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SUPERCOMPACTNESS AND NORMAL SUPERCOMPACTNESS

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ABSTRACT

A space is called supercompact if it has an open subbase such that every cover consisting of elements of the subbase has a subcover consisting of two elements. A space is called normally supercompact if it is has a normal open subbase with the property. In this paper we prove that: (1). In a continuous image of a closed G_δ-set of a supercompact space, a point is a cluster point of a countable set if and only if it is the limit of a nontrivial sequence; which answer questions asked by J. van Mill et al. (2). A space is normally supercompact if and only it homomeomorphism to a certain poset with the Lawson topology.

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In this paper, we consider Hausdoff spaces only and if not otherwise stated, subbase means subbase for closed sets. Let $\mathscr{S}$ be a closed family of a space X, we say that

$\mathscr{S}$ is *linked* if $S \cap S' \neq \phi$ for any $S, S' \in \mathscr{S}$;

$\mathscr{S}$ is *binary* if every linked subfamily has nonempty intersection; and

$\mathscr{S}$ is *normal* if for every pair of $S, S' \in \mathscr{S}$, $S \cap S' = \phi$ implies that there exist $T, T' \in \mathscr{S}$ such that

$$S \cap T = \phi = S' \cap T' \text{ and } T \cup T' = X.$$

A space is called *normally supercompact* if it has a normal binary subbase[10]. A space is called *supercomact* if it has a binary subbase[8]. It is trivial that every supercompact space is compact and every normally supercompact space is supercompact. S^1 is supercompact but not normally supercompact[10]. Many compact spaces, but not all, are supercompact. For example, all compact metric spaces are supercompace[5,13]; all continuous images of compact ordered spaces are supercompact[4]. On the other hand, closed G_δ-sets of supercompact spaces are not supercompact in general[3], nor continuous images of supercompact spaces[12]. Moreover, M.G.Bell gave an example of a non-supercompact dyadic space (=a continuous image of 2^κ)[2]. Without loss of generality we can assume that every (normally) supercompact space has a (normal) binary subbase which is closed with respect to arbitrary intersection and hence, by the Haudsorffness, every sigular point set is in the subbase.

§1.

In 1982, E.K. van Douwen and J.van Mill proved in [6] that in a continuous image of a supercompact space, at least one cluster point of a countable set is the limit of a nontrivial convergent sequence in the whole space; and at most countable many cluster points are not so. The result suggested to them the following question:

Question 1.1.[6] Let Y be a continuous image of a supercompact space (or just a supercompact space). If K is a countable subset of Y, then is every cluster point of K the limit of a nontrivial convergent sequence ?

Applying the result mentioned above, J. van Mill and C.F. Mills proved in [11] that under a set theoretical hypothesis, every infinite continuous image of a closed G_δ-set of a supercompact space contains a nontrivial convergent sequence. Then, they asked if the set theoretical hypothesis may be dropped.

Question 1.2.[11] If Y is an infinite continuous image of a closed G_δ-set of a supercompact space, then does Y contain a nontrivial convergent sequence?

In this sectoon, we prove the next theorem which answers the above two questions affirmatively.

Theorem 1.1. Let Y be a continuous image of a closed G_δ-subset of a supercompact space, and K a countable subset of Y. Then every cluster point of K is the limit of a nontrivial convergent sequence in Y.

To show the theorem, we first give some lemmas. The first two lemmas can be directly proved. Let N be the natural numbers set.

Lemma 1.1. Let $f: X \longrightarrow Y$ be a continuous mapping from a compact space X onto a space Y and $\{A_n \subset X: n \in N\}$ a decreasing sequence of closed sets of X. If $\bigcap_{n \in N} A_n \subset f^{-1}(y)$ for some $y \in Y$, then $f(a_n) \longrightarrow y$ for any $a_n \in A_n$.

Now let $\mathcal{S}$ be a subbase (note that subbase means subbase for closed) for a compact X. We fix a point $p \in X$. For $A \subset X$, let

$$J(A) = \bigcap \{S \in \mathcal{S}: p \in S \text{ and } S \cap A \neq \phi\}.$$

If $A=\{a\}$, we write $J(a)$ instead of $J(\{a\})$.

Lemma 1.2. Let $\mathcal{S}$ be a subbase for a compact space X and F a closed subset of X, U an open subset. If $F \subset U$, then there exist $S_1, S_2, \ldots, S_n \in \mathcal{S}$ such that $F \subset S_1 \cup S_2 \cup \ldots \cup S_n \subset U$. In particular, if $x \in U$ for some point $x \in X$, then there exist $S_1, S_2, \ldots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap S_2 \cap \ldots \cap S_n$ and $x \in \operatorname{int}(S_1 \cup S_2 \cup \ldots \cup S_n) \subset S_1 \cup S_2 \cup \ldots \cup S_n \subset U$.

Lemma 1.3. Let A, $B\subset X$. If for every $S\in\mathcal{S}$ with $p\in S$, $S\cap A\neq\phi$ implies $S\cap B\neq\phi$, then $p\in\bar{A}$ implies $p\in\bar{B}$. In particular, if $p\in\bar{A}$, then $J(A)=\{p\}$.

Proof. If $p\notin\bar{B}$, then by Lemma 1.2 there exist $S_1, S_2, \ldots, S_n\in\mathcal{S}$ such that $p\in S_1\cap S_2\cap\ldots\cap S_n$ and

$$p\in\operatorname{int}(S_1\cup S_2\cup\ldots\cup S_n)\subset S_1\cup S_2\cup\ldots\cup S_n\subset X\setminus\bar{B}. \qquad (1)$$

Since $p\in\bar{A}$, there exists S_i such that $S_i\cap A\neq\phi$. Hence $S_i\cap B\neq\phi$, which contradicts to (1). Now for any point $q\in J(A)$, we have $p\in\overline{\{q\}}=\{q\}$ since $S\cap A\neq\phi$ implies $q\in S$ for every $S\in\mathcal{S}$ with $S\ni p$. Hence $p=q$.

Lemma 1.4. Let E, $Z\subset X$ be closed sets and $C=\{c_n: n\in N\}\subset Z$ a countable set. If $p\in E\cap\bar{C}$ and $E\cap C=\phi$, then one of the following statements holds:

(A): There exists an increasing sequence $\{A_n: n\in N\}$ of subsets of C such that $Z\cap J(A_n)\not\subset E$ for all $n\in N$ but $Z\cap\bigcap_{n\in N}J(A_n)\subset E$.

(B): There exists a sequence $\{A_n:n\in N\}$ of subsets of C such that $C=\bigcup_{n\in N}A_n$ and $Z\cap J(A_n)\not\subset E$ but $Z\cap J(A_n)\cap J(A_m)\subset E$ for all n, $m\in N$, $n\neq m$.

Proof. Suppose that there exists no sequence of subsets of C satisfying the conditions in (A). Then we construct $\{A_n: n\in N\}$ so that for all $n\in N$ and $c\in C\setminus\bigcup_{i\leq n}A_i$

$$Z\cap J(A_n)\not\subset E,$$

$$Z\cap J(A_n)\cap J(c)\subset E,$$

$$c_{k(n)} \in A_n \subset C \setminus \bigcup_{i<n} A_i,$$

where $k(n)$ is the least k satisfying $c_k \in C \setminus \bigcup_{i<n} A_i$.

In fact, if $\{A_i: i<n\}$ have been defined satisfying the required conditions then $\bigcup_{i<n} A_i \neq C$ since, otherwise, $p \in \bar{A}_i$ for some $i<n$ and hence Lemma 1.3 implies that $Z \cap J(A_i) \subset J(A_i) = \{p\} \subset E$, which contradict to the assumption. Since C is countable, $c_{k(n)} \in C \setminus (E \cup \bigcup_{i<n} A_i)$ and (A) does not hold, there exists a maximal subset A_n of $C \setminus \bigcup_{i<n} A_i$ such that $c_{k(n)} \in A_n$ and $Z \cap J(A_n) \not\subset E$. Then for all $c \in C \setminus \bigcup_{i \leq n} A_i$, we have $Z \cap J(A_n) \cap J(c) \subset E$. The inductive definition is completed. It is clear that the sequence $\{A_n: n \in N\}$ satisfies the required conditions in (B).

Proof of Theorem 1.1. Suppose that Y and $K \subset Y$ satisfy the conditions in Theorem and $y \in \bar{K} \setminus K$. Let X be a supercompact space with a binary subbase $\mathcal{S}$ and $Z \subset X$ a closed G_δ-set, and let $f: Z \longrightarrow Y$ be a continuous mapping from Z onto Y. Then there exists a countable set $C \subset Z$ and $p \in Z$ such that $f(C) = K$ and $p \in \bar{C} \cap f^{-1}(y)$. Clearly, $E = f^{-1}(y)$, Z and C satisfy the requests in the last lemma. Hence there exists a sequence $\{A_n: n \in N\}$ of subsets of C satisfying the conditions in (A) or (B). Choose $z_n \in Z \cap J(A_n) \setminus f^{-1}(y)$. Then $\{f(z_n): n \in N\}$ is a sequence in Y and $f(z_n) \neq y$ for all $n \in N$. If (A) holds, then Lemma 1.1 implies $f(z_n) \longrightarrow y$. If (B) holds, then, by Lemma 1.3, we have that

$$p \in \overline{\{z_n: n \in N\}}. \qquad (2)$$

$$Z \cap J(z_n) \cap J(z_m) \subset f^{-1}(y) \qquad (3)$$

for all $n \neq m$. To complete the proof of the theorem, it suffices to show the following lemma:

Lemma 1.5. If $D=\{z_n: n\in N\}\subset Z\setminus f^{-1}(y)$ satisfies (2) and (3), then there exists a subsequence $\{z_{n_k}, k\in N\}$ of $\{z_n, n\in N\}$ such that $f(z_{n_k}) \longrightarrow y$.

