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Some Problems in Value Distribution and Hyperbolic Manifolds

Dedicated to Professor S. Kobayashi

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We will discuss open problems in the Nevanlinna theory, the theory of hyperbolic manifolds, and Diophantine geometry. Some of them are already posed ones and known, and the others may be new.

§1. Nevanlinna Theory

1.1. Transcendental Bezout problem

The transcendental Bezout problem, say, on $\mathbf{C}^n$ asks if it is possible to estimate the growth of the intersection of two analytic (effective) cycles, X_1 and X_2 by the growths of $X_i, i=1,2$. In general, the answer is negative; M. Cornalba and B. Shiffman [CS] constructed an example of $X_i, i=1,2$ in $\mathbf{C}^2$ such that the orders of $X_i, i=1,2$ are 0, but that of $X_1 \cap X_2$ can be arbitrarily large. On the other hand, W. Stoll [S] established an *average Bezout theorem* as follows. Let $X_i, i=1,...,q$ be effective divisors defined by entire functions $F_i(z), i=1,...,q$ on $\mathbf{C}^n$ with $F_i(O)=1$. One says that X_i or $F_i(z)$, $i=1,...,q$ define a complete intersection $Y=\cap_{i=1}^q X_i$ if Y is of pure dimension $n-q$, or empty, and that $F_i(z), i=1,...,q$ define a stable complete intersection if $F_{it}(z)=F_i(t_1z_1,...,t_nz_n), i=1,...,q$ define complete intersections for all $t=(t_1,...t_n)$ with $0<t_j\leq 1$. Put $Y_t=\cap_{i=1}^q\{F_{it}(z)=0\}$ (with multiplicities). Let $N(r;Y)$ denote the ordinary counting function of Y and set

$$\hat{N}(r,Y)=\int_0^1\cdots\int_0^1 N(r;Y_t)dt_1\cdots dt_n.$$

Let $M(r;F_i)$ denote the maximum modulus function of $F_i(z)$. Then W. Stoll [S] proved

1.1.1 Theorem. *For any $\theta>1$ there is a positive constant C_θ such that*

$$\hat{N}(r,Y)\leq C_\theta\prod_{i=1}^q \log M(\theta r;F_i).$$

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In the proof the following type of estimate plays an essential role:

$$\int_0^1 \cdots \int_0^1 \log \frac{1}{|F_t(z)|} dt_1 \cdots dt_n \leq C_\theta \log M(\theta r; F)$$

for an entire function $F(z)$ and $\|z\| < r$.

Any probabilistic measure μ in the unit disk would give a similar result if the above type of estimate holds. Therefore it is interesting to ask

1.1.2 Problem. *Characterize what kind of measures can be applied to get an average Bezout estimate?*

1.2. Nevanlinna's inverse problem

For a meromorphic function F on $\mathbf{C}$ we have Nevanlinna's defect relation:

$$\sum_{a \in \mathbf{P}^1(\mathbf{C})} \delta_F(a) \leq 2.$$

The defect $\delta_F(a)$ has a property such that $0 \leq \delta_F(a) \leq 1$ and $\delta_F(a) = 1$ if F omits the value a. As a consequence, there are at most countably many $a \in \mathbf{P}^1(\mathbf{C})$ such that $\delta_F(a) > 0$; such a is called Nevanlinna's exceptional value. Conversely, for a given (at most) countably many numbers $0 < \delta_i \leq 1$ with $\sum \delta_i \leq 2$ and points $a_i \in \mathbf{P}^1(\mathbf{C})$. D. Drasin [D] proved the existence of a meromorphic function F such that $\delta_F(a_i) = \delta_i$. It is known that the defect relation holds for a linearly non-degenerate meromorphic mapping $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ with respect to hyperplanes in general position (H. Cartan, L. Ahlfors, W. Stoll), and for a dominant meromorphic mapping $f : \mathbf{C}^m \to V$ into a projective manifold V with respect to hypersurfaces with simple normal crossings (P. Griffiths et al.). W. Stoll asked

1.2.1 Problem. *Does Nevanlinna's inverse problem hold for $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ with respect to hyperplanes in general position, or for a dominant meromorphic mapping $f : \mathbf{C}^m \to V$ into a projective manifold with respect to hypersurfaces with simple normal crossings?*

This may be a hard problem, but the following will be easier.

