数理解析研究所講究録
第818巻 1993年 42-59

ω_1-Souslin trees under countable support iterations

Tadatoshi MIYAMOTO
Nanzan University 宮元忠敏（南山大・経営）

Abstract

We show the property " is proper and preserves every ω_1-Souslin tree " iterates under countable support. As an example we show Con(SAD + $\neg$ SH) via a countable support iteration from [1].

Introduction

In [1], it is shown that the forcing axiom SAD is consistent via an iterated Souslin forcing. It is also shown that the forcing axiom does not imply the nonexistence of ω_1-Souslin trees by constructing a pair of an ω_1-Souslin tree and an iterated Souslin forcing in such a way that the ω_1-Souslin tree remains to be an ω_1-Souslin tree in the generic extensions via the Souslin forcing. In [2], a general theory on countable support iterations is developed and stronger versions of SAD are shown to be consistent.

We show countable support iterations for getting SAD preserve every ω_1-Souslin tree in the ground model. This note is organized as follows: In §0, we deal with various preliminaries. In §1, we consider preservations of ω_1-Souslin trees under proper and strongly proper preorders. In §2, we present an argument on σ-Baire under countable support iterations from [2]. In §3, we exhibit Con(SAD + $\neg$ SH) via a countable support iteration.

§0. Preliminary

(0.0) Definition. A triple $(P, \leq, 1)$ is a preorder iff $\leq$ is a reflexive and transitive binary relation on P with a greatest element 1. The symbol $\dot{G}$ usually denotes the canonical P-name for a P-generic filter over the ground model V. For an element x in V, we usually use x itself to denote its P-name. The preorder is *separative* iff for any $p, q \in P$ $q \Vdash_P$"$p \in \dot{G}$" implies $q \leq p$. We consider separative preorders in this note and so a preorder is always a separative one. For a formula φ, we simply write $\Vdash_P$"φ" instead of $1 \Vdash_P$"φ". A subset D of P is *predense below* q in P iff $q \Vdash_P$"$D \cap \dot{G} \neq \emptyset$".

For a set x, let $TC(x)$ denote the transitive closure of x. For a regular cardinal θ, let $H_\theta = \{x : | TC(x) | < \theta\}$. A countable subset N of H_θ is a *countable elementary substructure* of H_θ iff the structure $(N, \in)$ is an elementary substructure of $(H_\theta, \in)$. For a regular cardinal θ and a countable elementary substructure N of H_θ with $(P, \leq, 1) \in N$, a condition q in P is (P, N)-*generic* iff for any dense subset $D \in N$ of P $D \cap N$ is predense below q. Let $\mathrm{Gen}(P, N) = \{G \subset P \cap N : G$ is directed, upward closed in $P \cap N$ with respect to $\leq$ and for any open dense subset $C \in N$ of P $G \cap C \neq \emptyset\}$. For $p \in P \cap N$, let $\mathrm{Gen}(P, N, p) = \{G \in \mathrm{Gen}(P, N) \mid p \in G\}$. A condition r in P is a *lower bound* of $G \in Gen(P, N)$ iff for all $g \in G$ $r \leq g$. For a P-generic filter G over V and a P-name τ, $\tau[G]$ denotes the interpretation of τ by G. But $\{\tau[G] \mid \tau$ is a P-name and $\tau \in N\}$ is denoted by $N[G]$ which is a countable elementary substructure of $H_\theta^{V[G]}$. Let $\left((P_\alpha, \leq_\alpha, 1_\alpha)_{\alpha \leq \nu}, (\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)_{\alpha < \nu}\right)$ be a countable support iteration. For $p \in P_\alpha$, we donote $\{\beta < \alpha \mid p(\beta) \neq \dot{1}_\beta\}$ by $\mathrm{supp}(p)$ and so $|\mathrm{supp}(p)| \leq \omega$.

We pick up a couple of definitions and theorems from [2].

(0.1) Definition. A preorder $(P, \leq, 1)$ is *proper* iff for all sufficiently large regular cardinal θ and all countable elementary substructure N of H_θ with $(P, \leq, 1) \in N$, we have $\forall p \in P \cap N \exists q \leq p$ q is (P, N)-generic. For a countable ordinal ρ, a preorder $(P, \leq, 1)$ is *ρ-proper* iff for all sufficiently large regular cardinal θ and all continuously increasing countable elementary substructures $\langle N_k \mid k \leq \rho \rangle$ of H_θ s.t. $(P, \leq, 1) \in N_0$ and $\langle N_k \mid k \leq i \rangle \in N_{i+1}$ for all $i < \rho$, we have $\forall p \in P \cap N_0 \exists q \leq p \forall k \leq \rho$ q is (P, N_k)-generic. A preorder $(P, \leq, 1)$ is *strongly proper* iff for all sufficiently large regular cardinal θ, all countable elementary substructure N of H_θ with $(P, \leq, 1) \in N$ and all $\langle D_n \mid n < \omega \rangle$ s.t. D_n is a dense subset of $P \cap N$ for all $n < \omega$, we have $\forall p \in P \cap N \exists q \leq p \forall n < \omega$ q is predense below D_n.

$\dashv$

Let $\left((P_\alpha, \leq_\alpha, 1_\alpha)_{\alpha \leq \nu}, (\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)_{\alpha < \nu}\right)$ be a countable support iteration s.t. for all $\alpha < \nu \Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)$ is proper". Let θ be a sufficientry large regular cardinal and N be a countable elementary substructure of H_θ with $(P_\nu, \leq, 1_\nu) \in N$.

(0.2) Iteration Lemma for Proper. Let $\beta \leq \alpha \leq \nu$, $\beta \in N$ and $\alpha \in N$, then for any $x \in P_\beta$ and any P_β-name τ if x is (P_β, N)-generic and $x \Vdash_{P_\beta}$ "$\tau \in P_\alpha \cap N$ and $\tau \lceil \beta \in \dot{G}_\beta$", then there is $x^* \in P_\alpha$ s.t. $x^* \lceil \beta = x$, x^* is (P_α, N)-generic, $x^* \Vdash_{P_\alpha}$ "$\tau[\dot{G}_\alpha \lceil \beta] \in \dot{G}_\alpha$" and $supp(x^*) \cap [\beta, \alpha) \subseteq N$.

In particular, for any $x \in P_\beta$ and any $p \in P_\alpha \cap N$ if x is (P_β, N)-generic and $x \leq_\beta p \lceil \beta$, then there is $x^* \in P_\alpha$ s.t. $x^* \lceil \beta = x$, x^* is (P_α, N)-generic, $x^* \leq_\alpha p$ and $supp(x^*) \cap [\beta, \alpha) \subseteq N$.

$\dashv$

(0.3) Iteration Theorem for Proper. If $\left((P_\alpha, \leq_\alpha, 1_\alpha)_{\alpha \leq \nu}, (\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)_{\alpha < \nu}\right)$ is a countable support iteration s.t. for all $\alpha < \nu \Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)$ is proper", then $(P_\alpha, \leq_\alpha, 1_\alpha)$ is proper for all $\alpha \leq \nu$. Furthermore, under CH, if $\nu = \omega_2$ and for all $\alpha < \omega_2$ $\Vdash_{P_\alpha}$ "$| \dot{Q}_\alpha | \leq 2^\omega$", then P_α has a dense subset of size at most ω_1 for all $\alpha < \omega_2$.

$\dashv$

Let ρ be a countable ordinal and $\left((P_\alpha, \leq_\alpha, 1_\alpha)_{\alpha \leq \nu}, (\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)_{\alpha < \nu}\right)$ be a countable support iteration s.t. for all $\alpha < \nu \Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)$ is ρ-proper". Let θ be a sufficiently large regular cardinal and $\langle N_k \mid k \leq \rho \rangle$ be a continuously increasing countable elementary substructures of H_θ s.t. $(P_\nu, \leq_\nu, 1_\nu) \in N_0$ and $\langle N_k \mid k \leq i \rangle \in N_{i+1}$ for all $i < \rho$.

(0.4) Iteration Lemma for ρ-proper. Let $\eta \leq \zeta \leq \rho$, $\beta \leq \alpha \leq \nu$, $\beta \in N_\eta$ and $\alpha \in N_\eta$, then for any $x \in P_\beta$ and any P_β-name τ if x is (P_β, N_k)-generic for all k with $\eta \leq k \leq \zeta$ and $x \Vdash_{P_\beta}$ "$\tau \in P_\alpha \cap N_\eta$ and $\tau \lceil \beta \in \dot{G}_\beta$", then there is $x^* \in P_\alpha$ s.t. $x^* \lceil \beta = x$, x^* is (P_α, N_k)-generic for all k with $\eta \leq k \leq \zeta$, $x^* \Vdash_{P_\alpha}$ "$\tau[\dot{G}_\alpha \lceil \beta] \in \dot{G}_\alpha$" and $supp(x^*) \cap [\beta, \alpha) \subseteq N_\zeta$.

In particular, for any $x \in P_\beta$ and any $p \in P_\alpha \cap N_\eta$ if x is (P_β, N_k)-generic for all k with $\eta \leq k \leq \zeta$ and $x \leq_\beta p \lceil \beta$, then there is $x^* \in P_\alpha$ s.t. $x^* \lceil \beta = x$, x^* is (P_α, N_k)-generic for all k with $\eta \leq k \leq \zeta$, $x^* \leq_\alpha p$ and $supp(x^*) \cap [\beta, \alpha) \subseteq N_\zeta$.

$\dashv$

(0.5) Iteration Theorem for ρ-proper. If $\big((P_\alpha, \leq_\alpha, 1_\alpha)_{\alpha\leq\nu}, (\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)_{\alpha<\nu}\big)$ is a countable support iteration s.t. for all $\alpha < \nu \Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)$ is ρ-proper", then $(P_\alpha, \leq_\alpha, 1_\alpha)$ is ρ-proper for all $\alpha \leq \nu$.

$\dashv$

(0.6) Iteration Theorem for Strongly Proper.
If $\big((P_\alpha, \leq_\alpha, 1_\alpha)_{\alpha\leq\nu}, (\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)_{\alpha<\nu}\big)$ is a countable support iteration s.t. for all $\alpha < \nu$ $\Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)$ is strongly proper", then $(P_\alpha, \leq_\alpha, 1_\alpha)$ is strongly proper for all $\alpha \leq \nu$.

$\dashv$

The following is from [1] with minor changes.

(0.7) Definition. For $\alpha < \omega_1$, a *normal tree* U of height α means

(1) $U \subseteq {}^{\alpha>}\omega$.
(2) U is downward-closed in ${}^{\alpha>}\omega$ with respect to $\subseteq$.
(3) For any $\beta < \alpha$ $U \cap {}^{\beta}\omega \neq \emptyset$.
(4) If $\beta < \gamma < \alpha$ and $x \in U \cap {}^{\beta}\omega$, then there is $y \in U \cap {}^{\gamma}\omega$ with $x \subset y$.

