ON CERTAIN CLASSES OF MEROMORPHICALLY P-VALENT STARLIKE FUNCTIONS

Nak Eun Cho (釜山水産大学)
Shigeyoshi Owa (近畿大・理工 尾和重義)

Abstract. Let $M_{n+p-1}(\alpha)$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \ldots + a_{k+p-1}z^k + \ldots \quad (p \in N = \{1, 2, 3, \ldots\})$$

that are regular in the annulus $D = \{z : 0 < |z| < 1\}$ and satisfy

$$\text{Re}\left\{\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\right\} < -\alpha$$

for $0 \leq \alpha < p$ and $|z| < 1$, where

$$D^{n+p-1}f(z) = \frac{1}{z^p} \left(\frac{z^n+2p-1f(z)}{(n+p-1)!}\right)^{(n+p-1)}.$$

We prove that $M_{n+p}(\alpha) \subset M_{n+p-1}(\alpha)$, where $n$ is any integer greater than $-p$. We also consider some integrals of functions in the class $M_{n+p-1}(\alpha)$.

1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \ldots + a_{k+p-1}z^k + \ldots, \quad (1.1)$$

* 1990 Mathematics Subject Classification : 30C45.
which are regular in the annulus \( D = \{ z : 0 < |z| < 1 \} \), where \( p \) is a positive integer. The Hadamard product or convolution of two functions \( f \) and \( g \) in \( \sum_p \) will be denoted by \( f * g \). Let

\[
D^{n+p-1}f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z), \quad (z \in D)
\]

or, equivalently,

\[
D^{n+p-1}f(z) = \frac{1}{z^p} \left( \frac{z^{n+2p-1}f(z)}{(n+p-1)!} \right)^{(n+p-1)}
\]

\[
= \frac{1}{z^p} + (n+p)a_0 \frac{1}{z^{p-1}} + \frac{(n+p+1)(n+p)}{2!}a_1 \frac{1}{z^{p-2}} + \ldots + \frac{(n+k+2p-1)(n+p)}{(k+p)!}a_{k+p-1}z^k + \ldots (z \in D),
\]

where \( n \) is any integer greater than \(-p\).

In this paper, among other things, we shall show that a function \( f(z) \) in \( \sum_p \), which satisfies one of the conditions

\[
\text{Re} \left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} \right\} < -\alpha \quad (z \in U = \{ z : |z| < 1 \}),
\]

where \( n \) is any integer greater than \(-p\), is meromorphically \( p \)-valent starlike in \( U \). More precisely, it is proved that, for the classes \( M_{n+p-1}(\alpha) \) of functions in \( \sum_p \) satisfying (1.3),

\[
M_{n+p}(\alpha) \subseteq M_{n+p-1}(\alpha) \quad (0 \leq \alpha < p).
\]

Since \( M_0(\alpha) \) equals \( \sum^*(\alpha) \)(the class of meromorphically \( p \)-valent starlike functions of order \( \alpha \) in \( U \) [4]), the starlikeness of members of \( M_{n+p-1}(\alpha) \) is a consequence of (1.4).

2. Properties of the class \( M_{n+p-1}(\alpha) \)
In proving our main results, we shall need the following lemma due to Jack [3].

**Lemma.** Let \( w \) be non-constant regular in \( U = \{ z : |z| < 1 \} \), \( w(0) = 0 \). If \( |w| \) attains its maximum value on the circle \( |z| = r < 1 \) at \( z_0 \), we have \( z_0w'(z_0) = kw(z_0) \), where \( k \) is a real number, \( k \geq 1 \).

**Theorem 1.** \( M_{n+p}(\alpha) \subset M_{n+p-1}(\alpha) \) for each integer \( n \) greater than \(-p\).

**Proof.** Let \( f(z) \in M_{n+p}(\alpha) \). Then

\[
\Re\left\{ \frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} \right\} < -\alpha. \tag{2.1}
\]

We have to show that (2.1) implies the inequality

\[
\Re\left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} \right\} < -\alpha. \tag{2.2}
\]

Define \( w(z) \) in \( U \) by

\[
\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = -\frac{p + (2\alpha - p)w(z)}{1 + w(z)}. \tag{2.3}
\]

Clearly, \( w(z) \) is regular and \( w(0) = 0 \). Using the identity

\[
z(D^{n+p-1}f(z))' = (n + p)D^{n+p}f(z) - (n + 2p)D^{n+p-1}f(z), \tag{2.4}
\]

the equation (2.3) may be written as

\[
\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{n + p + (n + 3p - 2\alpha)w(z)}{(n + p)(1 + w(z))}. \tag{2.5}
\]

Differentiating (2.5) logarithmically, we obtain

\[
\frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} = -\frac{p + (2\alpha - p)w(z)}{1 + w(z)} + \frac{2(p - \alpha)zw'(z)}{(1 + w(z))(n + p + (n + 3p - 2\alpha)w(z))}. \tag{2.6}
\]
We claim that $|w(z)| < 1$ in $U$. For otherwise (by Jack's lemma) there exists $z_0$ in $U$ such that

$$z_0 w'(z_0) = kw(z_0),$$

where $|w(z_0)| = 1$ and $k \geq 1$. The equation (2.6) in conjunction with (2.7) yields

$$\frac{z_0(D^{n+p}f(z_0))'}{D^{n+p}f(z_0)} = \frac{-p + (2\alpha - p)w(z_0)}{1 + w(z_0)} + \frac{2(p - \alpha)kw(z_0)}{(1 + w(z_0))(n + p + (n + 3p - 2\alpha)w(z_0))},$$

(2.8)

Thus

$$\text{Re}\left\{ \frac{z_0(D^{n+p}f(z_0))'}{D^{n+p}f(z_0)} \right\} \geq -\alpha + \frac{p - \alpha}{2(n + 2p - \alpha)} \geq -\alpha,$$

(2.9)

which contradicts (2.1). Hence $|w(z)| < 1$ in $U$ and from (2.3) it follows that $f(z) \in M_{n+p-1}(\alpha)$.