Proof. Since Z is a G_δ-set, let $Z=\bigcap\{U_k: k\in N\}$ for open subsets U_k $(k\in N)$ of X with $U_{k+1}\subset U_k$. Then, by Lemma 1.2, for every $k\in N$ there exist $S_1, S_2, \ldots, S_m\in\mathcal{S}$ such that

$$p\in S_1\cap S_2\cap \ldots\cap S_m$$

and

$$p\in \operatorname{int}(S_1\cup S_2\cup\ldots\cup S_m)\subset S_1\cup S_2\cup\ldots\cup S_m\subset U_k.$$

Since $p\in\bar{D}$, there exists S_i such that $S_i\cap D$ is infinite. Thus $\{n: J(z_n)\subset U_k\}$ is infinite for $k\in N$. Therefore, we can inductively define $\{n_k: k\in N\}$ such that $n_1<n_2<\ldots$ and for $k\in N$

$$J(z_{n_k})\subset U_k. \tag{4}$$

Then $f(z_{n_k})\longrightarrow y$. In fact, otherwise, there exists an open set $V\ni y$ in Y such that $\{k: f(z_{n_k})\notin V\}$ is infinite. It follows from $f^{-1}(y)\subset f^{-1}(V)$ and Lemma 1.2 that there exist $T_1, T_2, \ldots$ $T_m\in\mathcal{S}$ such that

$$X\setminus f^{-1}(V)\subset T_1\cup T_2\cup\ldots\cup T_m\subset X\setminus f^{-1}(y). \tag{5}$$

Since $\{k: f(z_{n_k})\notin V\}$ is infinite there exists T_i such that $\{k: z_{n_k}\in T_i\}$ is infinite. Thus, we have

$$\bigcap\{J(z_{n_k}): z_{n_k}\in T_i\}$$

$$\subset \bigcap \{U_k : z_{n_k} \in T_i\}$$
$$= Z$$

Hence, it follows from (3) and (5) that

$$T_i \cap \bigcap \{J(z_{n_k}) : z_{n_k} \in T_i\}$$
$$= T_i \cap Z \cap \bigcap \{J(z_{n_k}) : z_{n_k} \in T_i\}$$
$$\subset T_i \cap f^{-1}(y)$$
$$= \phi .$$

On the other hand the family

$$\{T_i\} \cup \{J(z_{n_k}) : z_{n_k} \in T_i\}$$

is a linked subfamily of $\mathcal{S}$. Hence,

$$T_i \cap \bigcap \{J(z_{n_k}) : z_{n_k} \in T_i\} \neq \phi$$

since $\mathcal{S}$ is binary (This is the only point in the proof where we use the fact that $\mathcal{S}$ is binary). Now a contradiction occurs.

Remark. For a nonisolated point $y \in Y$, let

$$t(y) = \min\{|A| : A \subset Y \setminus \{y\} \text{ and } \bar{A} \ni y\}.$$

In Theorem, we have proved that in certain spaces Y, if $t(y)$ is countable, then y is the limit of a nontrivial sequence in Y. In fact, it is not difficult to extend the result to a general case. We call $Z \subset X$ to be a G_μ-set if $Z = \bigcap \{U_\xi : \xi < \mu\}$ for a decreasing open family $\{U_\xi : \xi < \mu\}$. Then we have

Let Y *be a continuous image of a closed* G_μ*-set of a supercompact space and* $y \in Y$ *a nonisolated point. If* $\mu \leq cf(t(y))$, *then* y *is the limit of a nontrivial* α*-sequence in* Y *for some* *limit ordinal* $\alpha \le t(y)$.

From the statement the following corollary is obtained:

Corollary 1.1. If Y is a continuous image of a supercompact space, then every nonisolated point in Y is the limit of a nontrivial linear net.

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§2.

Let P be a partially ordered set (poset for short) and $A \subset P$, we denote the supremum of A, if it exists, by $\sup A$ or $\sup_P A$. If $A=\{a_1, a_2, \ldots, a_n\}$, then we write $a_1 \vee a_2 \vee \ldots \vee a_n$ instead of $\sup A$. Similary, for infimum, by $\inf A$ or $a_1 \wedge a_2 \wedge \ldots \wedge a_n$. The greatest element and the least element of P, if they exist, are denoted by $\top$ and $\bot$, respectively. Below, we always assume that in a poset, every directed set has supremum. For $a,b \in P$, a is *way-below* to b, which is denoted by $a \ll b$ if for every directed set $D \subset P$ with $\sup D \ge b$, there exists $d \in D$ such that $d \ge a$. If $a \ll a$, then a is called *compact* in P. For $A \subset P$, let $\downarrow A=\{x \in P: x \le a$ for some $a \in A\}$. For $a \in P$, let $\downarrow a = \downarrow\{a\}$ and $\twoheaddownarrow a=\{x \in P: x \ll a\}$. Dually we define $\uparrow A$, $\uparrow a$ and $\twoheaduparrow a$. P is called *continuous poset* if for every $x \in P$, $\twoheaddownarrow x$ is directed and $x = \sup \twoheaddownarrow x$. Furthermore, if P is complete, then P is called a *continuous lattice*. It can be proved that a complete lattice is continuous if and only if it satisfies the distributive law for arbitrary infimums and directed supremums[7,p.58].