We use *height*(U) to denote the height of U so height$(U) = \alpha$. A *normal subtree* W of U means $W \subseteq U$ and W is a normal tree with height(W) =height(U). For β <height(U), let $U\lceil\beta = U \cap {}^{\beta>}\omega$ and $U_\beta = U \cap {}^{\beta}\omega$. So $U\lceil\beta$ is a normal tree of height β and U_β is the β-th level of U. A normal subtree W of U is *closed under taking the immediate successors* iff whenever β <height(W), $x \in W_\beta$ and $x^\frown\langle n\rangle \in U$, we have $x^\frown\langle n\rangle \in W$.

Let $\Omega = \{\alpha < \omega_1 \mid \alpha \text{ is a limit ordinal}\}$. An *array of directed sets* is a sequence $D = \langle D_{\alpha,f} \mid \alpha \in \Omega,\ f \in {}^{\alpha>}\omega\rangle$ s.t. for all $\alpha \in \Omega$ and all $f \in {}^{\alpha>}\omega$ $D_{\alpha,f}$ is a countably complete directed subsets of ${}^{\alpha}\omega$ (i.e. for all non-empty $X \subseteq D_{\alpha,f}$ s.t. $|X| \leq \omega$, we have $\bigcap X \in D_{\alpha,f}$). A normal tree U of height ω_1 is *appropriate* for the array of directed sets D iff

(1) If $\alpha \in \Omega$ and $f \in U\lceil\alpha$, then there is $A \in D_{\alpha,f}$ s.t. whenever $h \in A$ is such that $f \subset h$ and $\forall\xi < \alpha$ $h\lceil\xi \in U$, then $h \in U$.
(2) If $\alpha \in \Omega$ and W is a normal subtree of $U\lceil\alpha$ closed under taking the immediate successors, then for any $f \in W$ and any $B \in D_{\alpha,f}$ there is $h \in B$ s.t. $f \subset h$ and $\forall\xi < \alpha$ $h\lceil\xi \in W$.

We sometimes refer to a normal tree of height ω_1 appropriate for an array of directed sets D as a *tree appropriate* for D. The forcing axiom *SAD* denotes the conjunction of the following statements.

(1) GCH.
(2) Every constructible cardinal is a cardinal.
(3) For every cardinal κ, cf(κ) =cf$^L(\kappa)$.
(4) Every countable sequence of ordinals is constructible.
(5) If D is a constructible array of directed sets, then every tree appropriate for D has a cofinal branch through it.

§1. Preserving ω_1-Souslin Trees

For the rest of this note a Souslin tree means an ω_1-Souslin tree.

(1.1) Proposition. Let $(P, \leq, 1)$ be a proper preorder and $(T, <_T)$ be a Souslin tree. The following are equivalent.

(1) $\Vdash_P$"$(T, <_T)$ remains to be a Souslin tree".

(2) For all sufficiently large regular cardinal θ and all countable elementary substructure N of H_θ with $(P, \leq, 1)$, $(T, <_T) \in N$, let $\delta = N \cap \omega_1$, then for any $(q, t) \in P \times T_\delta$ if q is (P, N)-generic, then (q, t) is $(P \times T, N)$-generic.

(3) For all sufficiently large regular cardinal θ and all countable elementary substructure N of H_θ with $(P, \leq, 1), (T, <_T) \in N$, let $\delta = N \cap \omega_1$, then $\forall p \in P \cap N \exists q \leq p \forall t \in T_\delta$ (q, t) is $(P \times T, N)$-generic.

Proof. (1) implies (2): Fix an arbitrary regular cardinal θ s.t. $P, T \in H_\theta$ and a countable elementary substructure N of H_θ with $(P, \leq, 1), (T, <_T) \in N$. Suppose $(q, t) \in P \times T_\delta$ and q is (P, N)-generic. Let A be a maximal antichain of $P \times T$ with $A \in N$. Given an arbitrary P-generic filter G_P over the ground model V with $q \in G_P$ and an arbitrary T-generic filter G_T over $V[G_P]$ with $t \in G_T$. We want to show $(G_P \times G_T) \cap A \cap N \neq \emptyset$. Let $B = \{s \in T \mid \exists x \in G_P\ (x, s) \in A\}$ in $V[G_P]$. Then B is a maximal antichain of T and $B \in N[G_P]$. Since T remains to be a Souslin tree, B is a countable subset of T. Since $N[G_P]$ is a countable elementary substructure of $H_\theta^{V[G_P]}$, there is an enumeration of B in $N[G_P]$. Since q is (P, N)-generic, we get $B \subset N[G_P] \cap T = N \cap T = T\lceil\delta$. Since $t \in T_\delta$, there is $s \in B$ with $s <_T t$. So we have $x \in G_P$ s.t. $(x, s) \in A$. We may assume $x \in G_P \cap N[G_P] = G_P \cap N$ and so $(x, s) \in (G_P \times G_T) \cap N$.

(2) implies (3): By assumption $(P, \leq, 1)$ is proper. So for all sufficiently large regular cardinal θ and all countable elementary substructure N of H_θ with $(P, \leq, 1), (T, <_T) \in N$, given $p \in P \cap N$ there is $q \leq p$ s.t. q is (P, N)-generic. Now by (2) for any $t \in T_\delta$ (q, t) is $(P \times T, N)$-generic.

(3) implies (1): Suppose $\Vdash_P$"$\dot{A}$ is a maximal antichain of T" and $p \in P$. Let $B = \{(x, s) \in P \times T \mid x \Vdash_P \text{“}\check{s} \in \dot{A}\text{”}\}$. Then B is a predense subset of $P \times T$. Fix a sufficiently large regular cardinal θ and a countable elementary substructure N of H_θ with $p, B, (P, \leq, 1), (T, <_T) \in N$. By (3), we have $q \leq p$ s.t. for all $t \in T_\delta$ (q, t) is $(P \times T, N)$-generic. So $B \cap N$ is predense below (q, t) for all $t \in T_\delta$. We conclude $q \Vdash_P$"$\forall t \in T_\delta \exists s <_T t\ s \in \dot{A}$". Hence $q \Vdash_P$"$\dot{A} \subseteq T\lceil\delta$".

$\dashv$

(1.2) Lemma. Let $\big((P_\alpha, \leq_\alpha, 1_\alpha)_{\alpha \leq \nu}, (\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)_{\alpha < \nu}\big)$ be a countable support iteration and $(T, <_T)$ be a Souslin tree. If ν is a limit ordinal and for all $\alpha < \nu$ $\Vdash_{P_\alpha}$"$(T, <_T)$ remains to be a Souslin tree and $(\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)$ is proper", then $\Vdash_{P_\nu}$"$(T, <_T)$ remains to be a Souslin tree".

Proof. Suppose $p \in P_\nu$ and $\Vdash_{P_\nu}$"$\dot{A}$ is a maximal antichain of T". Let $B = \{(x, s) \in P_\nu \times T \mid x \Vdash \text{“}\check{s} \in \dot{A}\text{”}\}$. Fix a sufficiently large regular cardinal θ and a countable elementary substructure N of H_θ with $p, (P_\nu, \leq_\nu, 1_\nu), (T, <_T), B \in N$. Fix $\langle \alpha_n \mid n < \omega \rangle$ s.t. $\alpha_0 = 0$, $\alpha_n \in \nu \cap N$ and $\alpha_n < \alpha_{n+1}$ for all $n < \omega$ and $\sup\{\alpha_n \mid n < \omega\} = \sup(\nu \cap N)$. Let $\delta = N \cap \omega_1 < \omega_1$ and $\langle t_n \mid n < \omega \rangle$ enumerate T_δ. We construct $\langle \dot{x}_n \mid n < \omega \rangle$ and $\langle q_n \mid n < \omega \rangle$ s.t. for all $n < \omega$

(1) $\dot{x}_0$ is the P_0-name $\check{p}$.

(2) $q_0 = \emptyset \in P_0$.

(3) $\dot{x}_n$ is a P_{α_n}-name.

(4) q_n is (P_{α_n}, N)-generic.

(5) $q_n \Vdash_{P_{\alpha_n}}$ "$\dot{x}_n \in P_\nu \cap N$ and $\dot{x}_n \lceil \alpha_n \in \dot{G}_{\alpha_n}$".

(6) $q_{n+1} \lceil \alpha_n = q_n$.

(7) $q_{n+1} \Vdash_{P_{\alpha_{n+1}}}$ "$\dot{x}_{n+1} \leq_\nu \dot{x}_n[\dot{G}_{\alpha_{n+1}} \lceil \alpha_n]$ and $\exists s <_T t_n\ (\dot{x}_{n+1}, s) \in \check{B}$".

The construction is by recursion on $n < \omega$. For $n = 0$, let $\dot{x}_0, q_0$ be as specified. Now suppose we have $\dot{x}_n$ and q_n. Since (4) and (5) hold, we have $q_{n+1} \in P_{\alpha_{n+1}}$ s.t. $q_{n+1} \lceil \alpha_n = q_n$, q_{n+1} is $(P_{\alpha_{n+1}}, N)$-generic and $q_{n+1} \Vdash_{P_{\alpha_{n+1}}}$ "$\dot{x}_n[\dot{G}_{\alpha_{n+1}} \lceil \alpha_n] \lceil \alpha_{n+1} \in \dot{G}_{\alpha_{n+1}}$" by (0.2) iteration lemma for proper. Since $\Vdash_{P_{\alpha_{n+1}}}$ "$(T, <_T)$ remains to be a Souslin tree", we know (q_{n+1}, t_n) is $(P_{\alpha_{n+1}} \times T, N)$-generic by (1.1) proposition.

Now in order to get a $P_{\alpha_{n+1}}$-name $\dot{x}_{n+1}$, let us fix an arbitrary $P_{\alpha_{n+1}}$-generic filter $G_{\alpha_{n+1}}$ over V with $q_{n+1} \in G_{\alpha_{n+1}}$. Let $G_{\alpha_n} = G_{\alpha_{n+1}} \lceil \alpha_n$. We know G_{α_n} is a P_{α_n}-generic filter over V with $q_n \in G_{\alpha_n}$. Let $x_n = \dot{x}_n[G_{\alpha_n}]$. Then $x_n \in P_\nu \cap N$ and $x_n \lceil \alpha_{n+1} \in G_{\alpha_{n+1}}$ hold. Let $D = \{(a, s) \in P_{\alpha_{n+1}} \times T \mid a$ and $x_n \lceil \alpha_{n+1}$ are incompatible in $P_{\alpha_{n+1}}\} \cup \{(a, s) \in P_{\alpha_{n+1}} \times T \mid \exists x \in P_\nu (x \leq_\nu x_n,\ (x, s) \in B$ and $x \lceil \alpha_{n+1} = a)\}$. Then D is a predense subset of $P_{\alpha_{n+1}} \times T$ and $D \in N$. Hence $D \cap N$ is predense below (q_{n+1}, t_n). For convenience sake, let us fix a T-generic filter G_T over $V[G_{\alpha_{n+1}}]$ with $t_n \in T$. Then there is $(a, s) \in D \cap N \cap (G_{\alpha_{n+1}} \times G_T)$. Since $a \in G_{\alpha_{n+1}}$ and $x_n \lceil \alpha_{n+1} \in G_{\alpha_{n+1}}$, there must be $x \in P_\nu$ s.t. $x \leq_\nu x_n$, $(x, s) \in B$ and $x \lceil \alpha_{n+1} = a$. Since $(P_\nu, \leq_\nu, 1_\nu), x_n, s, B, \alpha_{n+1}$ and a are all in N, we may assume $x \in N$. Since $s \in N \cap G_T$ and $t_n \in G_T$, we have $s <_T t_n$. Let $\dot{x}_{n+1}$ be a $P_{\alpha_{n+1}}$-name of this x. This completes the construction.