Theorem 2. Let $f(z) \in \sum_p$ satisfy the condition

$$\text{Re}\left\{ \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} \right\} < -\alpha + \frac{p - \alpha}{2(c + p - \alpha)} \quad (z \in U),$$

(2.10)

for a given integer $n > -p$ and $c > 0$. Then

$$F_c(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1}f(t)dt$$

(2.11)

belongs to $M_{n+p-1}(\alpha)$.

Proof. Let $f(z) \in M_{n+p-1}(\alpha)$. Define $w(z)$ in $U$ by

$$\frac{z(D^{n+p-1}F_c(z))'}{D^{n+p-1}F_c(z)} = -\frac{p + (2\alpha - p)w(z)}{1 + w(z)},$$

(2.12)

Clearly, $w(z)$ is regular and $w(0) = 0$. Using the identity

$$z(D^{n+p-1}F_c(z))' = cD^{n+p-1}f(z) - (c + p)D^{n+p-1}F_c(z),$$

(2.13)
The equation (2.12) may be written as

\[
\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} = \frac{p + (2\alpha - p)w(z)}{1 + w(z)} + \frac{2(p - \alpha)zw'(z)}{(1 + w(z))(c + (c + 2p - 2\alpha)w(z))}.
\]

(2.14)

We claim that \(|w(z)| < 1\) in \(U\). For otherwise (by Jack's lemma) there exists \(z_0\) in \(U\) such that

\[z_0w'(z) = kw(z_0),\]

where \(|w(z_0)| = 1\) and \(k \geq 1\). Combining (2.14) and (2.15), we obtain

\[
\frac{z_0(D^{n+p-1}f(z_0))'}{D^{n+p-1}f(z_0)} \geq -\alpha + \frac{p - \alpha}{2(c + p - \alpha)} \geq -\alpha,
\]

(2.16)

which contradicts (2.10). Hence \(|w(z)| < 1\) in \(U\) and from (2.12) it follows that \(F(z) \in M_{n+p-1}(\alpha)\).

Similarly, from Theorem 2, we have

**Corollary.** Let \(f(z) \in M_{n+p-1}(\alpha)\). Then \(F_c(z)\) defined by (2.11) belongs to the class \(M_{n+p-1}(\alpha)\).

**Remarks.**

(1). A result of Bajpai[1] turns out to be a particular case of the above Theorem 2 when \(p = 1, n = 0, \alpha = 0\) and \(c = 1\).

(2). For \(p = 1, n = 0\) and \(\alpha = 0\), the above Theorem 2 extends a result of Goel and Sohi[2].

**Theorem 3.** Let \(f(z) \in M_{n+p-1}(\alpha)\). Then \(F_{n+p}(z)\) defined by (2.11) with \(c = n + p\) belongs to the class \(M_{n+p}(\alpha)\).

**Proof.** For the function \(F_{n+p}(z)\) defined by (2.11) with \(c = n + p\), we have

\[
cD^{n+p-1}f(z) = (n + p)D^{n+p}F_{n+p}(z) - (n + p - c)D^{n+p-1}F_{n+p}(z)
\]

(2.17)
Taking \( c = n + p \) in the above relation (2.17), we obtain

\[
D^{n+p-1}f(z) = D^{n+p}F_{n+p}(z). \tag{2.18}
\]

This implies that \( F_{n+p} \) belongs to the class \( M_{n+p-1}(\alpha) \).

**Theorem 4**  Let \( F_c(z) \in M_{n+p-1}(\alpha) \) and let \( f(z) \) be defined as (2.11). Then \( f(z) \in M_{n+p-1}(\alpha) \) in \(|z| < R_c\), where

\[
R_c = \frac{-(p - \alpha + 1) + \sqrt{(p - \alpha + 1)^2 + c(c + 2(p - \alpha))}}{c + 2(p - \alpha)}. \tag{2.19}
\]

**Proof.** Since \( F_c(z) \in M_{n+p-1}(\alpha) \), we can write

\[
\frac{z(D^{n+p-1}F_c(z))'}{D^{n+p-1}F_c(z)} = -(\alpha + (p - \alpha)u(z)), \tag{2.20}
\]

where \( u(z) \in P \), the class of functions with positive real part in \( U \) and normalized by \( u(0) = 1 \). Using the equation (2.13) and differentiating (2.20), we obtain

\[
-\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} + \frac{\alpha}{p - \alpha} = u(z) + \frac{zu'(z)}{(c+p) - (\alpha + (p - \alpha) u(z))}. \tag{2.21}
\]

Using the well known estimates, \( \frac{|zu'(z)|}{Reu(z)} \leq \frac{2r}{1-r^2} (|z| = r) \) and \( Reu(z) \leq \frac{1+r}{1-r} (|z| = r) \), the equation (2.21) yields

\[
Re\left\{ -\frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)} + \frac{\alpha}{p - \alpha} \right\} \geq Reu(z)\left\{ 1 - \frac{2r}{(1-r^2)(c+p-(\alpha+(p-\alpha)\frac{1+r}{1-r}))} \right\}. \tag{2.22}
\]

Now the right hand side of (2.22) is positive provided \( r < R_c \). Hence \( f(z) \in M_{n+p-1}(\alpha) \) for \(|z| < R_c\).
References


Nak Eun Cho
Department of Applied Mathematics
College of Natural Sciences
National Fisheries University of Pusan
Pusan 608-737
Korea

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan