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Fermion Fock space on S^3

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S^3上のフェルミオン・フォック空間の構成

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1. Preliminaries

a Here we give a brief résume of [K1] to fix the notations. Let $M = \mathrm{C}^2 \coprod_v \widehat{\mathrm{C}}^2 \simeq S^4$; $w = v(z) = -\frac{\overline{z}}{|z|^2}$, and $E \simeq S^3$ be the equator . Let S (resp. S^+ and S^-) be the spinor bundle (resp. of positive chirality and of negative chirality) on M. The inner product of two spinors $\phi, \varphi \in \Gamma(S^{\pm})$ is defined by $< \phi(z), \varphi(z) >= \phi_1(z)\overline{\varphi}_1(z) + \phi_2(z)\overline{\varphi}_2(z)$. We denote by γ_0 Clifford mutiplication of the radial vector field $\mathbf{n}$ on M . γ_0 switches S^+ and S^-. Transition for spinors is given by $\widehat{\varphi}(v(z)) = -\overline{(\gamma_0\varphi)}(z)$. Let H (resp. H^*) be the space of square integrable spinors on E of positive (resp. negative) chirality. From the definition $< \phi, \psi >= 0$ for all $\phi \in H$ and $\psi \in H^*$.

The Dirac operator is of the form $\mathcal{D} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}$; $D : \Gamma(S^+) \to \Gamma(S^-)$. Let $\mathcal{P}$ be Hamiltonian on E. We have the radial decomposition of Dirac operator:

$$D = \gamma_0(\mathbf{n} - \mathcal{P}), \quad D^\dagger = (\mathbf{n} + \mathcal{P})\gamma_0.$$

The eigenvalues of $\mathcal{P}$ are $\pm(r+\frac{3}{2})$, $r = 0, 1, 2, \cdots$ with multiplicity $(r+1)(r+2)$. A complete orthonormal system of eigenspinors in H ; $\{\phi^q_{k,r-k}, \pi^{r-k,k}_q\}_{r,q,k}$ was given explicit forms in [K1];

$$\mathcal{P}\phi^q_{k,r-k} = (r + \frac{3}{2})\phi^q_{k,r-k} \qquad \mathcal{P}\pi^{r-k,k}_q = -(r + \frac{3}{2})\pi^{r-k,k}_q.$$

$$\phi^q_{k,r-k} = \left(\frac{q!k!(r-k)!}{(r+1-q)!}\right)^{-\frac{1}{2}} \begin{pmatrix} q2^{-q+1}h^{q-1}_{k,r-k} \\ \\ -2^{-q}h^q_{k,r-k} \end{pmatrix}$$

$$\pi^{r-k,k}_q = \left(\frac{q!k!(r-k)!}{(r+1-q)!}\right)^{-\frac{1}{2}} \begin{pmatrix} 2^{-q}\widehat{h}^{r-k+1,k}_q \\ \\ 2^{-q}\widehat{h}^{r-k,k+1}_q \end{pmatrix},$$

where

$$2^{-q}h^q_{k,r-k}(z_1,z_2) = (-\overline{z}_2\frac{\partial}{\partial z_1} + \overline{z}_1\frac{\partial}{\partial z_2})^q(z_1^k z_2^{r-k}),$$
$$2^{-q}\hat{h}^{r-k,k}_q(z_1,z_2) = (\overline{z}_2\frac{\partial}{\partial \overline{z}_1} - z_1\frac{\partial}{\partial z_2})^q(\overline{z}_1^k z_2^{r-k}).$$

Let H_+ (resp. H_-) be the subspace of H spanned by $\phi^q_{k,r-k}$'s (resp. $\pi^{r-k,k}_q$) . We put $H^*_\pm = \gamma_0 H_\pm$.

b For a triplet $\lambda = \{\pm r;\, k,p\}$, $0 \le r$, $0 \le k \le r$, $0 \le p \le r+1$, we put $-\lambda = \{\mp r,\, r-k,\, r+1-p\}$. Lexicographic order for the triplets $\lambda = \{s,\, p,\, k\}$ is defined by $\lambda \ge \lambda'$ if either (i)$s \ge s'$, or (ii)$s = s'$, $k \ge k'$, or (iii) $s = s'$, $k = k'$ and $p \ge p'$. Hence $\lambda \ge \lambda'$ implies $-\lambda \le -\lambda'$. The smallest positive is $o_+ = (\frac{3}{2}, 0, 0)$ while the largest negative is $o_- = (-\frac{3}{2}, 0, 1)$. Let $\alpha(p)$ denote the triplet at the p-th place after o_+ if p is non-negative (resp. at the p-th place before o_- if p is negative).

We denote by $\mathcal{Z}$ (resp. $\mathcal{Z}_{\ge 0}$ and $\mathcal{Z}_{<0}$) the set of all triplets λ (resp. $\lambda \ge o_+$ and $\lambda \le o_-$). We put also $\mathcal{Z}_{\le\alpha} = \{\beta \in \mathcal{Z};\, \beta \le \alpha\}$ for $\alpha \in \mathcal{Z}$.

A subset S of $\mathcal{Z}$ is called Maya diagram if both $S \cap \mathcal{Z}_{\ge 0}$ and $S^c \cap \mathcal{Z}_{<0}$ are finite set. The integer $\chi(S) = \#(\mathcal{Z}_{\ge 0} \cap S) - \#(\mathcal{Z}_{<0} \cap S^c)$ is called *charge* of S. For each Maya diagram S with $\chi(S) = p$ there corresponds a unique increasing function $s : \mathcal{Z}_{\le\alpha(p)} \to \mathcal{Z}$ such that (1) $s(\nu) = \nu$ for sufficiently small ν and (2) Image$(s) = S$. The degree of a Maya diagram S is the number $d(S) = \sum_\nu (s(\nu) - \nu)$.

2 Extensions and duality

a Let $R = \{z \in \mathrm{C}^2;\ |z| < 1\}$ and $\hat{R} = \{w \in \widehat{\mathrm{C}}^2;\ |w| < 1\}$. Let

$$\mathcal{N}(R) = \{\phi \in \Gamma(R, S^+);\ \ \phi \text{ has } L^2\text{-boundary value on } |z| = 1, D\phi = 0\},$$
$$\mathcal{N}^\dagger(R) = \{\psi \in \Gamma(R, S^-),\ \ \phi \text{ has } L^2\text{-boundary value on } |z| = 1, D^\dagger\psi = 0\,\}.$$

$\mathcal{N}(\hat{R})$ and $\mathcal{N}^\dagger(\hat{R})$ are defined similarly.

We have proved in [K1] :

Theorem 1.

(1) $H_+ \cong \mathcal{N}(R), \qquad H_- \cong \mathcal{N}(\hat{R}),$
(2) $H_-^* \cong \mathcal{N}^\dagger(R)_0 \qquad H_+^* \cong \mathcal{N}^\dagger(\hat{R})_0,$

where 0 *indicates that the spinors in brace are* 0 *at* $0 \in \mathrm{C}^2$ *or at* $\hat{0} \in \widehat{\mathrm{C}}^2$.

For instance, the isomorphism $H_+^* \to \mathcal{N}^\dagger(\widehat{\mathrm{C}}^2)_0$ is given as follows: Let $\psi = \gamma_0\phi \in H_+^*$. We shall show that there is a $\widehat{\Psi} \in \mathcal{N}^\dagger(\hat{R})_0$ such that $\widehat{\Psi}(w) = \hat{\psi}(w)$ for $|w| = 1$, where $\hat{\psi}(v(z)) = -\overline{\gamma_0\psi}(z)$. Let $\phi = \sum_{\lambda>0} a_\lambda \phi_\lambda \in H_+$ be the eigenfunction expansion.
Put $\Phi(z) = \sum a_\lambda |z|^{-(\lambda-\frac{3}{2})}(\frac{2}{1+|z|^2})^{\frac{3}{2}}\phi_\lambda(\frac{z}{|z|})$. The expression on $\widehat{\mathrm{C}}^2$ becomes

$$\widehat{\Phi}(w) = \sum a_\lambda |w|^{(\lambda+\frac{3}{2})}(\frac{2}{1+|w|^2})^{\frac{3}{2}}\widehat{\phi}_\lambda(\frac{w}{|w|}),$$

$\widehat{\Phi}(v(z)) = -\overline{\gamma_0\Phi(z)}$. $\widehat{\Phi}$ is valued in Δ^- . We can verify that $\widehat{\Psi} = \overline{\gamma}_0\widehat{\Phi} \in \mathcal{N}^\dagger(\hat{R})_0$ and $\widehat{\Psi}(w) = \hat{\psi}(w)$ for $|w| = 1$.