Now we introduce a new concept. Let P be a poset. $D\subset P$ is called *relatively directed* if for every pair $a,b\in D$, there exists $x\in P$ such that $x\geq a,b$. It is trivial that every set is relatively directed in a poset containing the greatest element and every directed set is relativly directed in any poset. A poset L is called a *completely distributive poset* (*CDP* for short) if

(CDP 1) every nonempty set has a infimum;

(CDP 2) every relatively directed set has a supremum; and

(CDP 3) for every family $\{A_i: i\in I\}$ of relatively directed subsets of L, we have

$$\inf_{i\in I} \sup A_i=\sup\{ \inf_{i\in I}\{f(i)\}: f\in \prod_{i\in I} A_i\}.$$

It is trivial that completely distributive lattices (CDL for short) are exactly CDP's with the greatest elements. Clearly, a subset A of a CDP P is relatively directed if and only if $a\vee b$ exists in P for every pair of a,b in A.

Lemma 2.1. Let L be a CDP and $L^*=L\cup\{\top\}$, where $\top$ is the added greatest element in L. Then L is a continuous poset and L^* is a continuous lattice in which $\top$ is compact.

Proof. It is followed from the defintion and Theorem 2.3 in [7, p.58].

Remark. L^* is not necessarly a CDL, see the later example.

Lemma 2.2. For every CDP L and $x\in L$, $\downarrow x\subset L$ is a CDL and is closed with arbitrary supremums and arbitrary infimums in L.

Proof. It is trivial.

For a poset P, $m\in P$ is called a *molecule* (In [7] it is called co-prime) if for every $a, b\in P$, $m\le a\vee b$ implies that $m\le a$ or $m\le b$. The set of all molecules in P is denoted by $M(P)$.

Lemma 2.3. For every CDP L, the following statements hold.

(1). $M(L)$ is a continuous poset and for every $x\in L$,

$$x=\sup\{m\in M(L):\ m\ll x\}.$$

(2). For any $m\in M(L)$ and $a,b\in L$, if $m\ll a\vee b$, then $m\ll a$ or $m\ll b$.

Proof. (1). First, we note that $M(\downarrow x)=M(L)\cap\downarrow x$ for all $x\in L$. In fact, if $m\in M(\downarrow x)$ and $a,b\in L$ such that $m\le a\vee b$ then $m\le x\wedge(a\vee b)$ $=(x\wedge a)\vee(x\wedge b)$ and hence $m\le x\wedge a\le a$ or $m\le x\wedge b\le b$. Thus $m\in M(L)\cap\downarrow x$. The inversion is trivial. Secondly, it is followed from the above lemma and 3.15 Theorem in [7. p.72] that for all $x\in L$

$$x=\sup_{\downarrow x}\{m\in M(\downarrow x):\ m\ll x\}$$
$$=\sup_L\{m\in M(L):\ m\ll x\}.$$

In particular, for all $m\in M(L)$, $m=\sup_{M(L)}\Downarrow m\cap M(L)$. It follows that $M(L)$ is a continuous poset.

(2). By (1) we have $a\vee b=\sup\{x\vee y:\ x\ll a$ and $y\ll b\}$. Because

$\Downarrow a$ and $\Downarrow b$ are directed and $m \ll a \vee b$ there exist $x \ll a$ and $y \ll b$ such that $m \leq x \vee y$. It is followed from $m \in M(L)$ that $m \leq x \ll a$ or $m \leq y \ll b$.

Let P be a poset. Set

$\sigma(P)=\{U \subset P:\ U=\uparrow U$ and $P \setminus U$ is closed with directed sups$\}$.

Then $\sigma(P)$ is a topology on P (non-Hausdorff unless in some speical case) which is called *Scott topology*[7]. Moreover, it is proved that

(1). If P is a continuous poset then $\{\Uparrow x:\ x \in P\}$ is an open base for $\sigma(P)$, [7, p.107].

(2). P is a continuous poset if and only if $\sigma(P)$ with the inclusion relation is a CDL and then $M(\sigma(P))$ isomorphis to P[9].

The *Lawson topology* $\lambda(P)$ (see [7]) on P is the topology generated by $\sigma(P) \cup \{P \setminus \uparrow x:\ x \in P\}$ as an open subbase. The topological space $(P, \lambda(P))$ is denoted by ΛP. Many well-known topologies are the Lawson topologies on natural orders. For example, the product topology on I^m is the Lawson topology on the pointwise order, and more generally the interval topology generated by $\{\downarrow x:\ x \in L\} \cup \{\uparrow x:\ x \in L\}$ as a subbase on a CDL L is the Lawson topology, see [7, p.167 and p.204]; for a locally compact space, the Vietoris topology on the all closed sets is the Lawson topology on the inversely inclusion order[7, p.284].

Remark. Unlike CDL, it is not necessary that the Lawson topology and the interval topology coincide for a CDP.

Example. Let $L=\{0,1,2,...\}$ and for $a,b\in L$, define $a\le b$ if and only if $a=b$ or $a=0$. Clearly, L is a CDP, and hence ΛL is Hausdorff (see the following lemma) but the interval topology is not Hausdorff.

Lemma 2.4. For every CDP L, ΛL is a compact Hausdorff space.

Proof. It is followed from Lemma 2.1 and [4, p.146] that ΛL^* is a compact Hausdorff space and ΛL is a closed subspace since τ is compact.

Our main theorem in this section is the following one.

Theorem 2.1. A space X is normally supercompact if and only if X is homeomorphic to ΛL for a CDP L.

