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Hyperbolicity in Projective Spaces

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To Professor Shoshichi Kobayashi
on the occasion of his sixtieth birthday

In 1970 Sh. Kobayashi posed the following problems [Ko1]:

Let D be a generic hypersurface of degree d in $\mathbf{P}^n$, *where d is large enough with respect to n.*

I *Is it true that D is hyperbolic?*

II *Is it true that the complement* $\mathbf{P}^n \setminus D$ *is hyperbolic and, moreover, hyperbolically embedded into* $\mathbf{P}^n$ *? Is it true for* $d \geq 2n+1$ *?*

For $n = 2$ the answer to **I** is classically known to be positive (starting with $d = 4$), while for $n \geq 3$ the problem is open.

The answer to **II** is unknown even for $n = 2$. It is positive for $n = 1, d \geq 3$, and this is equivalent to the Montel Theorem.

Here we present a survey around the Kobayashi's Problems. Of course, it does not pretend neither to be exhausted, nor to be original.

I The compact case

Let $\mathbf{P}_{n,d} = \mathbf{P}^N$, where $N = \binom{n+d}{n} - 1$, be the projective space whose points parametrize hypersurfaces of degree d in $\mathbf{P}^n$ (not necessarily reduced).

Let $H_{n,d} \subset \mathbf{P}_{n,d}$ be the subset corresponding to hyperbolic hypersurfaces. To precise the meaning of "genericity" in **I** one could ask *whether $H_{n,d}$ contains a Zariski open subset of $\mathbf{P}_{n,d}$ for $d >> n$? Or, more generally, whether the complement $\mathbf{P}_{n,d} \setminus H_{n,d}$ is contained in a countable union of hypersurfaces in $\mathbf{P}_{n,d}$ for $d >> n$?*

It is known that $H_{n,d}$ is open (but probably empty) in the classical Hausdorff topology of $\mathbf{P}_{n,d}$ for any $n, d \in \mathbf{N}$. This follows from the Brody's Stability Theorem [Br], or, to be more precise, from the following version of it [Za1,4]:

Theorem I.1 *Let M be a complex manifold and X a compact analytic subset of M. If X is hyperbolic, then there exists a neighborhood U of X in M, which is hyperbolically embedded into M. Therefore, any compact analytic subset X' in M close enough to X is hyperbolic as well.*

In particular, if $f : M \to S$ is a proper holomorphic surjection onto a complex space S, then the subset of points in S that correspond to the hyperbolic fibres of f is open.

We give here a sketch of the proof.

Let h be a fixed Hermitian metric on M. An entire curve $f : \mathbf{C} \to M$ is called a *Brody curve* iff f is a contraction with respect to the Euclidean metric in $\mathbf{C}$ and the metric h on M (i.e. $| df(z) |_h \leq 1 \forall z \in \mathbf{C}$), and $| df(0) |_h = 1$.

Let the disc Δ_r of radii r in $\mathbf{C}$ be endowed with the metric rh_r, where h_r is the Poincaré metric in Δ_r. It is easily seen that the Euclidean metric in $\mathbf{C}$ is the limit of the metrics rh_r as $r \to \infty$. A holomorphic curve $f : \Delta_r \to M$ is called a *Bloch-Brody curve* iff f is a contraction with respect to the metrics rh_r in Δ_r and h in M, and $| df(0) |_h = 1$. By the Arzelà-Askoli Theorem any sequence $f_n : \Delta_n \to M$ of Bloch-Brody curves, whose images are contained in the same relatively compact subset of M, has a subsequence converged to a Brody curve $f : \mathbf{C} \to M$.