Let $q = \bigcup\{q_n \mid n < \omega\}^\frown 1_\nu \lceil [\sup(\nu \cap N), \nu)$. Then $q \in P_\nu$. We claim $q \Vdash_{P_\nu}$ "$\forall n < \omega \exists s \in \dot{A}\, s <_T t_n$" and so $q \Vdash$ "$\dot{A} \subseteq T \lceil \delta$". To this end let G_ν be an arbitrary P_ν-generic filter over V with $q \in G_\nu$. Put $G_{\alpha_n} = G_\nu \lceil \alpha_n$ and $x_n = \dot{x}_n[G_{\alpha_n}]$ for each $n < \omega$.

Since $q_n \in G_{\alpha_n}$ holds for all $n < \omega$, we have

(8) $x_0 = p$.

(9) $x_n \in P_\nu \cap N$ and $x_n \lceil \alpha_n \in G_{\alpha_n}$.

(10) $x_{n+1} \leq_\nu x_n$ and $\exists s <_T t_n\ (x_{n+1}, s) \in B$.

Since $x_n \in P_\nu \cap N$, we know $\mathrm{supp}(x_n) \subseteq P_\nu \cap N$ for all $n < \omega$. We conclude $x_n \in G_\nu$ for all $n < \omega$. Therefore for all $n < \omega$ there is $s \in \dot{A}[G_\nu]$ with $s <_T t_n$. Since G_ν is an arbitrary P_ν-generic filter over V with $q \in G_\nu$, we have $q \leq_\nu p$.

$\dashv$

(1.3) Theorem. Let $((P_\alpha, \leq_\alpha, 1_\alpha)_{\alpha \leq \nu}, (\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)_{\alpha < \nu})$ be a countable support iteration of arbitrary length ν. If for all $\alpha < \nu \Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)$ is proper and preserves every Souslin tree", then $(P_\nu, \leq_\nu, 1_\nu)$ is proper and preserves every Souslin tree.

Proof. Immediate from (1.2) lemma.

$\dashv$

(1.4) Note. There is a countable support iteration $((P_n)_{n \leq \omega}, (\dot{Q}_n)_{n < \omega})$ s.t. every Souslin tree remains to be a Souslin tree in the generic extensions via P_n for all $n < \omega$. But P_ω collapses ω_1.

$\dashv$

(1.5) Proposition. Every Souslin tree remains to be a Souslin tree in the generic extensions via the following notions of forcing.

(1) Strongly proper preorders.

(2) Preorders which appear in the forcing axiom SAD.

Proof. For (1): Suppose $(P, \leq, 1)$ is a strongly proper preorder and $(T, <_T)$ is a Souslin tree. Fix a sufficiently large regular cardinal θ and a countable elementary substructure N of H_θ with $(P, \leq, 1), (T, <_T) \in N$. Let $p \in P \cap N$ and $\delta = N \cap \omega_1$. By (1.1) proposition it suffices to find $q \leq p$ s.t. for all $t \in T_\delta$ (q, t) is $(P \times T, N)$-generic. Let $\langle D_n \mid n < \omega \rangle$ enumerate dense subsets of $P \times T$ which are in N. For each $(t, n) \in T_\delta \times \omega$, let $E^t_n = \{x \in P \cap N \mid \exists s <_T t \; (x, s) \in D_n\}$. Since $(T, <_T)$ is a Souslin tree, we know E^t_n is a dense subset of $P \cap N$. Since $(P, \leq, 1)$ is strongly proper there is $q \leq p$ s.t. for all $(t, n) \in T_\delta \times \omega$ E^t_n is predense below q. We conclude $D_n \cap N$ is predense below (q, t) for all $(t, n) \in T_\delta \times \omega$.

For (2): Let U be a normal tree of height ω_1 which is appropriate for some array of directed sets and $(T, <_T)$ be a Souslin tree. Suppose $p \in U$ and $\Vdash_U$"$\dot{A}$ is a maximal antichain of T" We want to find $q \in U$ s.t. $q \supseteq p$ and $q \Vdash_U$"$\dot{A}$ is countable". Fix a sufficiently large regular cardinal θ and $\langle N_n \mid n < \omega \rangle$ s.t. $p, U, (T, <_T), \dot{A} \in N_0$, $N_n \in N_{n+1}$ and N_n is a countable elementary substructure of H_θ for all $n < \omega$. Let $N = \bigcup\{N_n \mid n < \omega\}$, $\delta = N \cap \omega_1$ and $\delta_n = N_n \cap \omega_1$ for each $n < \omega$. Then $\delta_n < \omega_1$, $\delta_n < \delta_{n+1}$ for all $n < \omega$ and $\delta = \sup\{\delta_n \mid n < \omega\}$. Let $\langle t_n \mid n < \omega \rangle$ enumerate T_δ and for each $n < \omega$ s_n be the unique $z \in T_{\delta_n}$ with $z <_T t_n$. Note $s_n \in N_{n+1}$ holds for all $n < \omega$. We construct $\langle W^n \mid n < \omega \rangle$ s.t. for all $n < \omega$

1. W^n is a normal subtree of $U \lceil \delta_n + 1$ and $\mid W^n \mid = \omega$.
2. $W^n \lceil \delta_n \subseteq U \cap N_n$.
3. $\forall u \in W^n \lceil \delta_n \forall k < \omega \; (u^\frown\langle k \rangle \in U$ implies $u^\frown\langle k \rangle \in W^n)$.
4. $\forall z \in W^n_{\delta_n} \exists x \subset z \exists s <_T s_n \; x \Vdash_U$"$\check{s} \in \dot{A}$".
5. $W^n \in N_{n+1}$ and so $W^n_{\delta_n} \subset U_{\delta_n} \cap N_{n+1}$.
6. $W^{n+1} \lceil \delta_n + 1 = W^n$.

The construction is by recursion on $n < \omega$. We first construct W^0. Since T is a Souslin tree, we know $\{x \in U \cap N_0 \mid \exists s <_T s_0 \; x \Vdash_U$"$\check{s} \in \dot{A}$"$\}$ is a dense subset of $U \cap N_0$.

Now for each $y \in U \cap N_0$ we associate $\hat{y} \in U_{\delta_0}$ s.t. there is $x \in U \cap N_0$ s.t. $y \subseteq x \subset \hat{y}$ and for some $s <_T s_0$ $x \Vdash$ "$\check{s} \in \dot{A}$". This is possible because $U \cap N_0$ is a normal subtree of $U \lceil \delta_0$ closed under taking the immediate successors and it is assumed that U is appropriate for some array of directed sets. Let $W^0 = (U \cap N_0) \cup \{\hat{y} \mid y \in U \cap N_0\}$. Then this W^0 satisfies condition 1 through 4 for $n = 0$. Since relevant parameters are all in N_1, we may assume $W^0 \in N_1$.

Suppose we have gotten W^n. We know $\{x \in U \cap N_{n+1} \mid \exists s <_T s_{n+1}\ x \Vdash \text{"}\check{s} \in \dot{A}\text{"}\}$ is a dense subset of $U \cap N_{n+1}$ as before. Now for each $y \in U \cap N_{n+1}$ s.t. there is $z \in W^n_{\delta_n}$ with $y \supseteq z$, we associate $\hat{y} \in U_{\delta_{n+1}}$ s.t. there is $x \in U \cap N_{n+1}$ s.t. $y \subseteq x \subset \hat{y}$ and for some $s <_T s_{n+1}$ $x \Vdash$ "$\check{s} \in \dot{A}$". This is possible because $W^n \cup \{y \in U \cap N_{n+1} \mid \exists z \in W^n_{\delta_n}\ y \supseteq z\}$ is a normal subtree of $U \lceil \delta_{n+1}$ closed under taking the immediate successors. Let $W^{n+1} = W^n \cup \{y, \hat{y} \mid y \in U \cap N_{n+1} \text{ and } \exists z \in W^n_{\delta_n}\ y \supseteq z\}$. Then this W^{n+1} satisfies condition 1 through 4 for $n+1$ and condition 6. Since relevant parameters are all in N_{n+2}, we may choose W^{n+1} in N_{n+2}. This completes the construction of $\langle W^n \mid n < \omega\rangle$.

Let $W = \bigcup\{W^n \mid n < \omega\}$. Then W is a normal subtree of $U \lceil \delta$ closed under taking the immediate succsessors. Since $p \in W$ there is $q \in U_\delta$ s.t. $q \supset p$ and for all $n < \omega$ $q \lceil \delta_n \in W^n$. It is clear by the construction that for each $n < \omega$ there is $s <_T s_n <_T t_n$ s.t. $q \Vdash$ "$\check{s} \in \dot{A}$". Since $\{t_n \mid n < \omega\} = T_\delta$ and $q \Vdash$ "$\dot{A}$ is a maximal antichain of $\check{T}$", we conclude $q \Vdash$ "$\dot{A} \subseteq T\lceil\delta$".

$\dashv$

(1.6) Note. 1. There is a preorder which is not strongly proper but SAD is applicable. For each $\alpha \in \Omega$, let $\eta_\alpha : \omega \to \alpha$ be an increasing and cofinal function such that for all $n < \omega$ $\eta_\alpha(n)$ is a successor ordinal. Let $E = \{\{\eta_\alpha(n), \alpha\} \mid n < \omega \text{ and } \alpha \in \Omega\}$. Then (ω_1, E) is a Hajnal-Mate graph (see [1]). Now force a coloring $f : \omega_1 \to \omega$ s.t. $\{x_1, x_2\} \in E$ implies $f(x_1) \neq f(x_2)$. This p.o.set is an example.

2. There is a preorder which is strongly proper but SAD is not applicable. Consider the perfect p.o.set.

3. There is a preorder which is strongly proper and SAD is applicable. For each $\alpha \in \Omega$, let $f_\alpha : \alpha \to \omega$ be an arbitrary function. Force a function $f : \omega_1 \to \omega$ s.t. for all $\alpha \in \Omega$ $f\lceil\alpha \neq f_\alpha$. This p.o.set is an example.

$\dashv$

(1.7) Corollary. Countable support iterations of strongly proper preorders preserve every Souslin tree.

Proof. Since strongly proper preorders are iterable under countable support by (0.6) iteration theorem for strongly proper. This is immediate from (1.5) proposition.

$\dashv$

§2. σ-Baire

In this section we review an argument on σ-Baire under countable support iterations from [2].