Proof. *Necessity.* Let X be a normally supercompact space with a normal binary subbase $\mathcal{S}$. As mentioned above, we can assume that $\mathcal{S}$ is closed with arbitrary intersection. Moreover, we suppose that $\phi\notin\mathcal{S}$ but $X\in\mathcal{S}$. For $A\subset X$, let

$$I(A)=\{S\in\mathcal{S}:\ S\supset A\}.$$

If $A=\{a,b\}$, then $I(A)$ is denoted by $I(a,b)$. For a fixed point $\perp\in X$, the following partial order can be defined:

$$x\le y \text{ if and only if } I(\perp,x)\subset I(\perp,y).$$

Then we have(see [10]):

Fact 1. For every $x\in X$, $\downarrow x=I(\perp,x)\in\mathcal{S}$;

Fact 2. For every $x, y \in X$, if $x \leq y$ then $[x, y] = I(x, y)$.

Fact 3. For every nonempty set $A \subset X$, $\inf A$ exists and

$$I(A) \cap \bigcap \{\downarrow a:\ a \in A\} = \{\inf A\}.$$

Fact 4. For every $S \in \mathcal{S}$, $S = \downarrow S$ if and only if $S \ni \perp$.

Lemma 2.5. For every relatively directed set $A \subset X$, $\sup A$ exists.

Proof. Case 1. $A=\{a,b,c\}$ is a set of three points. Then the family $\{I(a\vee b,\ b\vee c),\ I(b\vee c,\ c\vee a),\ I(c\vee a,\ a\vee b)\}$ is a linked subfamily of $\mathcal{S}$. Hence by $\mathcal{S}$ being binary there exists $x \in I(a\vee b,\ b\vee c) \cap I(b\vee c,\ c\vee a) \cap I(c\vee a,\ a\vee b)$. Now we have only to verify that $a, b, c \leq x$. Otherwise, for example, $a \not\leq x$, then there S_1, $S_2 \in \mathcal{S}$ such that

$$a \notin S_1,\ \downarrow x \cap S_2 = \phi \ \text{ and } \ S_1 \cup S_2 = X.$$

Then there exist at least two elements in the set $\{a\vee b,\ b\vee c,\ c\vee a\}$ which belong to S_1 and hence, there exists at least one element in the set which is greater than a and belongs to S_1. Because $S_1 \supset \downarrow x \ni \perp$ we have that $S_1 \ni a$, which contradicts to the assumpations.

Case 2. A is finite. Suppose that $n>3$ and the statement hold for all A with $|A|=n-1$. Now let $A=\{a_1,a_2,\ldots,a_n\}$ be a relatively directed set. Set $B=\{a_1\vee a_2,\ a_3,\ldots,\ a_n\}$. Then $|B|=n-1$ and B is relatively directed by Case 1. Thus $\sup A = \sup B$ exists.

Case 3. For general case. By Case 2, we assume that A is directed. Because X is compact the net $\{a,\ a \in A\}$ has a cluster point x. Without loss of generality, suppose that

$x=\lim\{a, a\in A\}$. Then we have $x=\sup A$. In fact, if there exists $a_0\in A$ such that $a_0\not\leq x$, then $a_0\notin{\downarrow}x$ and hence, by the normality of $\mathcal{S}$, there exist $S_1, S_2\in\mathcal{S}$ such that

$$a_0\notin S_1,\ {\downarrow}x\cap S_2=\phi \text{ and } S_1\cup S_2=X.$$

Then for every $a\in A\cap{\uparrow}a_0$, we have $a\notin S_1$(otherwise $a_0\leq a\in S_1\supset{\downarrow}x\ni\perp$ and hence $a_0\in S_1$) and hence $x=\lim\{a, a\in A\}=\lim\{a, a\in A\cap{\uparrow}x\}\in S_2$ since S_2 is closed. A contradiction. On the other hand, if $y\in X$ such that $y\geq a$ for all $a\in A$, then $A\subset{\downarrow}y$. Hence $x=\lim A\in{\downarrow}y$ since ${\downarrow}y=I(\perp, y)$ is closed, that is, $a\leq b$.

Lemma 2.6. Let $\{A_i:i\in I\}$ be a family of relatively directed sets. Then

$$\inf_{i\in I}\sup A_i=\sup\{\inf_{i\in I}\{f(i)\}: f\in \prod_{i\in I} A_i\}.$$

Proof. Let $a_i=\sup A_i$ and $a=\inf\{a_i: i\in I\}$, $b=\sup\{\inf_{i\in I}\{f(i)\}: f\in \prod_{i\in I} A_i\}$. It is trivial that $a\geq b$. Now suppose that $a\neq b$. Then ${\downarrow}b\cap\{a\}=\phi$. By the normarity of $\mathcal{S}$, there exist $S_1, S_2\in\mathcal{S}$ such that

$$a\notin S_1,\ S_2\cap{\downarrow}b=\phi \text{ and } S_1\cup S_2=X.$$

Then for every $i\in I$, we have $a_i\notin S_1$ since $a\leq a_i$.

Case 1. $A_i\subset S_1$ for some $i\in I$. We consider the family

$$\mathcal{S}_0=\{S_1\}\cup\{I(x, a_i): x\in A_i\}.$$

Then $\mathcal{S}_0$ is a linked subfamily of $\mathcal{S}$ and hence $\bigcap\mathcal{S}_0\neq\phi$. But for every $y\in\bigcap\mathcal{S}_0$ we have $x\leq y\leq a_i$ for all $x\in A_i$ by Fact 2 and hence, from the definition of a_i, $a_i=y\in S_1$. A contradition.