Let $\{U_n\}$ be a fundamental sequence of (relatively compact) neighborhoods of the hyperbolic compact analytic subset $X \subset M$. Suppose that there is no $n \in \mathbf{N}$ such that U_n is hyperbolically embedded into M. That means that the inequality $K_{U_n} \geq ch$ for the Kobayashi-Royden pseudometric K_{U_n} on U_n does not hold for any constant $c > 0$; in particular, it does not hold for $c = \frac{1}{n}$. By the definition of the Kobayashi-Royden pseudometric there exists a sequence $h_n : \Delta_n \to U_n$ of holomorphic curves such that $| dh_n(0) | > 1$. By the Brody's Reparametrization Lemma [Br] there exists a sequence of Bloch-Brody curves $f_n : \Delta_n \to U_n$, where $f_n(z) = h_n \circ \alpha_n(r_n z)$ for some $r_n < 1$ and $\alpha_n \in \mathrm{Aut}(\Delta_n)$. Passing to a converged subsequence, one can obtain a limiting Brody curve $f : \mathbf{C} \to \bigcap U_n = X$, that contradicts to the assumption of hyperbolicity of X. ○

So, the hyperbolicity of a hypersurface in $\mathbf{P}^n$ is stable with respect to small deformations of the coefficients of the defining equation. More generaly, the set of points of a Hilbert scheme, which correspond to hyperbolic projective varieties, is open in the usual topology. We do not know *when this set is non-empty; whether, being non-empty, it must contain a Zariski open subset, or at least an algebraic subvariety of small enough codimension.*

For $n = 3$ R. Brody and M. Green [BrGre] gave examples of one-parametric families of hyperbolic surfaces in $\mathbf{P}^3$ of any even degree $d = 2k \geq 50$. Namely, the surfaces

$$D_{d,t} = \{{x_0}^{2k} + {x_1}^{2k} + {x_2}^{2k} + {x_3}^{2k} + t(x_0x_1)^k + t(x_0x_2)^k = 0\}$$

(deformations of the Fermat surfaces $F_{3,d} = D_{d,0}$) are hyperbolic for all but a finite number of values of $t \in \mathbf{C}$. This means that for $d = 2k \geq 50$ the set $H_{3,d}$ is non-empty and contains a quasi-projective rational curve $C = \{D_{d,t}\}$ (together with some small classical neighborhood of it, as follows from the Stability Theorem).

It is unknown *whether for any $n \geq 4$ there exists a hyperbolic hypersurface in $\mathbf{P}^n$*. J. Noguchi (private communication) supposed that the Brody-Green construction should be available also in some higher dimensions, at least for $n = 4$.

Notice that the Newton polyhedron of the Fermat hypersurface $F_{n,d}$ of degree d in $\mathbf{P}^n$ is the standard simplex in $\mathbf{R}^{n+1}$; the monomials in the Fermat equation correspond to its verticies. Additional monomials in the Brody-Green example correspond to the middle points of some edges of this simplex (so, the defining polynomials are fewnomials: they contain few monomials with respect to their degrees).

Definition. Let us say that a hypersurface $D = \{p(x_0, ..., x_n) = 0\}$ of degree d in $\mathbf{P}^n$ is *k-almost simplicial* if any monomial of p corresponds to a lattice point in $\mathbf{R}^{n+1}$ with one of coordinates $\geq d - k$ (that means that this point is placed in a k-neighborhood of some vertex of the n-simplex $\{x_0 + ... + x_n = d\}$ in ${\mathbf{R}_+}^{n+1}$).

The following statement belongs to A. Nadel [Na]; its proof is based on the Y.-T. Siu's version of the value distribution theory for holomorphic curves in a complex manifold endowed with a meromorphic connection.

Theorem I.2 *For arbitrary $e \geq 3$ in the projective space of all k-almost simplicial surfaces in $\mathbf{P}^3$ of degree $d = 6e + 3 > 4k + 10$ there exists a quasiprojective subvariety of the dimension $4\begin{pmatrix} k+4 \\ 4 \end{pmatrix} - 1$, which consists of hyperbolic smooth surfaces. In particular, $H_{3,d}$ is non-empty for any $d = 6e + 3 \geq 21$.*

Definition. Let us say that a complex Hermitian manifold (X, h) is *Brody hyperbolic* iff it does not contain any Brody curve $\mathbf{C} \to X$, and *Picard hyperbolic* iff it does not contain any non-constant entire curve $\mathbf{C} \to X$.