We define a pairing of H and H^* by

$$(\psi|\phi) = \int_E < \phi, \gamma_0\psi > \sigma(dz) \qquad \text{for } \phi \in H \text{ and } \psi \in H^*.$$

Theorem 1 and Stokes' theorem yield that $H_\pm$ and $H_\mp^*$ are annihilated mutually by this pairing. On the other hand, $H_\pm$ and $H_\pm^*$ are respectively in duality . This is proved by Hahn-Banach's extension theorem.

A coupling between $\mathcal{N}(R)$ and $\mathcal{N}^\dagger(\hat{R})_0$ is defined by

$$-\int_E \Phi(z)\cdot\widehat{\Psi}(\nu(z))\sigma(dz) = \int_E < \Phi, \gamma_0\Psi > \sigma(dz),$$

for $\Phi \in \mathcal{N}(R)$ and $\widehat{\Psi} \in \mathcal{N}^\dagger(\hat{R})_0$. Also the coupling of $\Psi \in \mathcal{N}(\hat{R})$ and $\Phi \in \mathcal{N}^\dagger(R)_0$ is defined by the same integral.

The duality between $H_\pm$ and $H_\pm^*$ in the above and Theorem 1 prove the following:

Theorem 2.

(1) *The dual of* $\mathcal{N}(R)$ *is isomorphic to* $\mathcal{N}^{\dagger}(\hat{R})_0$.

(2) *The dual of* $\mathcal{N}(\hat{R})$ *is isomorphic to* $\mathcal{N}^{\dagger}(R)_0$.

3 Fockspace on E

a Let

$$e_\lambda = \begin{cases} \phi^p_{k,r-k} \in H_+ & \text{if } \lambda \geq o_+ \\ \pi^{r-k,k}_p \in H_- & \text{if } \lambda \leq o_- \end{cases}.$$

We define the conjugation by $e^{*\lambda} = \gamma_0\, e_{-\lambda}$. It follows that $e^{*\lambda} \in H^*_-$ if $\lambda \geq 0$ and $e^{*\lambda} \in H^*_+$ if $\lambda < 0$. We have $(e^{*\lambda} | \, e_\mu) = \delta_{-\lambda,\mu}$. In particular $(e^{*\,o_+} | \, e_{o_-}) = 1$.

For a Maya-diagram S we put $e_S = \wedge e_\lambda = e_{\max S} \wedge \cdots$, the wedge being taken on decreasing order. We denote in prticular $|\alpha >= e_{\mathcal{Z}_{\alpha-}} = e_\alpha \wedge \cdots$.

The *Fock space* of charge p and total Fock space are introduced as follows:

$$\mathcal{F}_p = \Pi_{\{S;\, \chi(S)=p\}} \mathrm{C}\, e_S \qquad \mathcal{F} = \oplus_p \mathcal{F}_p.$$

$\mathcal{F}_p$ is given a filtration by the degree of Maya-diagramm introduced in setion 1 and this filtration endows $\mathcal{F}_p$ with a complete vector space topology.

For a Maya-diagram S we put $e^*_S = \wedge_{-\mu \in S}\, e^{*\mu} = \cdots \wedge e^{*\,-\max S}$, the wedge being taken on decreasing order. We denote $< \alpha | = e^*_{\mathcal{Z}_{\alpha-}} = \cdots \wedge e^{*\,-\alpha}$.

The dual Fock space is defined as a direct sum with discrete topology:

$$\mathcal{F}^* = \bigoplus_S \mathrm{C}\, e^*_S.$$

The coupling $(\,|\,)$ of $H_\pm$ and $H^*_\pm$ extends to give a coupling between $\mathcal{F}$ and $\mathcal{F}^*$. We have $(\, e^*_S \,| e_{S'}\,) = \delta_{S,S'}$. In particular we have $< \alpha | \beta >= \delta_{\alpha,\beta}$.

Differentiation D_α by $\alpha \in H$ is defined on H by

$$D_\alpha \phi = (e^{*\,-\alpha} | \, \phi) = \int_E < \phi, \alpha > d\sigma \quad \text{for } \phi \in H.$$

It is extended to $\mathcal{F}$ by the rule

$$D_\alpha(\phi \wedge \psi) = D_\alpha\phi \wedge \psi + (-1)^{\deg \phi} \phi \wedge D_\alpha \psi$$

for $\phi, \psi \in \mathcal{F}$. D_α acts on $\mathcal{F}$ from the left as an inner derivation.

We also define the differentiation on H^* by

$$D^*_\alpha \phi^* = (\phi^* | e_\alpha), \qquad \text{for } \phi^* \in H^*.$$

It is extended to $\mathcal{F}^*$ by $D^*_\alpha(\phi^* \wedge \phi^*) = \phi^* \wedge D^*_\alpha \psi^* + (-1)^{\deg \psi^*} D^*_\alpha \phi^* \wedge \psi^*$ for $\phi^*, \psi^* \in \mathcal{F}^*$. D^*_α acts on $\mathcal{F}^*$ from the right.

b We define the following actions a_ν, $a^\dagger_\nu$ on $\mathcal{F}$ and $\mathcal{F}^*$:

$$\begin{aligned} a_\nu &= D_\nu, \quad a^\dagger_\nu = e_\nu \wedge \quad \text{left action on } \mathcal{F}, \\ a_\nu &= \wedge e^{*\,-\nu}, \quad a^\dagger_\nu = D^*_\nu \quad \text{right action on } \mathcal{F}^*, \end{aligned}$$

where exterior multiplications should be arranged in order. We have then the relations

$$\begin{aligned} \{a_\lambda, a_\nu\} &= 0, \quad \{a^\dagger_\lambda, a^\dagger_\nu\} = 0 \\ \{a^\dagger_\lambda, a_\nu\} &= \{a_\lambda, a^\dagger_\nu\} = \delta_{\lambda,\nu}. \end{aligned}$$

Hence $\{a_\nu,\ a^\dagger_\nu\}$ generate a Clifford algebra $\mathcal{A}$, which is called *fermion operator algebra* . $\mathcal{A}$ acts on $\mathcal{F}$ and $\mathcal{F}^*$.

Proposition 3.

(1)

$$a_\nu|\alpha> = 0 \quad \text{for } \nu > \alpha \qquad a^\dagger_\nu|\alpha>= 0 \quad \text{for } \nu \le \alpha$$

$$<\alpha|a_\nu = 0 \quad \text{for } \nu \le \alpha \qquad <\alpha|a^\dagger_\nu = 0 \quad \text{for } \nu > \alpha.$$

(2)

$$(e^*_S a_\alpha | e_{S'}) = (e^*_S | a_\alpha e_{S'})$$

$$(e^*_S | a^\dagger_\alpha e_{S'}) = (e^*_S a^\dagger_\alpha | e_{S'})$$

c We shall introduce the following field operators of fermion:

$$\varphi_+(z) = \sum_{\nu \ge o_+} \phi_\nu(z)\, a_\nu \qquad \varphi^\dagger_-(z) = \sum_{\nu \ge o_+} {}^t\overline{\phi_\nu(z)}\, a^\dagger_\nu$$

$$\varphi_-(w) = \sum_{\nu \le o_-} \widehat{\pi}_\nu(w)\, a_\nu \qquad \varphi^\dagger_+(w) = \sum_{\nu \le o_-} {}^t\overline{\widehat{\pi}_\nu(w)}\, a^\dagger_\nu.$$

From the above proposition we have;

$$\varphi_+(z)|\, o_- > = 0, \qquad \varphi_+^\dagger(z)|\, o_- >= 0$$

$$< o_-|\, \varphi_-(z) = 0, \qquad < o_-|\, \varphi_-^\dagger(z) = 0.$$

Proposition 4.

$$< \varphi^\dagger(x)\varphi(y) > =< o_-|\varphi^\dagger(x)\cdot\varphi(y)|o_- >= \sum_r \sum_{q=0}^{r+1} \frac{r+1}{q!} h^q_{r+1-q,q}(A,B)$$

$$< \varphi(x)\varphi^\dagger(y) > =< o_-|\varphi(x)\cdot\varphi^\dagger(y)|o_- >= \sum_r \sum_{q=0}^{r+1} \frac{r+2}{q!} h^q_{r-q,q}(C,D) \quad ,$$

$$< \varphi(x)\varphi(y) > =< \varphi^\dagger(x)\varphi^\dagger(y) >= 0$$

where

$$A = \overline{x}_1 y_1 + x_2\overline{y}_2 \qquad B = \overline{x}_1 y_2 - x_2\overline{y}_1$$
$$C = x_1\overline{y}_1 + x_2\overline{y}_2 \qquad D = x_1 y_2 - x_2 y_1.$$

References

[K] Kori, T., Dirac operators on S^4 and on S^3. In nite dimensional Grassmanian on S^3.
[K] Kori,T., Extension problems for spinors on S^4..