1.3. Order of convergence of Nevanlinna's defects

Given a divergent sequence $\{z_i\}_{i=1}^\infty$, we classically defines its order by the infimum of $\rho > 0$ such that $\sum_{i=1}^\infty |z_i|^{-\rho} < \infty$. Thus for a sequence $\{w_i\}_{i=1}^\infty$ converging to 0 we may define its order of convergence by the supremum of $\alpha > 0$ such that $\sum_{i=1}^\infty |w_i|^\alpha < \infty$. As seen in 1.2, there are at most countably many Nevanlinna's defects values a_i of a meromorphic function F on $\mathbf{C}$. W.K. Hayman [Ha] proved that the order of convergence of $\{\delta_F(a_i)\}$ is 1/3 for F of finite lower order λ; i.e.,

1.3.1 $$\sum \delta_F(a_i)^\alpha \leq A(\alpha, \lambda) < \infty$$

for $\alpha > 1/3$. Moreover, A. Weitsman [W] proved the above bound for $\alpha = 1/3$.

Let f denote a linearly non-degenerate meromorphic mapping $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$. Then V.I. Krutin' [Kr] proved that

1.3.2 Theorem. *Let f be of finite lower order λ and $\alpha > 1/3$. Then there is a constant $A(\alpha, \lambda) > 0$ such that*

$$\sum \delta_f(D_i)^\alpha \leq A(\alpha, \lambda) < \infty$$

for any family of hyperplanes D_i of $\mathbf{P}^n(\mathbf{C})$ in general position.

1.3.3 Conjecture. *The above estimate still holds for $\alpha = 1/3$.*

Now let f be a dominant meromorphic mapping $f : \mathbf{C}^m \to V$ as in 1.1, and D_i hypersurfaces of V with simple normal crossings.

1.3.4 Problem. *Does the estimate*

$$\sum \delta_f(D_i)^\alpha \leq A(\alpha, \lambda) < \infty$$

hold for $\alpha \geq 1/3$ and for f of finite lower order λ?

1.4. Order of a meromorphic mapping into a projective manifold

Let $f : \mathbf{C}^n \to V$ be a dominant meromorphic mapping into a projective manifold V of dimension n.

1.4.1 Conjecture. *If the order of f is less than 2, V is unirational.*

Note that if the order of f is less than 2, then any global holomorphic section of tensors of $\Omega^k(V)$, $k = 0, 1, ..., n$ must identically vanish; hence, V is rational for $n \leq 2$, and for $n = 3$ V is rationally connected by J. Kollár, Y. Miyaoka and S. Mori [KMM].

Here one also should remark that any non-constant holomorphic mapping of $\mathbf{C}^m$ into a complex torus has order ≥ 2. Similarly, one may ask

1.4.2 Conjecture. *If V admits a holomorphic curve $f : \mathbf{C} \to V$ of order less than 2, then V contains a rational curve.*

In other words, can one construct a holomorphic curve $g : \mathbf{C} \to V$ from f such that $T_g(r) = O(\log r)$?

1.5. Holomorphic curves

Let $f : \mathbf{C} \to V$ be an algebraically non-degenerate holomorphic curve into a projective manifold V, and D_i, $i = 1, ..., q$ hypersurfaces of V with simple normal crossings whose first Chern classes are the same $\omega > 0$. P. Griffiths [Gr] posed

1.5.1 Conjecture. *The following defect relation holds:*

$$\sum_{i=1}^{q} \delta_f(D_i) \leq \left[\frac{c_1(-K_V)}{\omega}\right],$$

where $\left[\frac{c_1(-K_V)}{\omega}\right] = \inf\{t \in \mathbf{R}; t\omega + c_1(K_V) > 0\}$.

Assume that V is an Abelian variety A. Then $c_1(K_A) = 0$. By making use of the solution of Bloch's conjecture (A. Bloch [B], T. Ochiai [O], M. Green and P. Griffiths [GG], and Y. Kawamata [K]), the above conjecture implies that

any non-constant holomorphic holomorphic curve into A can not miss a smooth ample divisor of A.