(2.1) Proposition. Let $(P, \leq, 1)$ be a preorder. For all sufficiently large regular cardinal θ and all countable elementary substructure N of H_θ with $(P, \leq, 1) \in N$ if we assume $\forall p \in P \cap N\ \exists G \in \mathrm{Gen}(P, N, p)$ G has a lower bound in P, then $(P, \leq, 1)$ is σ-Baire.

Proof. Given open dense subsets $\langle D_n \mid n < \omega \rangle$ of P. We want to show $\bigcap\{D_n \mid n < \omega\}$ is a dense subset of P. To this end fix an arbitrary $p \in P$. Now take a sufficiently large regular cardinal θ and a countable elementary substructure N of H_θ with $p, (P, \leq, 1), \langle D_n \mid n < \omega \rangle \in N$. By assumption we have $G \in \mathrm{Gen}(P, N, p)$ with a lower bound $q \in P$. Since $D_n \in N$, there is $x \in G \cap D_n$ and so $q \leq x$ for all $n < \omega$. Since D_n is open for all $n < \omega$, we conclude $q \in \bigcap\{D_n \mid n < \omega\}$. Since $p \in G$, we have $q \leq p$.

$\dashv$

(2.2) Lemma. Let $\left((P_\alpha, \leq_\alpha, 1_\alpha)_{\alpha \leq \nu}, (\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)_{\alpha < \nu}\right)$ be a countable support iteration such that ν is a limit ordinal and for all $\alpha < \nu$ $(P_\alpha, \leq_\alpha, 1_\alpha)$ is σ-Baire. Then $(P_\nu, \leq_\nu, 1_\nu)$ is σ-Baire provided that

1. For all $\alpha < \nu$ $\Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)$ is ρ-proper for all $\rho < \omega_1$".
2. For all sufficiently large regular cardinal θ and all (α, M_0, M_1, G, p) s.t.
   (1) $\alpha < \nu$.
   (2) M_0 and M_1 are countable elementary substructures of H_θ s.t. $(P_{\alpha+1}, \leq_{\alpha+1}, 1_{\alpha+1}) \in M_0 \in M_1$.
   (3) $p \in P_\nu \cap M_0$.
   (4) $G \in \mathrm{Gen}(P_\alpha, M_0, p\lceil\alpha) \cap M_1$ and G has a lower bound in P_α.
   There is $G^* \in \mathrm{Gen}(P_{\alpha+1}, M_0, p\lceil\alpha+1)$ s.t.
   (5) $G = \{x\lceil\alpha \mid x \in G^*\}$.
   (6) For any $r \in P_\alpha$ if r is a lower bound of G and is (P_α, M_1)-generic, then there is $r^* \in P_{\alpha+1}$ such that $r^*\lceil\alpha = r$ and r^* is a lower bound of G^*.

Proof. Fix a sufficiently large regular cardinal θ and a sequence $\langle N_i \mid i < \omega_1 \rangle$ of continuously increasing countable elementary substructures of H_θ s.t. $(P_\nu, \leq_\nu, 1_\nu) \in N_0$ and $\langle N_k \mid k \leq i \rangle \in N_{i+1}$ for all $i < \omega_1$. Notice that we have $(P_\alpha, \leq_\alpha, 1_\alpha) \in N_i$ for all $\alpha \in N_i \cap \nu$. Let ν^* be the order type of $(N_0 \cap \nu, \in)$ and $\langle \alpha_i \mid i \leq \nu^* \rangle$ enumerate $N_0 \cap (\nu + 1)$ in increasing order. Since $\mid N_0 \mid = \omega$, we have $\nu^* < \omega_1$. Notice that $\alpha_{\nu^*} = \nu$, $\alpha_{i+1} = \alpha_i + 1$ for all $i < \nu^*$ and $\sup\{\alpha_j \mid j < i\} \leq \alpha_i$ for all limit ordinal $i \leq \nu^*$.

Claim 1. We have $\varphi(j)$ for all $j \leq \nu^*$, where $\varphi(j)$ means

For any $i < j$, any $p \in P_\nu \cap N_0$ and any $G \in \mathrm{Gen}(P_{\alpha_i}, N_0, p\lceil\alpha_i) \cap N_{i+1}$ with a lower bound, we have $G^* \in \mathrm{Gen}(P_{\alpha_j}, N_0, p\lceil\alpha_j)$ s.t. $G = G^*\lceil\alpha_i$ $(= \{b\lceil\alpha_i \mid b \in G^*\})$ and the following condition (1) holds.

(1) If a lower bound a of G is (P_{α_i}, N_k)-generic for all k with $i + 1 \leq k \leq j$, then there is $a^* \in P_{\alpha_j}$ s.t. a^* is a lower bound of G^* and $a^*\lceil\alpha_i = a$.

We show claim 1 by induction on $j \leq \nu^*$. But we first observe

Claim 2. $\varphi(j)$ implies $\varphi'(j)$ for all $j \leq \nu^*$, where $\varphi'(j)$ means

For any $i < j$, any $p \in P_\nu \cap N_0$ and any $G \in \mathrm{Gen}(P_{\alpha_i}, N_0, p\lceil\alpha_i) \cap N_{i+1}$ with a lower bound, we have $G^* \in \mathrm{Gen}(P_{\alpha_j}, N_0, p\lceil\alpha_j) \cap N_{j+1}$ s.t. G^* has a lower bound in P_{α_j} and not only condition (1) above holds but also the following condition (2) is satisfied.

(2) If a lower bound a of G is (P_{α_i}, N_k)-generic for all k with $i+1 \leq k \leq j$ and $j+1 \leq k \leq l$ for some $l < \omega_1$, then there is $a^* \in P_{\alpha_j}$ s.t. a^* is a lower bound of G^*, $a^*\lceil\alpha_i = a$ and a^* is (P_{α_j}, N_k)-generic for all k with $j+1 \leq k \leq l$.

Proof. Suppose $i < j$, $p \in P_\nu \cap N_0$ and $G \in \mathrm{Gen}(P_{\alpha_i}, N_0, p\lceil\alpha_i) \cap N_{i+1}$ with a lower bound. Then by $\varphi(j)$ we have $G^* \in \mathrm{Gen}(P_{\alpha_j}, N_0, p\lceil\alpha_j)$ s.t. $G = G^*\lceil\alpha_i$ and (1) holds. Since relevant parameters are all in N_{j+1}. We may assume $G^* \in N_{j+1}$. We claim this G^* works. Now since $G \in N_{i+1}$ and G has a lower bound, we may take a lower bound of G in N_{i+1}. Once we take the lower bound of G in N_{i+1}, we may fix a condition a of P_{α_i} which sits below the lower bound taken and is (P_{α_i}, N_k)-generic for all k with $i+1 \leq k \leq j$. This is possible because P_{α_i} is ρ-proper for all $\rho < \omega_1$. Then by condition (1), there is a lower bound a^* of G^* in P_{α_j}. So G^* has a lower bound.

Now we establish condition (2). Suppose a is a lower bound of G in P_{α_i} and is (P_{α_i}, N_k)-generic for all k with $i+1 \leq k \leq j$ and $j+1 \leq k \leq l$ for some $l < \omega_1$. We claim that there is a P_{α_i}-name τ s.t. $a \Vdash$"$\tau \in P_{\alpha_j} \cap N_{j+1}$, $\tau\lceil\alpha_i \in \dot{G}_{\alpha_i}$ and τ is a lower bound of G^* in P_{α_j}". This is because given an arbitrary P_{α_i}-generic filter G_{α_i} over V with $a \in G_{\alpha_i}$. By (1) we have $y \in P_{\alpha_j}$ s.t. $y\lceil\alpha_i \in G_{\alpha_i}$ and y is a lower bound of G^* in P_{α_j}. Since relevant parameters involved are all in $N_{j+1}[G_{\alpha_i}]$ and $(N_{j+1}[G_{\alpha_i}], \in)$ is a countable elementary substructure of $(H_\theta^{V[G_{\alpha_i}]}, \in)$. We may assume that $y \in P_{\alpha_j} \cap N_{j+1}[G_{\alpha_i}] = P_{\alpha_j} \cap N_{j+1}$. Let τ be a P_{α_i}-name of this y. We now apply (0.4) iteration lemma for ρ-proper. Since a is (P_{α_i}, N_k)-generic for all k with $j+1 \leq k \leq l$ and $a \Vdash$"$\tau \in P_{\alpha_j} \cap N_{j+1}$ and $\tau\lceil\alpha_i \in \dot{G}_{\alpha_i}$". We have this time $a^* \in P_{\alpha_j}$ s.t. $a^*\lceil\alpha_i = a$, a^* is (P_{α_j}, N_k)-generic for all k with $j+1 \leq k \leq l$ and $a^* \Vdash_{P_{\alpha_j}}$"$\tau[\dot{G}_{\alpha_j}\lceil\alpha_i] \in \dot{G}_{\alpha_j}$". Since $a \Vdash_{P_{\alpha_i}}$"τ is a lower bound of G^* in P_{α_j}", we conclude $a^* \Vdash_{P_{\alpha_j}}$"$G^* \subseteq \dot{G}_{\alpha_j}$" and so a^* is a lower bound of G^* in P_{α_j}. This completes the proof of claim 2.

$\dashv$

We next observe that claim 1 and 2 imply

Claim 3. For any $p \in P_\nu \cap N_0$ there is $G^* \in \mathrm{Gen}(P_\nu, N_0, p)$ with a lower bound in P_ν and so $(P_\nu, \leq_\nu, 1_\nu)$ is σ-Baire by (2.1) propotision.

Proof. We simply take $i = 0$, $G = \{\emptyset\} \in \mathrm{Gen}(P_0, N_0, \emptyset) \cap N_1$ and $a = \emptyset$ in $\varphi'(\nu^*)$. We get G^* as claimed.

$\dashv$

Now we show claim 1 by induction on $j \leq \nu^*$.

Case 1: j is a successor ordinal, say, $j = j_0 + 1$.

Given $i < j_0 + 1$, $p \in P_\nu \cap N_0$ and $G \in \mathrm{Gen}(P_{\alpha_i}, N_0, p\lceil\alpha_i) \cap N_{i+1}$ with a lower bound.

Subcase 1: $i < j_0$.

By applying $\varphi'(j_0)$, we have $G^\dagger \in \mathrm{Gen}(P_{\alpha_{j_0}}, N_0, p\lceil\alpha_{j_0}) \cap N_{j_0+1}$ with a lower bound and $G^\dagger\lceil\alpha_i = G$ s.t.

(1) For any $a \in P_{\alpha_i}$ if a is a lower bound of G and a is (P_{α_i}, N_k)-generic for all k with $i+1 \leq k \leq j_0$, then there is $a^\dagger \in P_{\alpha_{j_0}}$ s.t. $a^\dagger$ is a lower bound of $G^\dagger$ and $a^\dagger\lceil\alpha_i = a$.