Case 2. Otherwise. For every $i\in I$, there exists $f(i)\in S_2\cap A_i$.

It follows that the family

$$\{S_2\}\cup\{\downarrow f(i):\ i\in I\}$$

is a linked subfamily of $\mathcal{S}$ hence there exists $y\in S_2\cap\bigcap_{i\in I}\downarrow f(i)$. Then $y\le\inf_{i\in I}\{f(i)\}\le b$. Thus we have $y\in S_2\cap\downarrow b$, which contradicts to the assumptions.

Lemma 2.7. The topology on X coincides with $\lambda(X,\le)$.

Proof. Because the two topologies are compact Hausdorff, we have only to verify that every element of $\mathcal{S}$ is closed in ΛX, that is, for every $S\in\mathcal{S}$ and every $x\notin S$, there exists a closed set T in ΛX such that $x\notin T\supset S$. Let $S\in\mathcal{S}$ and $x\notin S$. Then by the normality of $\mathcal{S}$ there exists $S_1,\ S_2\in\mathcal{S}$ such that

$$x\notin S_1,\ S\cap S_2=\phi \text{ and } S_1\cup S_2=X.$$

Case 1. $\bot\in S_2$. Then $S_2\supset\downarrow x$ and $\inf S\in S$ by Fact 3,4. It is followed from $S_2\cap S=\phi$ that $S\subset\uparrow\inf S\not\ni x$. Thus $T=\uparrow\inf S$ satisfies the required conditions.

Case 2. $\bot\in S_1$. Suppose that $\Downarrow x\subset S_1$. Then we have $x=\lim_X\Downarrow x\in S_1$ because $\Downarrow x$ is directed and S_1 is closed in X (see the proof of Lemma 5). A contradition. Thus $\Downarrow x\not\subset S_1$, that is, there exists $y\in\Downarrow x\setminus S_1$. It follows that $x\notin X\setminus\Uparrow y\supset S_1\supset S$ since $S_1=\downarrow S_1$. Thus $T=X\setminus\Uparrow y$ satisfies the required conditions since $\Uparrow y$ is open in ΛX.

Sufficiency: Let L be a CDP. we at first define a conect. $B\subset M(L)$ is called a *subase* for L if for every $x\in L$, $x=\sup\{b\in B:$

$b \ll x\}$. Then we have the following lemma:

Lemma 2.8. Let B be a subase for L. Then

$$\mathcal{S}_B = \{L \setminus \Uparrow b : b \in B\} \cup \{\uparrow b : b \in B\}$$

is a subbase for the topological space ΛL.

Proof. Let $x \in L$. Then for every $y \in \Uparrow x$, there exists $z \in L$ such that $x \ll z \ll y$[7,p.47]. Moreover, from the definition of subbase, there exist $b_1, b_2, \ldots, b_n \in B$ such that

$$x \ll z \leq b_1 \vee b_2 \vee \ldots \vee b_n \ll y.$$

(Note that $b_1, b_2 \ll y$ implies $b_1 \vee b_2 \ll y$.) Thus $y \in \Uparrow b_1 \cap \Uparrow b_2 \cap \ldots \cap \Uparrow b_n \subset \Uparrow x$. It follows that for every $x \in L$, $\Uparrow x$ is an union of forms $\Uparrow b_1 \cap \Uparrow b_2 \cap \ldots \cap \Uparrow b_n$ for $b_1, b_2, \ldots, b_n \in B$. Moreover, it is easy to verify that for all $x \in L$, $\uparrow x = \bigcap \{\uparrow b : b \in B \text{ and } b \leq x\}$. Thus $\mathcal{S}_B$ is a subbase for ΛL.

Now we consider the case $B = M(L)$. Then Lemma 2.3 implies that $M(L)$ is a subase for L. To complete the proof of the theorem we have to show that $\mathcal{S} = \mathcal{S}_{M(L)}$ is binary and normal. Let

$$\{L \setminus \Uparrow m_i : i = 1, 2, \ldots, n\} \cup \{\uparrow x_j : j = 1, 2, \ldots, l\}$$

be a linked finite subfamily of $\mathcal{S}$. (It is possible that n or l is zero.) Then $\{x_j : j \leq l\}$ is relative directed since $\uparrow x_j \cap \uparrow x_{j'} \neq \phi$ for $j, j' \leq l$ and hence $a = x_1 \vee x_2 \vee \ldots \vee x_l$ exists. ($a = \bot$ if $l = 0$). It is trivial that $a \in \bigcap \{\uparrow x_j : j \leq l\}$. Now we verify $a \in L \setminus \Uparrow m_i$ for all $i \leq n$. Otherwise, $m_i \ll a = x_1 \vee x_2 \vee \ldots \vee x_l$ for some $i \leq n$. Thus by Lemma 2.3 we have $m_i \ll x_j$ for some $j \leq l$, that is, $(L \setminus \Uparrow m_i) \cap \uparrow x_j = \phi$, which contradicts to the assumption. Because ΛL is compact, we have that $\mathcal{S}$ is binary. Last, we verify that $\mathcal{S}$ is normal. Let $m, x\in M(L)$ such that $(L\setminus\Uparrow m)\cap\uparrow x=\phi$. Then $m\ll x$ and hence, by [7], there exists $m'\in M(L)$ such that $m\ll m'\ll x$. Let $S_1=L\setminus\Uparrow m'$ and $S_2=\uparrow m'$. Then $S_1, S_2\in\mathcal{S}$ and