This is a part of the following conjecture due to Griffiths [Gr]

1.5.2 Conjecture. *Any non-constant holomorphic curve into A intersects an ample divisor D of A.*

Ax [Ax] confirmed this when f is a one-parameter subgroup, answering a question raised by S. Lang. M. Green [G2] proved that $A \backslash D$ is complete hyperbolic if D contains no translation of an Abelian subgroup. J. Noguchi [No1] proved Conjecture 1.5.2 in the case where D contains two distinct irreducible components which are ample and homologous to each other. His arguments were based on an inequality of the second main theorem type ([No1], [No3], [No4], [No6]):

1.5.3 Theorem. *Let V be an n-dimensional complex projective manifold, D a complex hypersurface of V and $\alpha : V \backslash D \to \mathcal{A}$ the quasi-Albanese mapping. Let $f : \mathbf{C} \to V$ be a holomorphic curve. Assume that the closure of $\alpha(V \backslash D)$ in $\mathcal{A}$ is of dimension n and of log-general type, and that $f(\mathbf{C})$ is non-degenerate with respect to the linear system of $H^0(V, \Omega^n(\log D))$. Then we have the following inequality of the second main theorem type:*

$$KT_f(r) \leq N(r, f^*D) + \text{small order term},$$

where K is a positive constant independent of f.

If $K > 1$ for an Abelian variety, then this implies Conjecture 1.5.2. Thus it is interesting to investigate K.

1.5.4 Problem([No2]). *Compute the above positive constant K.*

See [No9] for a new type application of the Nevanlinna calculus to a moduli problem. Cf. [GG], [No8], [LY] and [Lu] to see how the methods used in the Nevalinna theory are related to the topics dicussed in the next section.

§2. Hyperbolic Manifolds

2.1. Finiteness and rigidity theorems

Let X and Y be compact complex spaces. Assume that Y is hyperbolic. In 1974 S. Lang [L1] posed a conjecture to claim the finiteness of the number of surjective holomorphic mappings from X onto Y (cf. also Kobayashi [Ko2]). This has motivated many works. See Zaidenberg-Lin [ZL]. The first result in this direction was given by S. Kobayashi and T. Ochiai [KO]:

2.1.1 Theorem. *There are only finitely many surjective meromorphic mappings from a compact complex space onto a complex space of general type.*

At the Taniguchi Symposium, Katata 1978, T. Sunada asked the following problem:

2.1.2 Problem. *Let $f, g : M \to N$ be two holomorphic mappings from a compact complex manifold M onto another N of general type. If f and g are topologically homotopic, then $f \equiv g$.*

This is true for Kähler N with non-positive curvature and negative Ricci curvature (Hartman [H] and Lichnerovich's theorem), but still open for N with $K_N > 0$.

The above Lang's finiteness conjecture was affirmatively solved by Noguchi [No10] in 1992:

2.1.3 Theorem. *Let X and Y be as above. Then* $\mathrm{Mer}_{\mathrm{surj}}(X, Y)$ *is finite.*

It is interesting to recall the following conjecture also by T. Sunada [Su]:

2.1.4 Conjecture. *Let $f, g : X \to Y$ be two topologically homotopic surjective holomorphic mappings. Then $f \equiv g$.*

In the case of C-hyperbolic manifolds there are works by A. Borel and R. Narasimhan [BN] and Y. Imayoshi [I1], [I2], [I3]. H. Nakamura [Na] recently gave a partial answer to this conjecture for varieties of a special type, too.

Lately, Makoto Suzuki [SuzMk] proved the non-compact version of Theorem 2.1.3. In view of his result we may ask

2.1.5 Conjecture. i) *Let Y be a complete hyperbolic complex space with finite hyperbolic volume. Then* $\mathrm{Aut}(Y)$ *is finite.*

ii) *Let X be also a complete hyperbolic complex space with finite hyperbolic volume. Then* $\mathrm{Hol}_{\mathrm{dom}}(X, Y)$ *is finite.*

G. Arérous and S. Kobayashi [AK] proved that if M is a complete Riemannian manifold of non-positive curvature with finite volume, and if M admits no non-zero parallel vector field, then there are only finitely many isometries. The proof based firstly on the fact that $\mathrm{Is}(M)$ is compact. In the case of Conjecture 2.1.5 $\mathrm{Aut}(Y)$ and $\mathrm{Hol}_{\mathrm{dom}}(X, Y)$ are compact, too.

In the case of dimension 1, Theorem 2.1.3 is de Franchis' theorem, and we know a stronger theorem called Severi's theorem.

One may ask for a similar statement for compact hyperbolic complex spaces.

2.1.6 Conjecture. *We fix a compact complex space X and set*

$$\mathrm{Sev}(X) = \{(f, Y); Y \textit{ is hyperbolic and } f : X \to Y \textit{ is surjective, holomorphic}\}.$$

Then $\mathrm{Sev}(X)$ *is finite.*

Making use of the idea of the proof of Mordell's conjecture over function fields for hyperbolic spaces proved by Noguchi [No10], Theorem B (see 2.2), we see that any element of $\mathrm{Sev}(X)$ is rigid ([No10]).