The Big Picard Theorem can be reformulated by saying that $\mathbf{P}^1 \backslash \{3 \text{ points}\}$ is Picard hyperbolic. The Brody's Theorem [Br] states that for a compact manifold X all three notions of hyperbolicity (i.e. Kobayashi hyperbolicity, Brody hyperbolicity and Picard hyperbolicity) are equivalent.

M. Green [Gre4] remarked that a Brody curve $\mathbf{C} \to \mathbf{T}^n$ in a complex torus $\mathbf{T}^n = \mathbf{C}^n/\Lambda$, where Λ is a lattice of the maximal rank in $\mathbf{C}^n = \mathbf{R}^{2n}$, is lifted to an affine isometric embedding $\mathbf{C} \to \mathbf{C}^n$. Therefore, a closed subvariety $X \subset \mathbf{T}^n$ is (Brody) hyperbolic iff it does not contain any shifted subtorus. The same is valid for any compact complex parallelizable manifold [HuWi].

In more general setting Sh. Kobayashi [Ko2] established the following fact.

Theorem I.3 *Let (X, h) be a Hermitian manifold of nonpositive holomorphic sectional curvature and $f : \mathbf{C} \to X$ be a Brody curve. Then f is an isometric immersion, and its image is totally geodesic.*

Problem I.1 *Let the conditions of the above theorem be fulfilled. Is* *it true that the closure* $\overline{f(\mathbf{C})}$ *in* X *contains the image of a complex torus by a non-constant holomorphic mapping, or at least any compact complex submanifold of positive dimension?*

We remark that the rational curve $\mathbf{P}^1$ and the simple complex tori are the only known examples of compact complex manifolds with totally degenerate Kobayashi pseudodistances that are minimal in this class, i.e. that contain no closed subvarieties, which have this property to be *completely non-hyperbolic.* This motivates the following

Definition. A compact complex space is said to be *algebraically hyperbolic* if it contains no image of a complex torus by a non-constant holomorphic mapping.

In particular, such a variety contains no rational or elliptic curve. It is clear that a complex space is algebraically hyperbolic if it is hyperbolic.

Problem I.2 *Does algebraic hyperbolicity imply (Brody) hyperbolicity, at least for projective varieties? In other words, is it true that a compact complex space (a projective variety) which possesses a Brody curve, should contain the image of a complex torus under a non-constant holomorphic mapping?*

The following recent result of J.-P. Demailly and B. Shiffman [DemSh] can be considered as an approximation to the positive answer.

Theorem I.3 *Let* X *be a smooth projective variety,* S *a Stein manifold such that* $\dim S \leq \dim X$*,* $f : S \to X$ *a holomorphic mapping,* T *a finite subset of* S *and* m *a fixed natural number. Then there exists an exhausted sequence* $\Omega_1 \subset \ldots \subset \Omega_k \subset \ldots$ *of Runge domains in* S *and a sequence of* *holomorphic mappings* $f_k : \Omega_k \to X_k$ *such that, for any* $k \in \mathbf{N}$, $\dim X_k = \dim S$ *and at each point* $s \in T$ *the* m*-jet of* f_k *coinside with the* m*-jet of* f. *If* S *is an affine algebraic manifold, then* f_k *can be chosen to be regular.*

As a corollary one has the following 'more algebraic' definition of the Kobayashi-Royden pseudometric K_X of a projective variety X:

$$K_X(v) = \inf\{K_{\tilde{C}}(v) \mid v \in TC\},$$

where infinum is taken over all algebraic curves C in X which touch the vector $v \in TX$, and $K_{\tilde{C}}$ is the Poincaré metric of the normalization $\tilde{C}$ of C. Furthermore, the Kobayashi pseudodistance $k_X(x, y)$ on X coincides with its algebraic analogue $d_X(x, y)$ suggested by J. Noguchi; briefly speaking, the chains of holomorphic discs in the definition of the Kobayashi pseudodistance are replaced by chains of algebraic curves and hyperbolic metrics of these curves are used instead of the Poincaré metric in the disc).

