,K} \frac{\mathrm{sgn}\begin{pmatrix} 12\cdots n \\ L \quad I \end{pmatrix}}{z\binom{I}{K}} \lambda_{k_1}\cdots\lambda_{k_p} \right) \times \det\left(\int_{\sigma_\nu} U\cdot\varphi\langle LK\rangle\right).$$

By the perfect pairing of $H^n(X,\nabla_\omega)$ and $H_n(X,\mathcal{S}_\omega^\vee)$, we have showed that $\det(\int_{\sigma_\nu} U\cdot\varphi\langle LK\rangle) \ne 0$ and hence the Wronskian $W \ne 0$ if each $z\binom{I}{K} \ne 0$.

Remark Varchenko[V1,2] showed that $\det(\int_{\sigma_\nu} U\cdot\varphi\langle LK\rangle)$ can be written in closed form as a product of a generalized beta function and critical values of $f_j^{\lambda_j}$ on bounded chambers Δ.

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