Let $(f, Y) \in \mathrm{Sev}(X)$. Then the diameter and the volume of Y are bounded by those of X. In light of these facts, it is interesting to ask

2.1.7 Problem. *There is a positive constant $v(n)$ such that the hyperbolic volume* $\mathrm{Vol}(Y) \geq v(n)$ *for every hyperbolic irreducible complex space Y of dimension n.*

2.2. Hyperbolic fiber spaces and extension problems

In Noguchi [No10] (cf. also [No5]) the analogue of Mordell's conjecture over function fields for hyperbolic space which was conjectured by S. Lang [L1] was affirmatively solved:

2.2.1 Theorem. *Let R be a non-singular Zariski open subset of $\bar{R}$ with boundary ∂R and $(\mathcal{W}, \pi, R)$ a hyperbolic fiber space such that*

2.2.2 $(\mathcal{W}, \pi, R)$ *is hyperbolically imbedded into $(\overline{\mathcal{W}}, \bar{\pi}, \bar{R})$ along ∂R.*

Then $(\mathcal{W}, \pi, R)$ contains only finitely many meromorphically trivial fiber subspaces with positive dimensional fibers, and there are only finitely many holomorphic sections except for constant ones in those meromorphically trivial fiber subspaces.

It is a question if Condition 2.2.2 is really necessary. In the case of 1-dimensional base and fibers, this is automatically satisfied by a suitable compactification (see J. Noguchi [No7]). On the other hand, we know an example of hyperbolic fiber space $(\mathcal{W}, \pi, \Delta^*)$ over the punctured disk Δ^* such that even after a finite base change it has no compactification at the origin into which $(\mathcal{W}, \pi, \Delta^*)$ is hyperbolically embedded along (over) the origin (see Noguchi [No11]). Condition 2.2.2 was essentially used in the proof to claim the extension and convergence of holomorphic sections, so that the space of holomorphic sections forms a compact complex space.

2.2.3 Question. *Is there any example of a hyperbolic fiber space $(\mathcal{W}, \pi, R)$ of which holomorphic sections do not form a compact space.*

From the viewpoint of holomorphic extension problem it is interesting to ask

2.2.4 Problem. *Let $(\mathcal{W}, \pi, \Delta^*)$ be a hyperbolic fiber space which does not satisfy* 2.2.2 *at the origin. Is there a holomorphic section having an essential singularity at the origin?*

The following is based on the same thought.

2.2.5 Problem. *Let Y be a hyperbolic complex space (or a hyperbolic Zariski open subset of a compact complex space) which does not admit any relatively compact imbedding into another complex space so that Y is hyperbolically imbedded into it. Then, is there a holomorphic mapping $f : \Delta^* \to Y$ with essential singularity at the origin.*

In the case of a compact Riemann surface M, T. Nishino [Ni] generalized the one point singular set to the set of capacity zero:

2.2.6 Theorem. *Let $E \subset \Delta$ be a closed subset of capacity zero and $f : \Delta \backslash E \to M$ a holomorphic mapping. If the genus of M is greater than* 1, *then f has a holomorphic extension over Δ.*

Later, Masakazu Suzuki [SuzMs] extended this to the higher dimensional case:

2.2.7 Theorem. *Let M be a complex manifold whose universal covering is a polynomially convex bounded domain of $\mathbf{C}^m$. Let D be a domain of $\mathbf{C}^n$ and $E \subset D$ a pluripolar closed subset. Let $f : D\backslash E \to M$ be a holomorphic mapping. If the image $f(D\backslash E)$ is relatively compact in M, in particular if M is compact, then f extends holomorphically over D.*

Thus one may ask

2.2.8 Conjecture. *Let N be a compact Kähler manifold with negative holomorphic sectional curvature, or more generally a compact hyperbolic complex space. If E is a pluripolar closed subset of a domain $D \subset \mathbf{C}^n$, then any holomorphic mapping $f : D\backslash E \to N$ extends holomorphically over D.*

2.3. Hypersurfaces of $\mathbf{P}^n(\mathbf{C})$

S. Kobayashi [Ko1] claimed

2.3.1 Conjecture. *A generic hypersurface of large degree d of $\mathbf{P}^n(\mathbf{C})$ is hyperbolic.*

For example, in the case of $\mathbf{P}^3(\mathbf{C})$ the possible smallest degree $d = 5$, since the Fermat quartic of $\mathbf{P}^n(\mathbf{C})$ is a Kummer K3 surface. Thus we ask

2.3.2 Conjecture. *A generic hypersurface of degree* 5 *of $\mathbf{P}^3(\mathbf{C})$ is hyperbolic.*

By a small deformation of a Fermat variety, R. Brody and M. Green [BrG] constructed a hyperbolic hypersurface of $\mathbf{P}^3(\mathbf{C})$ of large even degree ≥ 50. A. Nadel [N] to 21.