An approach to Kobayashi's Problem **I** is to divide it into two parts: Problem I.2 on the equivalence of (Brody) hyperbolicity and algebraic hyperbolicity for projective varieties, as the first part, and as the second one the following

Problem I.3 *Is it true that a generic projective hypersurface of a large enough degree in* $\mathbf{P}^n$ *is algebraically hyperbolic?*

For $n = 3$ the positive answer follows from the next recent result of Geng Xu [Xu], that precises an earlier one of H. Clemens and proves a conjecture due to J. Harris.

Theorem I.4 *For any algebraic curve on a generic surface* $D \in \mathbf{P}_{3,d}$ *of* *degree $d \geq 5$ in $\mathbf{P}^3$ the following estimate holds:*

$$g(\tilde{C}) \geq \frac{d(d-3)}{2} - 2 \geq 3,$$

where $g(\tilde{C})$ is the genus of the normalization $\tilde{C}$ of C. This bound is sharp, and for $d \geq 6$ the curves of the minimal genus are sections of D by tritangent planes.

Therefore, for $d \geq 5$ a generic surface of degree d in $\mathbf{P}^3$ does not contain any rational or elliptic curve, and so it is algebraically hyperbolic.

Observe that on a smooth quartic surface in $\mathbf{P}^3$ and, moreover, on any $K3$-surface, there exist a rational curve and a linear pencil of elliptic curves (see [GreGri] and [MoMu]). Thus, such a surface is not algebraically hyperbolic. This shows that the above bound $d \geq 5$ is sharp.

The proof of Theorem I.4 involves the Brill-Noether Theorem, and thus the meaning of "genericity" in its formulation is more extended than the genericity in Zariski sense. Namely, let $AH_{n,d} \subset \mathbf{P}_{n,d}$ be the set of all algebraically hyperbolic hypersurfaces. Then by Theorem I.4 for $d \geq 5$ the complement $\mathbf{P}_{3,d} \setminus AH_{3,d}$ consists of a countable number of proper algebraic subvarieties of the $\mathbf{P}_{3,d}$. There is no information about their replacement. In particular, the following problem seems to be important.

Problem I.4 *Is the locus $\mathbf{P}_{3,d} \setminus AH_{3,d}$ closed in $\mathbf{P}_{3,d}$ in the usual topology?*

Suppose that this locus is not closed. Then there exists a sequence of non-algebraically hyperbolic surfaces D_k in $\mathbf{P}^3$ converged to an algebraically hyperbolic surface D_0. By the stability of hyperbolicity, D_0 is not (Brody) hyperbolic; indeed, otherwise for k large enough D_k should be hyperbolic as well, and therefore algebraically hyperbolic. So, if the answer to Problem

I.4 is negative, then also the answer to Problem I.3 is negative; indeed, such D_0 would be an example of an algebraically hyperbolic surface which is not hyperbolic (and therefore it contains a Brody entire curve $\mathbf{C} \to D_0$).

A generic (in Zariski sense) hypersurface of degree $d \leq 2n - 3$ in $\mathbf{P}^n$ contains a projective line (in particular, a smooth cubic surface in $\mathbf{P}^3$ contains exactly 27 lines), thus it is not algebraically hyperbolic.

Question. *What is the maximal number $d = d(n)$ such that $\mathbf{P}_{n,d} \setminus AH_{n,d}$ contains a Zariski open subset of $\mathbf{P}_{n,d}$?*

By the above remarks we have that $d(3) = 4$ and $d(n) \geq 2n - 3$.

It is worth mentioning here the well known problems: *Whether hyperbolicity (resp. algebraic hyperbolicity), or even measure hyperbolicity of a compact complex manifold implies that it is a projective variety of general type?*

The positive answer is known in the case of surfaces (see [GreGri], [MoMu]).

A weaker property that could serve as a bridge between hyperbolicity and algebraic hyperbolicity, is *algebraic degeneracy.*

Definition. One says that a complex space X has the property of *algebraic degeneracy* iff the image of any non-constant entire curve $\mathbf{C} \to X$ lies in a proper closed complex subspace of X. We mention *stong algebraic degeneracy,* if this subspace is the same for all such curves.

Perhaps, it is worth also to specify this notion by restricting the class of curves under consideration to Brody curves.