$$S_1\cup S_2=L,\ S_1\cap\uparrow x=\phi,\ S_2\cap(L\setminus\Uparrow m)=\phi.$$

Moreover, suppose that $x, x'\in L$ such that $\uparrow x\cap\uparrow x'=\phi$. Then

$$\bigcap\{\uparrow m:\ m\in M(L) \text{ and } m\ll x\}\cap\uparrow x'=\uparrow x\cap\uparrow x'=\phi.$$

Since $\mathcal{S}$ is binary, we have that $\uparrow m\cap\uparrow x'=\phi$ for some $m\ll x$. Now let $S_1=\uparrow m$ and $S_2=L\setminus\Uparrow m$. Then we have that

$$S_1\cup S_2=L \text{ and } S_1\cap\uparrow x'=\phi,\ S_2\cap\uparrow x=\phi.$$

Now some applications of the above theorem can be listed. First, we give charaterizations of CDP.

Let $I=[0, 1]$. Then for any cardinal number m, the cube I^m, with the pointwise order, is a CDL, For $a, b, c\in I^m$, let

$$tr(a, b, c)=(a\wedge b)\vee(b\wedge c)\vee(c\wedge a).$$

A set $A\subset I^m$ is called *third-convex* if $tr(a, b, c)\in A$ for all $a, b, c\in A$[10].

Theorem 2.2. For a poset L the following statements are equivalent:

(1). L is a CDP;

(2). L satisfies (CDP1) and (CDP2), and $\downarrow x$ is a CDL for all $x\in L$;

(3). There exists a CDL $L^\#$ such that $L\subset L^\#$ is closed for arbitrary infimums and relatively directed supremums, and $M(L)=M(L^\#)$.

(4). L is isomorphic to a subset L_0 of some cube which is closed for arbitrary infimums, and for any set $A \subset L_0$ if A is relative directed in L_0 then $\sup_{I^m} A \in L_0$.

(5). L is isomorphic to a subset of some cube which is third-convex and is closed with arbitrary infimums and directed supremums.

Proof. (1)⟶(2) and (3)⟶(4) can be obtained from Lemma 2.2 and [7,p204], respectively; (4)⟶(1) is trivial.

(2)⟶(1). First, for every $x \in L$ and every relative directed set $A \subset L$, because $\downarrow y$, where $y = \sup A$, is a CDL, we have

$$x \wedge \sup A = (x \wedge y) \wedge \sup A = \sup\{x \wedge y \wedge a : a \in A\} = \sup\{x \wedge a : a \in A\}.$$

Secondly, for every family $\{A_i : i \in I\}$ of relatively directed sets and a fixed element $i_0 \in I$, let $x = \sup A_{i_0}$. Then it is followed from $\downarrow x$ being a CDL that

$$\begin{aligned} &\inf_{i\in I}\{\sup A_i\} \\ &= \inf_{i\in I}\{(\sup A_i) \wedge x\} \\ &= \inf_{i\in I}\{\sup\{a \wedge x : a \in A_i\}\} \\ &= \sup\{\inf_{i\in I}\{f(i) \wedge x\} : f \in \prod_{i\in I} A_i\} \\ &= \sup\{\inf_{i\in I}\{f(i)\} : f \in \prod_{i\in I} A_i\} \end{aligned}$$

because $\inf_{i\in I}\{f(i)\} \le f(i_0) \le x$ for every $f \in \prod_{i\in I} A_i$.

(1)⟶(3). By Lemma 2.4 M(L) is a continuous poset and hence there exists a CDL $L^\#$ such that M(L) and $M(L^\#)$ are isomorphic, [9].(In fact, $L^\# = \sigma(L)$ as mentioned above.) Let $f_0 : M(L) \longrightarrow M(L^\#)$ be a isomorphism and $f : L \longrightarrow L^\#$ defined by

$$f(x)=\sup_{L^{\#}}\{f_0(m): m\leq x \text{ and } m\in M(L)\}.$$

Since $M(\downarrow x)=M(L)\cap\downarrow x$ we have that $f_0|M(\downarrow x): M(\downarrow x)\longrightarrow M(\downarrow f(x))$ is a isomorphism and hence $f|\downarrow x: \downarrow x\longrightarrow\downarrow f(x)$ is also a isomorphism for every $x\in L$ because $\downarrow x$ and $\downarrow f(x)$ are CDL's, [9]. It follows that $f:L\longrightarrow L^{\#}$ is embeding and preserves arbitrary infs and relatively directed sups.

(4)⟶(5). Suppose that $x, y, z\in L$. Let $A=\{x\wedge y, y\wedge z, z\wedge x\}$. Then A is relatively directed and hence $tr(x, y, z)=\sup A\in L$, that is, L is third-convex.