(2) For any $a \in P_{\alpha_i}$ if a is a lower bound of G and a is (P_{α_i}, N_k)-generic for all k with $i+1 \leq k \leq j_0$ and $j_0+1 \leq k \leq l$ for some $l < \omega_1$, then there is $a^\dagger \in P_{\alpha_{j_0}}$ s.t. $a^\dagger$ is a lower bound of $G^\dagger$, $a^\dagger\lceil\alpha_i = a$ and $a^\dagger$ is $(P_{\alpha_{j_0}}, N_k)$-generic for all k with $j_0+1 \leq k \leq l$.

Since θ is a sufficiently large regular cardinal and $(\alpha_{j_0}, N_0, N_{j_0+1}, G^\dagger, p)$ is s.t.

(1) $\alpha_{j_0} < \nu$.
(2) N_0 and N_{j_0+1} are countable elementary substructures of H_θ s.t. $(P_{\alpha_{j_0}+1}, \leq_{\alpha_{j_0}+1}, 1_{\alpha_{j_0}+1}) \in N_0 \in N_{j_0+1}$.
(3) $p \in P_\nu \cap N_0$.
(4) $G^\dagger \in \mathrm{Gen}(P_{\alpha_{j_0}}, N_0, p\lceil\alpha_{j_0}) \cap N_{j_0+1}$ with a lower bound in $P_{\alpha_{j_0}}$.

We apply the assumption of this lemma. So we have $G^* \in \mathrm{Gen}(P_{\alpha_{j_0+1}}, N_0, p\lceil\alpha_{j_0+1})$ s.t.

(5) $G^*\lceil\alpha_{j_0} = G^\dagger$ and so $G^*\lceil\alpha_i = G$.
(6) For all lower bound $a^\dagger$ of $G^\dagger$ if $a^\dagger$ is $(P_{\alpha_{j_0}}, N_{j_0+1})$-generic, then there is $a^* \in P_{\alpha_{j_0+1}}$ s.t. a^* is a lower bound of G^* and $a^*\lceil\alpha_{j_0} = a^\dagger$.

To show this G^* works for (1) in $\varphi(j)$, fix a lower bound a of G s.t. a is (P_{α_i}, N_k)-generic for all k with $i+1 \leq k \leq j_0+1$. Then there is $a^\dagger \in P_{\alpha_{j_0}}$ s.t. $a^\dagger$ is a lower bound of $G^\dagger$, $a^\dagger\lceil\alpha_i = a$ and $a^\dagger$ is $(P_{\alpha_{j_0}}, N_{j_0+1})$-generic by (2) in $\varphi'(j_0)$. Now by (6) just above, there is $a^* \in P_{\alpha_{j_0+1}}$ s.t. a^* is a lower bound of G^* and $a^*\lceil\alpha_{j_0} = a^\dagger$ and so $a^*\lceil\alpha_i = a$. This completes subcase 1.

Subcase 2: $i = j_0$ i.e. $j = i+1$.

This case is done by simply repeating a part of previous subcase. We are given $p \in P_\nu \cap N_0$ and $G \in \mathrm{Gen}(P_{\alpha_i}, N_0, p\lceil\alpha_i) \cap N_{i+1}$ with a lower bound in P_{α_i}. Since $(\alpha_i, N_0, N_{i+1}, G, p)$ is s.t.

(1) $\alpha_i \in \nu$.
(2) N_0 and N_{i+1} are countable elementary substructures of H_θ s.t. $(P_{\alpha_i+1}, \leq_{\alpha_i+1}, 1_{\alpha_i+1}) \in N_0 \in N_{i+1}$.
(3) $p \in P_\nu \cap N_0$.
(4) $G \in \mathrm{Gen}(P_{\alpha_i}, N_0, p\lceil\alpha_i) \cap N_{i+1}$ with a lower bound.

By assumption there is $G^* \in \mathrm{Gen}(P_{\alpha_{i+1}}, N_0, p\lceil\alpha_{i+1})$ s.t.

(5) $G^*\lceil\alpha_i = G$.
(6) For any lower bound a of G if a is (P_{α_i}, N_{i+1})-generic then there is $a^* \in P_{\alpha_{i+1}}$ s.t. a^* is a lower bound of G^* and $a^*\lceil\alpha_i = a$.

This completes subcase 2 and case 1.

Case 2: j is a limit ordinal.

Since j is a countable limit ordinal, we may fix a sequence $\langle j_n \mid n < \omega\rangle$ of ordinals s.t. $j_0 = i$, $j_n < j_{n+1}$ for all $n < \omega$ and $\sup\{j_n \mid n < \omega\} = j$. Note that $\sup\{\alpha_{j_n} \mid n < \omega\} \leq \alpha_j$. Suppose $p \in P_\nu \cap N_0$ and $G \in \mathrm{Gen}(P_{\alpha_i}, N_0, p\lceil\alpha_i) \cap N_{i+1}$. Let $\langle D_n \mid n < \omega\rangle$ be an enumeration of the open dense subsets of P_{α_j} which belong to N_0. We construct $\langle p_n \mid n < \omega\rangle$ and $\langle G^n \mid n < \omega\rangle$ s.t. for all $n < \omega$

(a) $p_0 = p\lceil\alpha_j$ and $G^0 = G$.

(b) $p_n \in P_{\alpha_j} \cap N_0$ and $G^n \in \mathrm{Gen}(P_{\alpha_{j_n}}, N_0, p_n\lceil\alpha_{j_n}) \cap N_{j_n+1}$ with a lower bound in $P_{\alpha_{j_n}}$.

(c) $p_{n+1} \in D_n \cap N_0$, $p_n \geq p_{n+1}$ and $G^{n+1}\lceil\alpha_{j_n} = G^n$.

(d) For any $x \in P_{\alpha_{j_n}}$ if x is a lower bound of G^n and is $(P_{\alpha_{j_n}}, N_k)$-generic for all k with $j_n + 1 \leq k \leq j_{n+1}$, then there is $y \in P_{\alpha_{j_{n+1}}}$ s.t. y is a lower bound of G^{n+1} and $y\lceil\alpha_{j_n} = x$.

(e) For any $x \in P_{\alpha_{j_n}}$ if x is a lower bound of G^n and is $(P_{\alpha_{j_n}}, N_k)$-generic for all k with $j_n + 1 \leq k \leq j_{n+1}$ and $j_{n+1} + 1 \leq k \leq l$ for some $l < \omega_1$, then there is $z \in P_{\alpha_{j_{n+1}}}$ s.t. z is a lower bound of G^{n+1}, $z\lceil\alpha_{j_n} = x$ and z is $(P_{\alpha_{j_{n+1}}}, N_k)$-generic for all k with $j_{n+1} + 1 \leq k \leq l$.

The construction is by a simultaneous recursion on $n < \omega$. Suppose we have constructed p_n and G^n s.t. (a) and (b) are satisfied. Let $D = \{x \in P_{\alpha_{j_n}} \mid x$ and $p_n\lceil\alpha_{j_n}$ are incompatible in $P_{\alpha_{j_n}}\} \cup \{x \in P_{\alpha_{j_n}} \mid \exists d \in D_n\ p_n \geq d$ and $d\lceil\alpha_{j_n} = x\}$. Then D is an open dense subset of $P_{\alpha_{j_n}}$ and $D \in N_0$. By (b) we have x in $D \cap G^n$. Since $p_n\lceil\alpha_{j_n} \in G^n$ and G^n is directed, there must be $d \in D_n$ s.t. $p_n \geq d$ and $d\lceil\alpha_{j_n} = x$. Since parameters $D_n, p_n, \geq, \alpha_{j_n}$ and x are all in N_0, we may assume $d \in D_n \cap N_0$. We put $p_{n+1} = d$. Since we have $G^n \in \mathrm{Gen}(P_{\alpha_{j_n}}, N_0, p_{n+1}\lceil\alpha_{j_n}) \cap N_{j_n+1}$ with a lower bound. We apply $\varphi'(j_{n+1})$. So there is $G^{n+1} \in \mathrm{Gen}(P_{\alpha_{j_{n+1}}}, N_0, p_{n+1}\lceil\alpha_{j_{n+1}}) \cap N_{j_{n+1}+1}$ s.t. G^{n+1} has a lower bound, $G^{n+1}\lceil\alpha_{j_n} = G^n$ and (d) and (e) are satisfied. This completes the construction of $\langle p_n \mid n < \omega\rangle$ and $\langle G^n \mid n < \omega\rangle$. Let $G^* = \{x \in P_{\alpha_j} \cap N_0 \mid \exists n < \omega\ p_n \leq x\}$. Since $p_n \in G^*$ for all $n < \omega$, we conclude $G^* \in \mathrm{Gen}(P_{\alpha_j}, N_0, p\lceil\alpha_j)$ and so $G^*\lceil\alpha_i$ is in $\mathrm{Gen}(P_{\alpha_i}, N_0, p\lceil\alpha_i)$. Since both $G^*\lceil\alpha_i$ and G are in $\mathrm{Gen}(P_{\alpha_i}, N_0, p\lceil\alpha_i)$ with $G^*\lceil\alpha_i \subseteq G$, we get $G^*\lceil\alpha_i = G$. Now given any $a \in P_{\alpha_i}$ s.t. a is a lower bound of G and is (P_{α_i}, N_k)-generic for all k with $i + 1 \leq k \leq j$. We must show there is $a^* \in P_{\alpha_j}$ s.t. a^* is a lower bound of G^* and $a^*\lceil\alpha_i = a$. To this end fix such a. We construct $\langle a_n \mid n < \omega\rangle$ s.t. for all $n < \omega$

(f) $a_0 = a$.

(g) $a_n \in P_{\alpha_{j_n}}$, a_n is a lower bound of G^n and is $(P_{\alpha_{j_n}}, N_k)$-generic for all k with $j_n + 1 \leq k \leq j$.

(h) $a_{n+1}\lceil\alpha_{j_n} = a_n$.

The construction is by recursion on $n < \omega$. Suppose we have constructed a_n s.t. (f) and (g) are satisfied. In (e), take $l = j$. Then we have $a_{n+1} \in P_{\alpha_{j_{n+1}}}$ s.t. a_{n+1} is a lower bound of G^{n+1}, $a_{n+1}\lceil\alpha_{j_n} = a_n$ and a_{n+1} is $(P_{\alpha_{j_{n+1}}}, N_k)$-generic for all k with $j_{n+1} + 1 \leq k \leq j$. This completes the construction of $\langle a_n \mid n < \omega\rangle$.

By (h) there is $a^* \in P_{\alpha_j}$ s.t. $a^* \lceil \alpha_{j_n} = a_n$ for all $n < \omega$ and $\mathrm{supp}(p^*) \subseteq \sup\{\alpha_{j_n} \mid n < \omega\}$. Since $p_n \in P_{\alpha_j} \cap N_0$, we have $\mathrm{supp}(p_n) \subseteq \alpha_j \cap N_0 \subseteq \sup\{\alpha_{j_n} \mid n < \omega\}$ for all $n < \omega$. Hence $a^* \leq p_n$ for all $n < \omega$. We conclude a^* is a lower bound of G^*. This finishes case 2 and the proof of Claim 1.