The Bloch Conjecture, proven by T. Ochiai, Y. Kawamata, and also by M. Green and P. Griffiths, R. Kobayashi (see [RKo] for the references), states that *an irregular projective variety X (i.e. a variety with the irregularity*

$q(X) = h^{1,0}(X) > \dim X$) *has the property of algebraic degeneracy.* The above restriction was weakened in the case of surfaces of general type to $q(X) \geq 2$ by C. Grant [Gra1] (see also [Gra2], [HuWi], [Lu] and St. Lu's report in this volume for some related results).

Another property, close to algebraic hyperbolicity, is finiteness of the number of non-hyperbolic (resp. non-algebraically hyperbolic) proper subvarieties. In the surface case this is finiteness of the number of rational and elliptic curves, that was proved by F. Bogomolov [Bo] for projective surfaces of general type under the assumption that the inequality for Chern numbers ${c_1}^2 > c_2$ holds (see also [Lu]). H. Clemens conjectured the finiteness of the number of rational curves of any given degree d on a generic quintic threefold in $\mathbf{P}^4$, that was verified by N. Katz for $d \leq 7$ (see [Xu]).

II The non-compact case

Denote by $HE_{n,d}$ the subset of $\mathbf{P}_{n,d}$ consisting of the hypersurfaces of degree d in $\mathbf{P}^n$ with hyperbolically embedded complements. Then $HE_{n,d}$ is non-empty for any $d \geq 2n + 1$; indeed, it contains the union $C_{n,d}$ of d hyperplanes in general position. This fact (modulo Kiernan's criterion of hyperbolic embeddedness [Ki2]) goes back to E. Borel, A. Bloch, A. Cartan and J. Dufresnoy (see [KiKo] for the references). It was reproved many times, for instance by M. Green [Gre2], E. Babets [Ba] and others.

The bound $d \geq 2n + 1$ for $HE_{n,d}$ being non-empty should be sharp. It is sharp for $n = 2$; indeed, M. Green remarked in [Gre3] that for any quartic curve C in $\mathbf{P}^2$ there exists a projective line l that intersects C not more than in two points (an inflection tangent to C, or a bitangent, or a tangent in a singular point, or a line passing through two singular points of C). Thus $\mathbf{P}^2 \setminus C$ is not hyperbolic; indeed, it contains $l \setminus C \supset \mathbf{P}^1 \setminus \{2 \text{ points}\}$, and so the Kobayashi pseudodistance $k_{\mathbf{P}^2\setminus C}$ is degenerate along $l \setminus C$.

We do not know whether for $d \leq 2n$ $HE_{n,d}$ is empty or not, but we know at least [Za3] that its complement $\mathbf{P}_{n,d} \setminus HE_{n,d}$ contains a Zariski open subset:

Proposition II.1 *For a generic (in Zariski sense) hypersurface D of degree $d \leq 2n$ in $\mathbf{P}^n$ and for any $k, 0 \leq k \leq d$, there exists a projective line l that intersects D in two points only with multiplicities k and $d-k$, respectively. Thus, the pseudodistance $k_{\mathbf{P}^n\setminus D}$ is degenerate along l. If $d = 2n$, then the number of such lines is finite.*

In contrast with the subset $H_{n,d}$ of the $\mathbf{P}_{n,d}$, the subset $HE_{n,d}$ is never open in the usual topology of $\mathbf{P}_{n,d}$. For instance, for any $d \geq 2n+1$ the totally reducible hypersurfaces $C_{n,d} \in HE_{n,d}$ considered above belong to the boundary of $HE_{n,d}$. This follows from the next simple observation [Za4]:

Proposition II.2 *Any hypersurface D_0 in $\mathbf{P}^n$ that contains a projective line l, can be approximated by a sequence of hypersurfaces $\{D_k\}$ such that $l \cap D_k$ consists of one point only. Thus $\mathbf{P}^n \setminus D_k$ is not hyperbolic, and so $D_0 \in \overline{\mathbf{P}_{n,d} \setminus HE_{n,d}}$.*

Nevertherless, in [Za4] a stability principle is obtained which can be applied in connection with Kobayashi's Problem II. Its proof follows the line of the proof of Theorem I.1. One of its consequencies is the following