(5)⟶(4). First, we note that for $a, b\in L$, if $a\vee_L b$ exists, then (5) can imply $a\vee_{I^m} b=tr(a, b, a\vee_L b)\in L$ and hence $a\vee_L b=a\vee_{I^m} b$. Secondly, for every relatively directed finite set A, by the inductive method for $|A|$, we have $\sup_{I^m}A\in L$. In fact, if $A=\{a, b, c\}$ is a relatively directed set of three points, then $\sup_{I^m}A=tr(a\vee b, b\vee c, c\vee a)\in L$. (Note that $a\vee_L b=a\vee_{I^m} b$.) If $A\ni a,b$ is a relatively directed set of n-points for $n>3$, then $\sup_{I^m}A=\sup_{I^m}((A\setminus\{a, b\})\cup\{a\vee b\})\in L$ by the inductive assumpation. Last, for any relatively directed set A, by the above fact and the assumpation in (5), we have $\sup_{I^m}A=\sup_{I^m}\{a_1\vee a_2\vee\ldots\vee a_n: a_i\in A$ for $i=1, 2, \ldots n\}\in L$.

Corollary 2.1. A topological space X is homeomorphic to a CDL with the interal topology if and only if there exists a binary normal subbase $\mathcal{S}$ for X and two points x, y in X such that X is the unique element in $\mathcal{S}$ which containes x and y.

Corollary 2.2. [10] Every normally supercompact space is a retract of its hyperspace of all closed sets.

Proof. It is a corollary of 3.9 Proposition in [7,p.285].

Corrllary 3.[10] Every connected normally supercompact space is generalized arcwise connected and locally connected.

Proof. The first statement is a corollary of well-known Koch's Arc Theorem (see [7,p.300]). In here we give a simple direct proof. Let L be a CDP. Since the set of all Scott-open filter sets (A set $U=\uparrow U\subset L$ is *filter* if it is closed with finite infs) is base for $\sigma(L)$[7,p.107], we have only to verify that $V=B\setminus(\uparrow x_1\cup\uparrow x_2\cup\ldots\cup\uparrow x_n)$ is generalized arcwise connected for all Scott-open filter sets B and any $x_1, x_2, \ldots, x_n\in L$. Suppose a, b∈V. Then $a\wedge b\in V$. Let C_a and C_b be two maximal chains in L such that $C_a\subset[a\wedge b, a]$ and $C_b\subset[a\wedge b, b]$. Then C_a, $C_b\subset V$ and $C_a\cap C_b=\{a\wedge b\}$. To complete the proof of this corollary we have only to verify that C_a and C_b is order dense, that is, for all x, $y\in C_a$, for example, and $x<y$, there exists $z\in C_a$ such that $x<z<y$. In fact, $y\not\leq x$ implies that there exists $m\in M(L)$ such that $m\ll y$ and $m\not\leq x$. Let $z_0=x\vee m$. (Note $x,m\leq y$). Then $x<z_0\leq y$. To show that C_a is order dense we have only to verify that $z_0\neq y$ since C_a is maximal. Otherwise, $m\ll z=x\vee m$ and hence, $m\ll m$ since $m\in M(L)$ and $m\not\leq x$. Thus m is a non-zero compact element in L, which implies that $\wedge L$ is not connected since

$\uparrow m = \Uparrow m$ is clopen.

Lemma 2.9. For a CDP L, we have

(1). ΛL is metric if and only if there exists a contable subbase in L. Hence, ΛL is metric if and only if $\Lambda L^{\#}$ is metric.

(2). ΛL is connected if and only if $\Lambda L^{\#}$ is connected.

Proof.(1).It can be directly showed by Lemma 2.8. (*cf.* [7, p.170])

(2). By the above corollary we know that ΛL is connected if and only if there no exist non-zero compact element in M(L). Moreover, M(L) and $M(L^{\#})$ are isomorphic.

Corollary 4. [10] Let X be a connected normal supercompact space and $x_0 \in X$. Then there exists a connected linearly compcat order space J and a continuous mapping $f: J \times X \longrightarrow X$ such that $f(\top_J, x) = x$ for all $x \in X$ and $f(\bot_J, x) = x_0$. Furthermore, if X is metrc then J=I.

Proof. Let $X = \Lambda L$ for a CDP L such that $x_0 = \bot_L$. Let J be a maximal chain in $L^{\#}$ and define $f: J \times X \longrightarrow X$ by

$$f(j, x) = j \wedge x.$$

Then f satisfies the required conditions. Furthermore, if X is metric, it is followed from the above lemma that so is J. Thus J=I.

Lemma 2.10. If $L \subset I^m$ is closed with abitrary infs and directed sups, then the topology as a subspace of I^m coincides with $\lambda(L)$.

Proof. It is direct.

Corollary 2.5. [14] If X is a normally supercompact space, then X can be embeded into I^m as a closed and thire-convex subset.

Conclusion: There exists a example to show that the hyperspace of normally supercompat space may be not supercompact[1]. Thus the continuous lattice with the Lawson topology may not be supercompact. But Coroloaries 2.2, 2.3 and 2.4 hold for continuous lattices with the Lawson topology, see [7], although the proof of Corollary 2.3 given in present paper is invalid for the general case.

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