$\dashv$

§3. Con(SAD + ¬SH)

(3.1) Proposition. Let U be a normal tree of height ω_1 appropriate for an array of directed sets D. For any (θ, N, p) if we assume

1. θ is a regular cardinal with $\theta > 2^{2^\omega}$.
2. N is a countable elementary substructure of H_θ with $U \in N$.
3. $p \in U \cap N$.

Then there is W such that, if $\delta = N \cap \omega_1$, then

1. W is a normal subtree of $U\lceil\delta$ and so height$(W) = \delta$.
2. $p \in W \subseteq U \cap N \subseteq U\lceil\delta$.
3. $\forall a \in W \forall n < \omega$ if $a^\frown\langle n\rangle \in U$, then $a^\frown\langle n\rangle \in W$.
4. For any $h \in {}^\delta\omega$ if $\forall\xi < \delta\ h\lceil\xi \in W$, then $\{h\lceil\xi \mid \xi < \delta\} \in \mathrm{Gen}(U, N)$.

Furthermore there is W^* such that

1. W^* is a normal subtree of $U\lceil\delta + 1$ s.t. $\mid W^* \mid= \omega$ and $W^*\lceil\delta = W$.
2. For all $h \in W^*_\delta$ $\{h\lceil\xi \mid \xi < \delta\} \in \mathrm{Gen}(U, N)$ and so h is (U, N)-generic.

Also for any non-empty countable subset X of $D_{\delta,p}$, there is $q \in U_\delta \cap \bigcap X$ s.t. $\{q\lceil\xi \mid \xi < \delta\} \in \mathrm{Gen}(U, N, p)$.

Proof. Suppose θ is a regular cardinal with $\theta > 2^{2^\omega}$ and N is a countable elementary substructure of H_θ with $U \in N$. Let $p \in U \cap N$.

Claim. We may fix a sequence $\langle N_n \mid n < \omega\rangle$ of countable elementary substructures of $H_{(2^\omega)^+}$ s.t.

1. $U, p \in N_0$.
2. $N_n \in N_{n+1}$ for all $n < \omega$.
3. $\bigcup\{N_n \mid n < \omega\} = N \cap H_{(2^\omega)^+}$.

Proof. Since N is an elementary substructure of H_θ and $\theta > 2^{2^\omega}$, we have $H_{(2^\omega)^+} \in N$ and so $N \cap H_{(2^\omega)^+}$ is a countable elementary substructure of $H_{(2^\omega)^+}$. Let $\langle x_n \mid n < \omega\rangle$ enumerate $N \cap H_{(2^\omega)^+}$. There is a countable elementary substructure N_0 of $H_{(2^\omega)^+}$ with $U, p, x_0 \in N_0$. Since parameters $H_{(2^\omega)^+}, U, p$ and x_0 are all in N, we may assume $N_0 \in N$. Now suppose we have $N_n \in N$ s.t. N_n is a countable elementary substructure of $H_{(2^\omega)^+}$ with $x_n \in N_n$. There is N_{n+1} s.t. N_{n+1} is a countable elementary substructure of $H_{(2^\omega)^+}$ with $N_n, x_{n+1} \in N_{n+1}$. Since parameters $H_{(2^\omega)^+}, N_n$ and x_{n+1} are all in N, we may assume $N_{n+1} \in N$. This way we get $\langle N_n \mid n < \omega\rangle$. Since $x_n \in N_n \in N$ and $\mid N_n \mid= \omega$ we have $N_n \subset N$ and so $N \cap H_{(2^\omega)^+} = \{x_n \mid n < \omega\} \subseteq \bigcup\{N_n \mid n < \omega\} \subseteq N \cap H_{(2^\omega)^+}$. This completes the proof of claim.

$\dashv$

Let $\langle D_n \mid n < \omega\rangle$ enumerate the open dense subsets of U which belong to N. Since $D_n \in N \cap H_{(2^\omega)^+} = \bigcup\{N_n \mid n < \omega\}$, we may assume $D_n \in N_n$ for all $n < \omega$. Let $\delta_n = N_n \cap \omega_1$ for each $n < \omega$ and let $\delta = N \cap \omega_1$. We construct $\langle W^n \mid n < \omega\rangle$ s.t. for all $n < \omega$

(1) W^n is a normal subtree of $U\lceil\delta_n + 1$, $\mid W^n \mid = \omega$ and $W^n \in N_{n+1}$.

(2) $W^n\lceil\delta_n \subseteq N_n$.

(3) For all $x \in W^n_{\delta_n}$ there is $y \in D_n \cap N_n$ s.t. $y \subset x$.

(4) For all $a \in W^n\lceil\delta_n$ and all $k < \omega$ if $a^\frown\langle k\rangle \in U$, then $a^\frown\langle k\rangle \in W^n$.

(5) $W^{n+1}\lceil\delta_n + 1 = W^n$.

The construction is by recursion on $n < \omega$. Since for any $z \in U \cap N_0$ there is $y \in D_0 \cap N_0$ s.t. $z \subseteq y$ and $U \cap N_0$ is a normal subtree of $U\lceil\delta_0$ closed under taking the immediate successors. So for all $y \in U \cap N_0$ there exists $x \in U_{\delta_0}$ s.t. $y \subset x$. We may construct W^0 in N_1 s.t. each condition (1) through (4) for $n = 0$ is satisfied. Suppose we got W^n. Since $W^n_{\delta_n} \in N_{n+1}$ and $\mid W^n_{\delta_n} \mid = \omega$, so $W^n_{\delta_n} \subset N_{n+1}$ holds. Since for any $z \in U \cap N_{n+1}$ there is $y \in D_{n+1} \cap N_{n+1}$ s.t. $z \subseteq y$. And for all $y \in U \cap N_{n+1}$ there exists $x \in U_{\delta_{n+1}}$ s.t. $y \subset x$. We may construct W^{n+1}. This completes the construction. Let $W = \bigcup\{W^n \mid n < \omega\}$. Since $\delta = \sup\{\delta_n \mid n < \omega\}$, we claim this W works.

We next construct W^*. Since U is appropriate for D and W is a normal subtree of $U\lceil\delta$ closed under taking the immediate successors. For any $f \in W$ we may associate $\hat{f} \in U_\delta$ s.t. for all $\xi < \delta$ $\hat{f}\lceil\xi \in W$ holds. Let $W^* = W \cup \{\hat{f} \mid f \in W\}$. This W^* works.

Lastly for any countable $X \subseteq D_{\delta,p}$ with $X \neq \emptyset$, we have $\bigcap X \in D_{\delta,p}$ and so there is $q \in U_\delta \cap \bigcap X$ s.t. $q \supset p$ and for all $\xi < \delta$ $q\lceil\xi \in W$ holds and so $\{q\lceil\xi \mid \xi < \delta\} \in \mathrm{Gen}(U, N, p)$. This completes the proof of (3.1).

$\dashv$

(3.2) Lemma. Let U be a normal tree of height ω_1 appropriate for an array of directed sets D. Then

1. $(U, \leq, \emptyset)$ is σ-Baire.
2. $(U, \leq, \emptyset)$ is ρ-proper for all $\rho < \omega_1$.

Proof. For 1. By (3.1) for all sufficiently large regular cardinal θ and all countable elementary substructure N of H_θ with $U \in N$, we know for all $p \in U \cap N$ there exists $G \in \mathrm{Gen}(U, N, p)$ s.t. G has a lower bound in U. By (2.1) U is σ-Baire.

For 2. Let θ be a sufficiently large regular cardinal and $p \in U$. Fix a continuously increasing countable elementary substructures $\langle N_\xi \mid \xi < \omega_1\rangle$ of H_θ s.t. $U, p \in N_0$ and $\langle N_\eta \mid \eta \leq \xi\rangle \in N_{\xi+1}$ for all $\xi < \omega_1$. Let $\delta_\xi = N_\xi \cap \omega_1$ for each $\xi < \omega_1$.

Claim. $\varphi(\xi)$ holds for all $\xi < \omega_1$, where $\varphi(\xi)$ means

For any $\eta < \xi$ and any W s.t.

1. W is a normal subtree of $U\lceil\delta_\eta+1$ and $p \in W \in N_{\eta+1}$.
2. $W\lceil\delta_\eta \subseteq U \cap N_\eta \subseteq U\lceil\delta_\eta$ and $\mid W \mid = \omega$.
3. $\forall a \in W\lceil\delta_\eta\ \forall k < \omega\ a^\frown\langle k\rangle \in U$ implies $a^\frown\langle k\rangle \in W$.
4. $\forall h \in W_{\delta_\eta}\ \forall\bar{\eta} \leq \eta\ h$ is $(U, N_{\bar{\eta}})$-generic.

There is W^* s.t.

1. W^* is a normal subtree of $U\lceil\delta_\xi+1$ and $W^* \in N_{\xi+1}$.
2. $W^*\lceil\delta_\xi \subseteq U \cap N_\xi \subseteq U\lceil\delta_\xi$ and $\mid W^* \mid = \omega$.
3. $\forall a \in W^*\lceil\delta_\xi\ \forall k < \omega\ a^\frown\langle k\rangle \in U$ implies $a^\frown\langle k\rangle \in W^*$
4. $\forall h \in W^*_{\delta_\xi}\ \forall\bar{\xi} \leq \xi\ h$ is $(U, N_{\bar{\xi}})$-generic.
5. $W^*\lceil\delta_\eta+1 = W$.

Proof. By induction on $\xi < \omega_1$. Fix $\eta < \xi < \omega_1$ and W as in the hypothesis.

Case 1: ξ is a successor ordinal.

Without loss of generality we may assume $\xi = \eta+1$. For each $x \in W_{\delta_\eta}$, since $x \in N_\xi$, we may apply (3.1) proposition for x by putting $N = N_\xi$. So there is W_x s.t.

1. W_x is a normal subtree of $U\lceil\delta_\xi+1$ and $x \in W_x$.
2. $W_x\lceil\delta_\xi \subseteq U \cap N_\xi$ and $\mid W_x \mid = \omega$.
3. $\forall a \in W_x\lceil\delta_\xi\ \forall k < \omega\ a^\frown\langle k\rangle \in U$ implies $a^\frown\langle k\rangle \in W_x$.
4. For all $h \in (W_x)_{\delta_\xi}$ h is (U, N_ξ)-generic.

Let $W^* = W \cup \bigcup\{y \in W_x \mid y \supseteq x \text{ and } x \in W_{\delta_\eta}\}$. Since parameters $U, \delta_\xi, N_\xi, \langle N_{\bar{\xi}} \mid \bar{\xi} \leq \xi\rangle, \delta_\eta$ and W are all in $N_{\xi+1}$, we may assume $W^* \in N_{\xi+1}$. This W^* works.

Case 2: ξ is a limit ordinal.