Theorem II.1 *Let M be a compact complex manifold and D a hypersurface in M. If D and $M \setminus D$ are both Brody hyperbolic, then $M \setminus D$ is hyperbolically embedded into M; moreover, all these properties are preserved by small deformations of the pair (M, D).*

Corollary *$HE_{n,d} \cap H_{n,d}$ is an open (but possibly empty) subset of $\mathbf{P}_{n,d}$ in the usual Hausdorff topology.*

It would be reasonable to suppose that *the intersection $HE_{n,d} \cap H_{n,d}$ contains a Zariski open subset of $\mathbf{P}_{n,d}$ if $d >> n$*, which would imply the positive answer to both of the Kobayashi's Problems.

To construct examples of hypersurfaces that belong to $HE_{n,d} \cap H_{n,d}$, one can use the following generalization of the Borel-Bloch-Cartan-Dufresnoy Theorem. It can be deduced from a result of M. Green [Gre2], and it was proven by E. Babets [Ba] by a different method.

Theorem II.2 *The complement of the union of $2n+1$ smooth hypersurfaces in $\mathbf{P}^n$ in general position is hyperbolically embedded into $\mathbf{P}^n$.*

In fact, this is true for any union of $2n + 1$ hypersurfaces such that the intersection of any $n+1$ of them is empty (A. Eremenko and M. Sodin [ErSo]; a simplified proof has been recently done by Min Ru). Using this theorem and Theorem II.1, one can easily obtain the following

Corollary *If $H_{n,k}$ is non-empty, then $HE_{n,d} \cap H_{n,d}$ is a non-empty open set for any $d \geq (2n+1)k$.*

Indeed, by Theorem II.2 the union of any $2n + 1$ smooth hyperbolic surfaces in general position belongs to $HE_{n,d} \cap H_{n,d}$.

In particular, from the existence of a hyperbolic surface in $\mathbf{P}^3$ of degree 21 [Na] it follows that $HE_{3,d} \cap H_{3,d}$ is non-empty for any $d \geq 147 = 7 \cdot 21$.

For $n = 2$, a more refined version of the Stability Principle, which uses *absorbing stratifications* [Za4], leads to the following

Theorem II.3 *For any $d \geq 5$ the open set $HE_{2,d} \cap H_{2,d}$ is non-empty, i.e. there exists a classicaly open set of smooth curves in $\mathbf{P}^2$ of degree d with* *hyperbolically embedded complements.*

The bound $d \geq 5$ here is sharp, as follows from the remark of M. Green mentioned above.

The first examples of smooth curves of any even degree $d \geq 30$ in $HE_{2,d}$ were constructed by K. Azukawa and M. Suzuki [AzSu] by the Brody-Green method [BrGre]. Remark that if B in $\mathbf{P}^2$ is a branching curve of a regular projection of some hyperbolic projective surface into $\mathbf{P}^2$, then the complement $\mathbf{P}^2 \setminus B$ is a base of a hyperbolic covering and so it is hyperbolic. But the class of such curves is rather restricted, as has been remarked by F. Bogomolov, B. Moishezon and M. Teicher; for instance, the number of cusps of the branching curve of a generic projection of a smooth projective surface onto $\mathbf{P}^2$ is divided by 3.

In a series of papers by M. Green, J. Carlson and M. Green, H. Grauert and U. Peternell (see [Za4] for the references) certain sufficient conditions were worked out that ensure, for an irreducible plane curve C of genus ≥ 2, the existence in the complement $\mathbf{P}^2 \setminus C$ of a complete Hermitian metric with holomorphic sectional curvature bounded from above by a negative constant (by Ahlfors Lemma this implies the hyperbolic embededdness of $\mathbf{P}^2 \setminus C$ into $\mathbf{P}^2$). Any curve satisfying these conditions is singular and of at least sixth degree; the only known examples are the dual curves to generic smooth plane curves of degrees $d \geq 4$.