G. Xu [X] recently proved an interesting theorem answering to a conjecture of J. Harris:

2.3.3 Theorem. *On a generic hypersurface of degree $d \geq 5$ in $\mathbf{P}^3(\mathbf{C})$, there is no curve with geometric genus $g \leq d(d-3)/2 - 3$. This bound is sharp. Moreover, if $d \geq 6$, this sharp bound can be achieved only by a tritangent hyperplane section.*

Now it is of interest to recall Bloch's conjecture [B]:

2.3.4 Conjecture. *Let f be a holomorphic curve from $\mathbf{C}$ into a hypersurface of $\mathbf{P}^3(\mathbf{C})$ of degree 5. Then f is algebraically degenerate.*

Note that Conjecture 1.5.1 implies this.

There is a corresponding conjecture for the complements of hypersurfaces of $\mathbf{P}^n(\mathbf{C})$ by S. Kobayashi [Ko1]:

2.3.5 Conjecture. *The complements of hypersurfaces of large degree of $\mathbf{P}^n(\mathbf{C})$ are hyperbolic.*

In the simplest case of $\mathbf{P}^2(\mathbf{C})$, it is known that d must be greater than 4 (M. Green [G1]), and that the complement of 5 lines in general position is hyperbolic and hyperbolically imbedded into $\mathbf{P}^2(\mathbf{C})$.

2.3.6 Conjecture. *The complement of a generic smooth curve of degree 5 of $\mathbf{P}^2(\mathbf{C})$ is hyperbolic and hyperbolically imbedded into $\mathbf{P}^2(\mathbf{C})$.*

We know at least the existence of such a curve of degree 5 by M.G. Zaidenberg [Z] (cf. also K. Azukawa and Masaaki Suzuki [AS] and A. Nadel [N] for examples of such smooth curves of larger degrees).

2.3.7 Theorem. *For each $d \geq 5$ there exists an irreducible smooth curve of degree d of $\mathbf{P}^2(\mathbf{C})$ whose complement is hyperbolic and hyperbolically imbedded into $\mathbf{P}^2(\mathbf{C})$.*

2.4. Non-algebraic hyperbolic manifold

So far, all known compact hyperbolic manifolds or complex spaces are algebraic. I have been once asked by I. Graham and lately by Y. Kawamata

2.4.1 Problem. *Is there any non-algebraic compact hyperbolic manifold?*

§3. Algebraic and arithmetic Kobayashi pseudodistances

Let V be an algebraic variety defined over a number field K, and $\sigma : K \hookrightarrow \mathbf{C}$ be an imbedding. Then V naturally carries a structure of a complex manifold denoted by V^σ. The following problem was given by S. Lang [L2].

3.1.1 Problem. *Let V be an algebraic variety defined over a number field K. If V^σ is hyperbolic, is V^τ hyperbolic for another $\tau : K \hookrightarrow \mathbf{C}$?*

Through a discussion on this problem at M.P.I., Bonn, S. Bando mentioned an idea to use chains of algebraic curves to connect two points on an algebraic variety instead of chains of holomorphic mappings from the unit disk. Here we explore this idea. Let M be a complex algebraic variety, and $P, Q \in M$. Let $\{(f_i, C_i, p_i, q_i)\}_{i=0}^{i=\ell}$ be a chain of smooth algebraic curves with algebraic morphisms $f_i : C_i \to M$ and points $p_i, q_i \in C_i$ so that

$$P = f_0(p_0), \quad f_{i-1}(q_{i-1}) = f_i(p_i), 1 \le i \le \ell, \quad f_\ell(q_\ell) = Q.$$

Let $d_{C_i}(p_i, q_i)$ denote the hyperbolic pseudodistance of the 1-dimensional complex space C_i, and set

$$D_M(P, Q) = \inf \left\{ \sum d_{C_i}(p_i, q_i) \right\},$$

where the infimum is taken over all possible such chains.*

Then we see

3.1.2 Theorem. i)$D_M \ge d_M$.

ii)*$D_M(P, Q)$ is a continuous function defining a pseudodistance on M.*

iii)*If $D_M(P, Q)$ is a distance, then it is an inner distance and its topology is the same as the underlying differential topology.*

iv)*(Distance decreasing principle) For an algebraic morphism $f : M \to N$, we have*

$$D_M(P, Q) \ge D_N(f(P), f(Q)).$$

So, we call $D_M(P, Q)$ the *algebraic hyperbolic pseudodistance* of M. For instance, $D_M(P, Q) \equiv 0$ for the complex projective space, the complex affine space, and Abelian varieties.