Fix an increasing sequence $\langle\xi_n \mid n < \omega\rangle$ of ordinals s.t. $\eta = \xi_0$ and $\sup\{\xi_n \mid n < \omega\} = \xi$. We construct $\langle W^n \mid n < \omega\rangle$ s.t. for all $n < \omega$

0. $W^0 = W$.
1. W^n is a normal subtree of $U\lceil\delta_{\xi_n}+1$ and $W^n \in N_{\xi_n+1}$.
2. $W^n\lceil\delta_{\xi_n} \subseteq U \cap N_{\xi_n} \subseteq U\lceil\delta_{\xi_n}$ and $\mid W^n \mid = \omega$.
3. $\forall a \in W^n\lceil\delta_{\xi_n}\ \forall k < \omega\ a^\frown\langle k\rangle \in U$ implies $a^\frown\langle k\rangle \in W^n$.
4. $\forall h \in W^n_{\delta_{\xi_n}}\ \forall i \leq \xi_n\ h$ is (U, N_i)-generic.
5. $W_{n+1}\lceil\delta_{\xi_n}+1 = W_n$.

This is done by applying $\varphi(\xi_n)$ for all $1 \leq n < \omega$. Let $W^\dagger = \bigcup\{W^n \mid n < \omega\}$. Then $W^\dagger$ is a normal subtree of $U\lceil\delta_\xi$ closed under taking the immediate successors. So for each $a \in W^\dagger$ we may associate $\hat{a} \in U_{\delta_\xi}$ s.t. $a \subset \hat{a}$ and for all $\alpha < \delta_\xi$ $\hat{a}\lceil\alpha \in W^\dagger$. Let $W^* = W^\dagger \cup \{\hat{a} \mid a \in W^\dagger\}$. Since parameters $U, \delta_\xi, N_\xi, \langle N_{\bar{\xi}} \mid \bar{\xi} \leq \xi\rangle, \delta_\eta$ and W are all in $N_{\xi+1}$. We may assume $W^* \in N_{\xi+1}$. This W^* works. This completes the proof of claim.

$\dashv$

For any ξ with $0 < \xi < \omega_1$ by (3.1) proposition, there is W s.t. W satisfies the assumption in $\varphi(\xi)$ with $\eta = 0$. So there is W^* as in $\varphi(\xi)$. In particular there is $q \in U$ s.t. $q \supset p$ and for all $\bar{\xi} \leq \xi$ q is $(U, N_{\bar{\xi}})$-generic. So U is ρ-proper for all countable ordinal ρ.

$\dashv$

(3.3) Lemma. Let $(P * \dot{U}, \leq, (1, \dot{1}))$ be a two-step iteration such that

1. $(P, \leq_P, 1)$ is σ-Baire and proper.
2. For some fixed array of directed sets D, we assume $\Vdash_P$"Either $\dot{U}$ is a tree appropriate for $\check{D}$ or $\dot{U} = \{\emptyset\}$".

Then for any $(\theta, N_0, N_1, (p, \tau), G)$ such that

1. θ is a sufficiently large regular cardinal.
2. N_0 and N_1 are countable elementary substructures of H_θ with $(P * \dot{U}, \leq, (1, \dot{1})), D \in N_0 \in N_1$.
3. $(p, \tau) \in (P * \dot{U}) \cap N_0$.
4. $G \in \mathrm{Gen}(P, N_0, p) \cap N_1$ with a lower bound in P.

There is $G^* \in \mathrm{Gen}(P * \dot{U}, N_0, (p, \tau))$ such that

1. $G = \{x \in P \cap N_0 \mid \exists \sigma\ (x, \sigma) \in G^*\}$ and G^* has a lower bound in $P * \dot{U}$.
2. For any $r \in P$ if r is a lower bound of G and (P, N_1)-generic, then there is π s.t. (r, π) is a lower bound of G^* in $P * \dot{U}$.

Proof. Since $\{x \in P \mid x \Vdash$"$\dot{U}$ is appropriate for $\check{D}$" or $x \Vdash$"$\dot{U} = \{\emptyset\}$"$\}$ is a dense open subset of P and belongs to N_0. We have two cases to consider.

Case 1: There is $g \in G$ s.t. $g \Vdash$"$\dot{U} = \{\emptyset\}$".

Let $G^* = \{(x, \sigma) \in (P * \dot{U}) \cap N_0 \mid \exists g \in G\ (g, \tau) \leq (x, \sigma)\}$. This G^* works.

Case 2: There is $g \in G$ s.t. $g \Vdash$"$\dot{U}$ is appropriate for $\check{D}$".

Since $G \in \mathrm{Gen}(P, N_0, p) \cap N_1$ with a lower bound, there is a lower bound $x \in N_1$ of G. Since $(P, \leq_P, 1)$ is proper, there is $y \in P$ s.t. $y \leq_P x$ and y is (P, N_1)-generic. Notice that since y is a lower bound of G, y is also (P, N_0)-generic. Hence it is possible to fix a P-generic filter G_P over V s.t.

1. $G_P \cap N_0[G_P] = G_P \cap N_0 = G$.
2. $N_0[G_P] \cap V = N_0$.
3. $N_1[G_P] \cap V = N_1$.

Let $\delta_0 = N_0 \cap \omega_1$. Then $\delta_0 = N_0[G_P] \cap \omega_1$ holds. We make use of G_P to define G^* for convenience sake. Since $N_0[G_P]$ is a countable elementary substructure of $H_\theta^{V[G_P]}$ with $\dot{U}[G_P] \in N_0[G_P]$ and $\tau[G_P] \in \dot{U}[G_P] \cap N_0[G_P]$. We may apply (3.1) proposition with $X = N_1 \cap D_{\delta_0, \tau[G_P]}$. So there is $q \in \dot{U}[G_P] \cap {}^{\delta_0}\omega \cap \bigcap X$ s.t.

4. $\{q\lceil\xi \mid \xi < \delta_0\} \in \mathrm{Gen}(\dot{U}[G_P], N_0[G_P], \tau[G_P])$.

For every $\xi < \delta_0$, since $q\lceil\xi \in N_0[G_P] \cap \dot{U}[G_0] \cap {}^{\xi}\omega = N_0 \cap \dot{U}[G_P] \cap {}^{\xi}\omega$ and $(P, \leq_P, 1)$ is σ-Baire, there is $\langle \tau_\xi \mid \xi < \delta_0 \rangle \in V$ s.t. for all $\xi < \delta_0$

5. $\tau_\xi \in N_0$.
6. $\Vdash_P$"$\tau_\xi \in \dot{U}$".
7. $\exists\, x_\xi \in G\ (= N_0[G_P] \cap G_P)$ s.t. $x_\xi \Vdash$"$\tau_\xi = q\lceil\xi$".
8. $\exists\, x \in G$ s.t. $x \Vdash$"$\mathrm{dom}(\tau) = \check{\alpha}$" for a unique $\alpha < \omega_1$, let $\tau_\alpha = \tau$.

Define $G^* = \{(x, \sigma) \in (P * \dot{U}) \cap N_0 \mid \exists g \in G\ \exists \xi < \delta_0\ (g, \tau_\xi) \leq (x, \sigma)\}$ in V.

Claim 1. $G^* \in \mathrm{Gen}(P * \dot{U}, N_0, (p, \tau))$.

Proof. It is clear that $G = \{x \in P \cap N_0 \mid \exists \sigma\ (x, \sigma) \in G^*\}$, $(p, \tau) \in G^* \subseteq (P * \dot{U}) \cap N_0$ and G^* is upward-closed in $(P * \dot{U}) \cap N_0$. To see G^* is directed, suppose $g_1, g_2 \in G$, $\xi_1, \xi_2 < \delta_0$. Since $x_{\xi_1} \Vdash$"$\tau_{\xi_1} = q\lceil\xi_1$", $x_{\xi_2} \Vdash$"$\tau_{\xi_2} = q\lceil\xi_2$" and $x_{\xi_1}, x_{\xi_2} \in G$, there is $g_3 \in G$ s.t. $g_3 \leq_P g_1, g_2, x_{\xi_1}, x_{\xi_2}$. We may assume $\xi_1 \leq \xi_2$ so $g_3 \Vdash$"$\tau_{\xi_2} \supseteq \tau_{\xi_1}$" and so $(g_3, \tau_{\xi_2}) \leq (g_1, \tau_{\xi_1}), (g_2, \tau_{\xi_2})$. To show G^* takes care of every open dense subset $C \in N_0$ of $P * \dot{U}$. We first note that $\{\dot{d}[G_P] \mid \exists\, d \in G_P\ (d, \dot{d}) \in C\}$ is an open dense subset of $\dot{U}[G_P]$ which is in $N_0[G_P]$. Since $\{q\lceil\xi \mid \xi < \delta_0\} \in \mathrm{Gen}(\dot{U}[G_P], N_0[G_P], \tau[G_P])$, there is $\xi < \delta_0$ s.t. for some $(d, \dot{d}) \in C \cap N_0[G_P] = C \cap N_0$, $d \in G = G_P \cap N_0$ and $q\lceil\xi = \dot{d}[G_P]$ hold. So there is $z \in G_P \cap N_0[G_P] = G$ s.t. $z \Vdash_P$"$q\lceil\xi = \dot{d}$". Since $d, x_\xi \in G$, we may assume $z \leq d, x_\xi$ and so $z \Vdash$"$\tau_\xi = \dot{d}$". Namely we got $(z, \tau_\xi) \in G^*$ s.t. $(z, \tau_\xi) \leq (d, \dot{d}) \in C \cap N_0$. This completes the proof of claim 1.

$\dashv$

Claim 2. For any $r \in P$ if r is a lower bound of G and is (P, N_1)-generic then $r \Vdash$"$\check{q} \in \dot{U}$". And so there is π s.t. (r, π) is a lower bound of G^* in $P * \dot{U}$.

Proof. Let $f \in {}^{\delta_0 >}\omega \cap N_0$ be s.t. there is $g \in G$ with $g \Vdash$"$\tau = f$". There is a P-name $\dot{A} \in N_1$ s.t.

$\Vdash_P$"If $\dot{U}$ is appropriate for $\check{D}$ and $\check{f} \in \dot{U}\lceil\delta_0$, then $\dot{A} \in \check{D}_{\delta_0, f}$ and for all $h \in \dot{A}$ if for all $\xi < \delta_0$ $h\lceil\xi \in \dot{U}$ and $h \supset \check{f}$ hold then $h \in \dot{U}$".

This is possible because parameters $\dot{U}, D, D_{\delta_0, f}, \delta_0, f$ and $(P, \leq_P, 1)$ are all in N_1. Since r is a lower bound of G and is (P, N_1)-generic we get $r \Vdash_P$"$\dot{A} \in N_1[\dot{G}] \cap D_{\delta_0, f} = N_1 \cap D_{\delta_0, f} = X$". Since $q \supset \tau[G_P] = f$, $q \in {}^{\delta_0}\omega \cap \bigcap X$ and $r \Vdash$"$\forall \xi < \delta_0\ q\lceil\xi \in \dot{U}$", we conclude $r \Vdash$"$\check{q} \in \dot{U}$". This completes the proof of claim 2, case 2 and (3.3).