By Green-Babets Theorem II.2 any union of 5 smooth curves in $\mathbf{P}^2$ in general position has the hyperbolically embedded complement. Therefore, for $d \geq 5$ the set $HE_{2,d}$ contains some quasiprojective varieties. For instance, the quasiprojective submanifold $M = \{C_{2,5}\} = \{$unions of 5 lines in general position in $\mathbf{P}^2\}$ of dimension 10 is contained in $HE_{2,5} \subset \mathbf{P}_{2,5} = \mathbf{P}^{20}$. Recently G. Dethloff, G. Schumacher and P.-M. Wong [DetSchWo] have shown that the complement to a union C of 4 plane curves in general position is hyperbolically embedded into $\mathbf{P}^2$, if the degree of C is at least 5 (see P.-M. Wong's report in this volume). This fact can also be obtained by using of a result of Y. Adachi and M. Suzuki [AdSu1]. Another result of [DetSchWo], conjectured by H. Grauert [Grau], is the hyperbolic embeddedness of the complements to three quadrics in $\mathbf{P}^2$ in general position (the latter condition can be formulated explicitly in this case; in other cases it means at least Zariski openess).

Let us mention a related criterion of hyperbolic embededdness of complements of curves [Za2].

Proposition II.3 *Let C be a closed curve in a smooth compact complex surface M. The complement $M \setminus C$ is hyperbolically embedded into M iff the curve $C \setminus \mathrm{Sing}(C)$ is hyperbolic and the complement $M \setminus C$ is Brody hyperbolic.*

The property of algebraic degeneracy of complements of curves was treated by T. Nishino and M. Suzuki [NiSu], Y. Adachi and M. Suzuki [AdSu1,2]. In particular, it is worth mentioning the following results.

Theorem II.4 ([NiSu]) *Let M and C be as above. If the logarithmic Kodaira dimension $\overline{k}(M \setminus C) = 2$, then any proper holomorphic mapping $f : \mathbf{C} \to M \setminus C$ is algebraically degenerate, i.e. the image $f(C)$ is contained in some closed curve E in M.*

Theorem II.5 ([AdSu1]) *Let a reducible curve C in $\mathbf{P}^2$ consists of at least 4 irreducible components, which do not belong to the same linear pencil.*

Then there exists a curve A in $\mathbf{P}^2$ such that the image of any non-constant entire curve $\mathbf{C} \to \mathbf{P}^2 \setminus C$ is contained in A. Thus, $\mathbf{P}^2 \setminus C$ has the property of strong algebraic degeneracy.

All possible exceptions here are completely classified. For some examples of degeneracy loci in complements of quartic curves see [Gre3].

Another degeneracy principle had been used in the Babets' proof of Theorem II.2 [Ba]. It states that,*with respect to an appropriate complete Hermitian metric in the complement of a divisor D of normal crossings type in a compact complex manifold M, any holomorphic differential in $M \setminus D$ with logarithmic poles along D is constant on any Brody curve $\mathbf{C} \to M \setminus D$.* See also [Na] for an algebraic degeneracy principle in the presence of an ample meromorphic connection in Siu sense.

The definition of algebraic hyperbolicity, after some evident changes, is available for affine algebraic or, more generally, quasiprojective varieties. This allows one to divide Problem II into two subproblems that correspond to Problems I.2 and I.3 above.

Problem II.1 *Let D be a hyperbolic hypersurface in $\mathbf{P}^n$ such that there exists a Brody curve $\mathbf{C} \to \mathbf{P}^n \setminus D$. Is it true that there exists a rational projective curve C in $\mathbf{P}^n$ which has not more than two places on D?*

Problem II.2 *Let $L_{n,d} \subset \mathbf{P}_{n,d}$ be the locus of those hypersurfaces D of degree d in $\mathbf{P}^n$, for which such rational curve C as above does exist. Is it true that the complement $\mathbf{P}_{n,d} \setminus L_{n,d}$ contains a Zariski open subset of $\mathbf{P}_{n,d}$ for $d >> n$? Is the locus $L_{n,d}$ Hausdorff closed in $\mathbf{P}_{n,d}$?*

Next we pass to the special subproblem of hyperbolicity of complements to hyperplanes in $\mathbf{P}^n$. For hyperplanes in general position the following result, due to H. Fujimoto [Fu] and M. Green [Gr], is well known; we formulate it together with some additional information obtained by P. Kiernan and Sh. Kobayashi [KiKo].