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* During the preparation of this paper the author lately learned that J.-P. Demailly and B. Shiffman (Algebraic approximations of analytic maps from Stein domains to projective manifolds, preprint) proved an approximation theorem for the Kobayashi-Royden infinitesimal form on a complex projective manifold by algebraic curves (possibly singular). So far, it is not clear if their result implies that $D_M(P, Q)$ is the same as the original Kobayashi hyperbolic pseudodistance.

Let V be an algebraic variety defined over a number field K, and $P, Q \in V(K)$. Set

$$D_{V_K}(P,Q) = \frac{1}{[K:\mathbf{Q}]} \sum_{\sigma} D_{V^\sigma}(P^\sigma, Q^\sigma),$$

where σ runs over all possible imbedding of K into $\mathbf{C}$. We call $D_{V_K}(P,Q)$ the *arithmetic hyperbolic pseudodistance*, which also satisfies the *distance decreasing principle.* for K-morphisms. If $L \supset K$ is a field extension, then

$$D_{V_K}(P,Q) = D_{V_L}(P,Q).$$

There is another way to define the *arithmetic hyperbolic pseudodistance* $\tilde{D}_{V_K}(P,Q)$. We use of only chains $\{(f_i, C_i, p_i, q_i)\}_{i=0}^{i=\ell}$, connecting P and Q such that all $f_i : C_i \to V$ are defined over K and $p_i, q_i \in C_i(K)$, and set

$$\tilde{D}_{V_K}(P,Q) = \inf\left\{\sum D_{C_{iK}}(p_i, q_i)\right\}.$$

Then $\tilde{D}_{V_K}(P,Q) \geq D_{V_K}(P,Q)$, and $\tilde{D}_{V_K}(P,Q)$ satisfies the *distance decreasing principle* not only for K*-morphisms* but also for *field extensions*

$$\tilde{D}_{V_K}(P,Q) \geq \tilde{D}_{V_L}(P,Q).$$

In what follows, $d_{V_K}(P,Q)$ stands for $\tilde{D}_{V_K}(P,Q)$ or $D_{V_K}(P,Q)$. If V is a curve of higher genus, $d_{V_K}(P,Q)$ is a distance. For this moment we do not know any substantial implication from this pseudodistance, but may consider many many problems!!! Some of them are

3.1.3 Problem. i) *The equivalence problems between all possible notion of hyperbolicities.*

ii) *The lower bound of $d_{V_K}(P,Q)$ in the case where $d_{V_K}(P,Q)$ is a distance.*

iii) *The relation between $d_{V_K}(P,Q)$ and the heights $h_K(P)$ of rational points $P \in V(K)$. For instance, is there a relation between $1/\inf\{d_{V_K}(P,Q); P,Q \in V(K)\}$ and $\sup\{h_K(P); P \in V(K)\}$?*

Here it is of some interest to recall the following result due to J. Kollár, Y. Miyaoka and S. Mori [KMM]:

3.1.4 Theorem. *Let k be an algebraically closed field of characteristic 0 whose order is uncountable. Let W be a proper algebraic variety over k. Then the following conditions are equivalent.*

(1) *Given finitely many arbitrary points $x_i \in W, i = 1, \ldots, m$ there is an irreducible rational curve containing all x_i.*

(2) *Given two arbitrary points $x_1, x_2 \in W$ there is an irreducible rational curve containing x_1 and x_2.*

(3) *For a sufficiently general $(x_1, x_2) \in W \times W$ there is an irreducible rational curve containing x_1 and x_2.*

If W is smooth, we have more equivalent conditions.

(4) *Given two arbitrary points $x_1, x_2 \in W$ there is a connected curve containing x_1 and x_2, which is a union of rational curves.*

(5) *There is a morphism $f : \mathbf{P}^1 \to W$ with ample f^*T_W.*

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