$\dashv$

(3.4) Lemma ($V = L$). Let $\langle D^\zeta \mid \zeta < \omega_2 \rangle$ enumerate the arrays of directed sets $D = \langle D_{\alpha, f} \mid \alpha \in \Omega,\ f \in {}^{\alpha >}\omega \rangle$ s.t. for all $\alpha \in \Omega$ and all $f \in {}^{\alpha >}\omega$ $\mid D_{\alpha, f} \mid \leq \omega_1$. Fix a function $\pi : \omega_2 \to \omega_2 \times \omega_2 \times \omega_2$ s.t.

1. If $\pi(\alpha) = (\zeta, \eta, \xi)$ then $\zeta, \eta, \xi \leq \alpha$.
2. For all $(\zeta, \eta, \xi) \in \omega_2 \times \omega_2 \times \omega_2$, $\{\alpha < \omega_2 \mid \pi(\alpha) = (\zeta, \eta, \xi)\}$ is cofinal in ω_2.

We can define a countable support iteration $((P_\alpha, \leq_\alpha, 1_\alpha)_{\alpha\leq\omega_2}, (\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)_{\alpha<\omega_2})$ and $\langle \tau_{\eta,\xi} \mid \eta, \xi < \omega_2 \rangle$ such that for all $\alpha < \omega_2$

(1) P_α is ρ-proper for all $\rho < \omega_1$.

(2) P_α is σ-Baire.

(3) P_α has a dense subset of size at most ω_1 and so has the ω_2-c.c..

(4) P_α preserves every cofinality and so cardinality.

(5) P_α preserves GCH.

(6) For all $\xi < \omega_2$ $\tau_{\alpha,\xi}$ is a P_α-name s.t. $\Vdash_{P_\alpha}$ "$\tau_{\alpha,\xi} \subseteq {}^{\omega_1>}\omega$".

(7) For all P_α-name τ, there is $\xi < \omega_2$ s.t. $\Vdash_{P_\alpha}$ "$\tau \subseteq {}^{\omega_1>}\omega$ implies $\tau = \tau_{\alpha,\xi}$".

(8) Let $\pi(\alpha) = (\zeta, \eta, \xi)$, then $\Vdash_{P_\alpha}$ "If $\tau_{\eta,\xi}[\dot{G}_\alpha\lceil\eta]$ is a tree appropriate for $\check{D}^\zeta$ then $\dot{Q}_\alpha = \tau_{\eta,\xi}[\dot{G}_\alpha\lceil\eta]$ else $\dot{Q}_\alpha = \{\emptyset\}$".

Proof. The construction is by recursion on $\alpha < \omega_2$. Suppose we have constructed $((P_\beta, \leq_\beta, 1_\beta)_{\beta\leq\alpha}, (\dot{Q}_\beta, \dot{\leq}_\beta, \dot{1}_\beta)_{\beta<\alpha})$ and $\langle \tau_{\eta,\xi} \mid \eta < \alpha,\ \xi < \omega_2 \rangle$. We want to get $\dot{Q}_\alpha$ and $\langle \tau_{\alpha,\xi} \mid \xi < \omega_2 \rangle$. Since P_α has a dense subset of size ω_1 and $\Vdash_{P_\alpha}$ "$\omega_1 = \omega_1^V$ and ${}^{\omega_1>}\omega = ({}^{\omega_1>}\omega)^V$", we may get $\langle \tau_{\alpha,\xi} \mid \xi < \omega_2 \rangle$ s.t. (6) and (7) are satisfied. If $\pi(\alpha) = (\zeta, \eta, \xi)$, then $\eta \leq \alpha$ and so we have a P_η-name $\tau_{\eta,\xi}$. Hence it makes sense to define $\dot{Q}_\alpha$ as in (8). Then we have by (3.2) lemma

$$\Vdash_{P_\alpha} \text{"} \mid \dot{Q}_\alpha \mid \leq \omega_1,\ \dot{Q}_\alpha \text{ is } \rho\text{-proper for all } \rho < \omega_1 \text{ and is } \sigma\text{-Baire"}.$$

So $P_{\alpha+1}$ also satisfies (1) through (5). All we left to show is that (1) through (5) hold for the limit ordinal α. But the iteration lemma for ρ-proper (0.4) takes care of (1). We combine (2.1),(2.2),(3.2) and (3.3) to get (2). The iteration theorem for proper (0.3) gives us (3). Now (4) and (5) follow from (1),(2) and (3). This completes the construction.

$\dashv$

(3.5) Theorem ($V = L$). Let $((P_\alpha, \leq_\alpha, 1_\alpha)_{\alpha\leq\omega_2}, (\dot{Q}_\alpha, \dot{\leq}_\alpha, \dot{1}_\alpha)_{\alpha<\omega_2})$ be as in (3.4). Then we have

(1) P_{ω_2} is ρ-proper for all $\rho < \omega_1$.

(2) P_{ω_2} is σ-Baire.

(3) P_{ω_2} has the ω_2-c.c. and has a dense subset of size ω_2.

(4) P_{ω_2} preserves every cofinality and so cardinality.

(5) P_{ω_2} preserves GCH.

(6) $\Vdash_{P_{\omega_2}}$ "SAD".

(7) For all ω_1-Souslin tree T $\Vdash_{P_{\omega_2}}$ "$\check{T}$ remains to be an ω_1-Souslin tree ".

Proof. We know (1) and (2) are dealt with in the same way as in (3.4). For (3), we make use of (3) of (3.4) and a usual Δ-system argument under CH. Hence (4) and (5) hold. We deduce (7) by putting together (1.3) and (1.5). So we concentrate on showing (6). Suppose D is an array of directed sets and $p \Vdash_{P_{\omega_2}}$ "$\dot{U}$ is appropriate for $\check{D}$". Since

P_{ω_2} has the ω_2-c.c., there is $(\eta, \xi) \in \omega_2 \times \omega_2$ s.t. $p \Vdash_{P_{\omega_2}}$ "$\dot{U} = \tau_{\eta,\xi}[\dot{G}_{\omega_2}\lceil\eta]$". For each $\delta \in \Omega$ and each $f \in {}^{\delta>}\omega$, let $\dot{A}_{\delta,f}$ be a P_{ω_2}-name s.t.

$p \Vdash_{P_{\omega_2}}$ "$\check{f} \in \dot{U}\lceil\delta$ implies ($\dot{A}_{\delta,f} \in \check{D}_{\delta,f}$ and for any $h \in \dot{A}_{\delta,f}$ if for all $\xi < \delta$ $h\lceil\xi \in \dot{U}$ and $h \supset \check{f}$, then $h \in \dot{U}$)".

By the ω_2-c.c., there is a countably complete directed subsets $D'_{\delta,f}$ of $D_{\delta,f}$ s.t. $|\, D'_{\delta,f} \,| \leq \omega_1$ and $p \Vdash_{P_{\omega_2}}$ "$\check{f} \in \dot{U}\lceil\delta$ implies $\dot{A}_{\delta,f} \in D'_{\delta,f}$". Choose $\zeta < \omega_2$ s.t. $D^\zeta = \langle D'_{\delta,f} \mid \delta \in \Omega,$ $f \in {}^{\delta>}\omega\rangle$. Let $\alpha < \omega_2$ be s.t. $\pi(\alpha) = (\zeta, \eta, \xi)$ and $\mathrm{supp}(p) \subset \alpha$. So $D'_{\delta,f} = (D^\zeta)_{\delta,f} \subseteq D_{\delta,f}$ for all $\delta \in \Omega$ and all $f \in {}^{\delta>}\omega$.

Claim. $p\lceil\alpha \Vdash_{P_\alpha}$ "$\tau_{\eta,\xi}[\dot{G}_\alpha\lceil\eta]$ is appropriate for $\check{D}^\zeta$".

Proof. Let G_α be an arbitrary P_α-generic filter over V with $p\lceil\alpha \in G_\alpha$. Since $\tau_{\eta,\xi}$ is a P_η-name and $\eta \leq \alpha$, it makes sense to consider $\tau_{\eta,\xi}[G_\alpha\lceil\eta]$ in $V[G_\alpha]$. Since $\mathrm{supp}(p) \subset \alpha$, we may fix a P_{ω_2}-generic filter G_{ω_2} s.t. $p \in G_{\omega_2}$ and $G_{\omega_2}\lceil\alpha = G_\alpha$. Let $G_\eta = G_\alpha\lceil\eta = G_{\omega_2}\lceil\eta$ and $U = \dot{U}[G_{\omega_2}] = \tau_{\xi,\eta}[G_\eta]$. Now since U is appropriate for D in $V[G_{\omega_2}]$, U is a normal tree of height ω_1 in $V[G_{\omega_2}]$. Since $({}^{\omega_1>}\omega)^{V[G_{\omega_2}]} = ({}^{\omega_1>}\omega)^V$, U is a normal tree of height ω_1 in $V[G_\alpha]$. Since $\dot{A}_{\delta,f}[G_{\omega_2}] \in (D^\zeta)_{\delta,f}$ for all $\delta \in \Omega$ and all $f \in U\lceil\delta$. For any $\delta \in \Omega$ and any $f \in U\lceil\delta$, there is A $(= \dot{A}_{\delta,f}[G_{\omega_2}])$ in $(D^\zeta)_{\delta,f}$ such that for all $h \in A$ if for all $\xi < \delta$ $h\lceil\xi \in U$ and $h \supset f$, then $h \in U$ in $V[G_\alpha]$. Next let $\delta \in \Omega$ and W be a normal subtree of $U\lceil\delta$ closed under taking the immediate successors in $V[G_\alpha]$. For any $f \in W$ and any $B \in (D^\zeta)_{\delta,f}$, since W is a normal subtree of $U\lceil\delta$ closed under taking the immediate successors in $V[G_{\omega_2}]$, there is $h \in B$ s.t. $h \supset f$ and for all $\xi < \delta$ $h\lceil\xi \in W$. This is true in $V[G_\alpha]$ as well by absoluteness. This completes the proof of claim.

Hence $p\lceil\alpha \Vdash_{P_\alpha}$ "$\dot{Q}_\alpha = \tau_{\eta,\xi}[\dot{G}_\alpha\lceil\eta]$". So $p\lceil\alpha + 1 \Vdash_{P_{\alpha+1}}$ "$\tau_{\eta,\xi}[\dot{G}_{\alpha+1}\lceil\eta]$ gains a cofinal path through it" and so $p \Vdash_{P_{\omega_2}}$ "$\dot{U}$ gains a cofinal path through it".

$\dashv$

(3.6) Note (CH). There is a notion of forcing P s.t.

1. P is strongly proper and so preserves every ω_1-Souslin tree in the ground model.
2. P is σ-Baire and so preserves CH.
3. The negation of $\diamondsuit$ holds in the generic extensions.

The construction of P is similar to (3.4) and (3.5).

References

[1] U.Avraham, K.Devlin and S.Shelah, The consistency with CH of some consequences of Martin's Axiom plus $2^{\aleph_0} > \aleph_1$, Israel Journal of Math., Vol. 31, No.1, 1978.

[2] S.Shelah, Propre Forcing, Lecture Notes in Math., Vol. 940, Springer-Verlag, 1982.