Theorem II.6 *Let D be a union of $n+k$ hyperplanes in general position in $\mathbf{P}^n$, where $k > 0$. Then the image of any non-constant entire curve $\mathbf{C} \to \mathbf{P}^n \setminus D$ is contained in a linear subspace of dimension $\leq [\frac{n}{k}]$. The bound here is sharp. In addition, the degeneracy locus is contained in a finite union of the 'diagonal linear subspaces' of dimension $n-k+1$, defined by D in a canonical way. Thus, $\mathbf{P}^n \setminus D$ has the property of strong algebraic degeneracy.*

For $k = 2$ this gives the estimate $[\frac{n}{2}]$ of the dimension of the degeneracy subspace, while from the Borel Lemma it follows just the linear degeneracy, which means that any non-constant entire curve in the complement to $n+2$ hyperplanes in $\mathbf{P}^n$ in general position is contained in a hyperplane. In fact, the latter is true without the assumption of general position [Gre1]. And for $k = n+1$ Theorem II.6 leads once again to the Borel-Bloch-Cartan-Dufresnoy Theorem.

The exactness of the bound $d \geq 2n+1$ for the hyperbolicity of $\mathbf{P}^n \setminus D$ is shown by the following result of V.E. Snurnitsyn [Sn], which proves a conjecture of P. Kiernan [Ki1].

Theorem II.7 *For any union D of $2n$ hyperplanes in $\mathbf{P}^n$ there exists a projective line l such that the intersection $D \cap l$ consists of not more than two points. Therefore, $\mathbf{P}^n \setminus D$ is not hyperbolic.*

Some examples, where the union of hyperplanes in non-general position has hyperbolically embedded complement, were given by P. Kiernan [Ki1].

In [Za2] the following conditions for a finite union D of hyperplanes in $\mathbf{P}^n$ were considered:

(a) *There exists no pair of points x, y in $\mathbf{P}^n$ such that each hyperplane in D passes through* at least *one of these points. In other words, there exists no projective line l $(l = (x, y))$, which intersects the union of those hyperplanes in D, that do not contain l, in not more than two points.*

(b) *There exists no pair of points (x, y) in $\mathbf{P}^n$ such that each hyperplane in D passes through* exactly *one of these points. In other words, there exists no projective line l $(l = (x, y))$ that intersects D in one or two points only.*

It is clear that if condition (b) fails, then the Kobayashi pseudodistance $k_{\mathbf{P}^n \setminus D}$ is degenerate along l, and if (a) is violated, its limit is degenerate along l. The following criteria were obtained in [Za2, Sect.3].

Theorem II.8 *Let D be as above. The complement $\mathbf{P}^n \setminus D$ is hyperbolically embedded in $\mathbf{P}^n$ iff condition (a) holds, and it is Picard hyperbolic iff condition (b) is fulfilled. Furthermore, for $n = 2$ (b) is equivalent to hyperbolicity of $\mathbf{P}^2 \setminus D$.*

The latter statement had been earlier conjectured by S. Iitaka.

Another criterion of Picard hyperbolicity of complements of hyperplanes has been recently obtained by Min Ru [Ru].

Theorem II.9 *The complement $\mathbf{P}^n \setminus D$ of a finite union D of hyperplanes in $\mathbf{P}^n$ is Picard hyperbolic iff for any linear subspace V in $\mathbf{P}^n$, which is not contained in D, the intersection $V \cap D$ contains at least three distinct hyperplanes of V that are linearly dependent.*

An algorithm that allows one to check the latter condition (which is obviously equivalent to condition (b)) is also given in [Ru]. We remark that to verify (b) one can apply an algorithm of passing from one pair of isolated intersection points of n hyperplanes in D (if there is any such pair) to another one, as it is done in the simplex method.

In conclusion, let us mention the Lang's Conjecture on equivalence of Picard hyperbolicity and mordelleness (see [La]), which was proven for complements of hyperplanes by P.-M. Wong and M. Ru [WoRu] under the assumption of general position, and by M. Ru [Ru] without this assumption.

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