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On Splitting Numbers

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Abstract. We shall discuss about splitting numbers of uncountable regular cardinals.

1. Question about splitting numbers

The author introduced a question about splitting numbers of uncountable regular cardinals at a talk in Aug. 4. 1992 at RIMS, Kyoto.

Let κ be an infinite cardinal. $\mathcal{S} \subseteq [\kappa]^\kappa (= \{X \subseteq \kappa : |X| = \kappa\})$ is called a splitting family on κ if for all $X \in [\kappa]^\kappa$ there exists an $A \in \mathcal{S}$ such that $|X \cap A| = |X \setminus A| = \kappa$. The splitting number of κ is the minimum cardinality of a splitting family on κ. We denote it by $s(\kappa)$.

In particular, $s(\omega)$ is the original splitting number, which is one of the so-called six cardinals and the following is well-known.

Fact. *[2]*

(1) $s(\omega) \geq \omega_1$.
(2) *Con(ZFC) implies* $Con(ZFC + s(\omega) \geq \omega_2)$.

What about uncountable regular cardinals ? In 1991, M. Motoyoshi studied splitting numbers of uncountable regular cardinals under the supervision of S. Kamo.

Fact. *[5] Let κ be an uncountable regular cardinal. Then κ is strongly inaccessible $\iff s(\kappa) \geq \kappa$.*

At that time I pointed out the following fact.

PROPOSITION. *Let κ be an uncountable regular cardinal. Then κ is weakly compact $\iff$ $s(\kappa) \geq \kappa^+$.*

PROOF: ($\Rightarrow$) Suppose κ is weakly compact. Assume for a contradiction that $\mathcal{S} = \{S_\alpha : \alpha < \kappa\}$ is a splitting family on κ. For each α, let ${S_\alpha}^0 = S_\alpha$ and ${S_\alpha}^1 = \kappa \setminus S_\alpha$. Put $T = \{f \in {}^{<\kappa}2 : |\kappa \cap \bigcap \{{S_\alpha}^{f(\alpha)} : \alpha \in \mathbf{dom} f\}| = \kappa\}$. Then $(T, \subseteq)$ makes a κ-tree. Hence there is a $g : \kappa \to 2$ such that for all $\alpha < \kappa$ $\;g\lceil\alpha \in T$. Consider the next two cases. (i) When the sequence $\langle \bigcap \{{S_\alpha}^{g(\alpha)} : \alpha < \xi\} : 0 < \xi < \kappa \rangle$ is eventually constant. (ii) Otherwise. In either cases, using the sequence above, we can get an $X \in [\kappa]^\kappa$ which witnesses that $\mathcal{S}$ is not a splitting family on κ, a contradiction.

($\Leftarrow$) Suppose κ is uncountable regular and $s(\kappa) \geq \kappa^+$. By Motoyoshi's result above, κ is strongly inaccessible. Assume for a contradiction that κ is not weakly compact. Let $(T, \prec)$ be a well-pruned κ-Aronszajn tree which witnesses this assumption. For each $t \in T$, let $S_t = \{u \in T : t \preceq u\}$ and put $\mathcal{S} = \{S_t : t \in T \text{ and } |S_t| = \kappa\}$. Fix an arbitrary $X \in [T]^\kappa$. Then it is easily verified that for some $\alpha < \kappa$, the α-th level of T has two distinct members, say u and v, so that both $X \cap S_u$ and $X \cap S_v$ has size κ ; if not, one can get a chain of length κ in T, contradicting that T is κ-Aronszajn. Since X was arbitrary, $\mathcal{S}$ makes a splitting family on T. However, the size of $\mathcal{S}$ is clearly at most κ. Therefore $s(\kappa) \leq \kappa$, a contradiction (Q.E.D.)

Question 1 Does $Con(ZFC + \exists$ a measurable cardinal) imply $Con(ZFC{+}\exists\kappa$ s.t. κ is an uncountable regular cardinal and $s(\kappa) \geq \kappa^{++})$?

REMARK: $Con(s(\kappa) \geq \kappa^{++}, \kappa$ uncountable regular) implies $Con(\exists$ a measurable). Suppose κ is an uncountable regular cardinal and $s(\kappa) \geq \kappa^{++}$. Let K be the Dodd-Jensen core model. There exists an $X \in [\kappa]^\kappa$ such that no $Y \in P(\kappa) \cap K$ can split X. Then $U = \{Y \in P(\kappa) \cap K : |X \setminus Y| < \kappa\}$ makes a $K{-}\kappa$-complete ultrafilter. Hence by Dodd-Jensen's result [1], there is an inner model of a measurable cardinal.

2. Answer(s)

In August 1992, S.Kamo and T.Miyamoto independently showed a slight stronger result than the following.

THEOREM. *[3], [4] $Con(ZFC + \exists$ a supercompact cardinal) implies*

$Con(ZFC + \exists\kappa$ *s.t.* κ *is an uncountable regular cardinal and* $s(\kappa) \geq \kappa^{++})$

After that, in October 1992, J. Zapletal in U. S. told us that he improved the lower bound.

Answer. *[6]* $Con(ZFC + \exists\kappa$ *s.t.* κ *is an uncountable regular cardinal and* $s(\kappa) \geq \kappa^{++})$ *implies* $Con(ZFC + \exists$ *a measurable cardinal* κ *such that* $o(\kappa) = \kappa^{++})$.

Thus we had negative answer for **Question 1**. However, I don't know the exact consistecy strength yet.

Question 2 Find a large cardinal axiom which is equi-consistent to the existence of uncountable regular κ s. t. $s(\kappa) \geq \kappa^{++}$.

References

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