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A Remark on Nowhere Dense Closed P-Sets

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Abstract. Using the methods from continua theory of R^*, we prove that NCF implies that ω^* can be covered by an increasing sequence of nowhere dense closed P-sets.

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Kunen, van Mill and Mills proved in [4] that no compact space of weight 2^{ω} can be covered by nowhere dense closed P-sets under CH. It was proved in [1] that, in the model obtained by adding ω_1 Cohen reals to a model of MA+$\neg$CH, ω^* can be covered by nowhere dense closed P-sets. It is not difficult to show that the axiom of near coherence of filters, abbreviated as NCF (See [2]), implies that ω^* can be covered by nowhere dense closed P-sets. Our purpose in this note is to strengthen the conclusion as follows:

Theorem 1. NCF *implies that* ω^* *can be covered by an increasing sequence* of *nowhere dense closed* P-*sets.*

Our way is to use the methods from continua theory of R^* to guarantee an induction construction going smoothly through the limit steps. Acturally, we shall prove, (See also Corollary 5.7 in [5]).

Theorem 2. NCF *is equivalent to that* $\beta[0,\infty)-[0,\infty)$ *can be covered by a strictly increasing sequence of subcontinua which are nowhere dense* P-*sets.*

It is not difficult to show that if ω^* can be covered by nowhere dense closed P-sets then so can R^* (See Corollary 4). But the author don't know whether or not the converse is true.

we refer to [2] for the background on NCF and [5] for continua theory of R^*.

Let Ω be the collection of all families of infinite discrete non-degenerate closed interval of the half real line $[0,\infty)$. For $\mathcal{I}\in\Omega$, we let $i:\omega\longrightarrow\mathcal{I}$ be the bijection such that $i(n)<i(n+1)$ for $n\in\omega$, where $i(n)<i(n+1)$ means that $r<s$ for all $(r,s)\in i(n)\times i(n+1)$. Let $\mathring{\mathrm{i}}:\cup\mathcal{I}\longrightarrow\omega$ be such that $\mathring{\mathrm{i}}(x)=n$ if and only if $x\in i(n)$. Let $\beta\mathring{\mathrm{i}}$ be the Stone-Čech extension of $\mathring{\mathrm{i}}$ from $cl_{\beta R}(\cup\mathcal{I})$ to $\beta\omega$. For $B\subset\omega^*$, we define

$$M(\mathcal{I},B)=\beta\mathring{\mathrm{i}}^{-1}(B)$$

and, if $B=\{u\}$, than $M(\mathcal{I},\{u\})$ is denoted by $M(\mathcal{I},u)$. It is well-known that $M(\mathcal{I},u)$ is a continuum for any $u\in\omega^*$. Moreover, a subcontinuum C of $\beta[0,\infty)-[0,\infty)$ is called a standard continuum if $C=M(\mathcal{I},u)$ for some $\mathcal{I}\in\Omega$ and $u\in\omega^*$. Note that every proper subcontinuum of $\beta[0,\infty)-[0,\infty)$ is nowhere dense since $\beta[0,\infty)-[0,\infty)$ is an indecomposable continuum.

Recall that a subset B of a space X is called a P-set provided that the intersection of countably many neighbourhoods of B is again a neighbourhood of B. A point x of X is called a P-point if the singleton $\{x\}$ is a P-set.

For an open set U of a metric space X, we let $O(U)=\{x\in\beta X:\exists F\in x(F\subset U)\}$. Then $\{O(U):U \text{ is open in } X\}$ is a base for βX. $[\omega]^{\omega}$ is the set of all infinite subsets of ω. As usual, $O(A)\cap\omega^*$ is denoted by A^* for $A\in[\omega]^{\omega}$.

For $\mathcal{I},\mathcal{I}'\in\Omega$, we say that $\mathcal{I}'$ is an expander of $\mathcal{I}$ if $\iota(n)$ is contained in the interior of $\iota'(n)$ for all $n\in\omega$.

Lamma 3. $B\subset\omega^*$ *is a nowhere dense closed* P-*set if and only if* $M(\mathcal{I},B)$ *is a nowhere dense closed* P-*set of* $\beta[0,\infty)-[0,\infty)$ *for* $\mathcal{I}\in\Omega$.

Proof. Assume that $M(\mathcal{I},B)$ is a nowhere dense closed P-set of $\beta[0,\infty)-[0,\infty)$. It is easily seen that B is nowhere dense closed in ω^*. Let $\mathcal{I}'\in\Omega$ be an expander of $\mathcal{I}$. Then $M(\mathcal{I},B)$ is a P-set of $cl_{\beta R}(\cup\mathcal{I}')$. Suppose that $\{A_n^*:n\in\omega\}$ is a family of countably many neighbourhoods of B. Then $\{\beta\iota'^{-1}(A^*):n\in\omega\}$ is a family of neighbourhoods of $M(\mathcal{I},B)$. Therefore, there is a basic open set $O(U)$ such that $M(\mathcal{I},B)\subset O(U)\cap R^*\subset\beta\iota'^{-1}(A^*_n)$ for all $n\in\omega$. Note that, for $u\in\omega^*$, $M(\mathcal{I},u)=\bigcap\{cl_{\beta R}(\cup\mathcal{F}):\iota^{-1}(\mathcal{F})\in u\}$. Therefore, for each $u\in B$, there is $A_u\in u$ such that $\bigcup\{\iota(n):n\in A_u\}\subset U$. Let $A=\{\iota^{-1}(I):I\in\mathcal{I}$ and $I\subset U\}$. Then $A_u\subset A$ for $u\in B$. So A^* is a neighbourhood of B. Since $O(U)\cap R^*\subset\beta\iota'^{-1}(A_n)$, we have that $A^* \subset A_n^*$ for all $n \in \omega$.

Assume that B is a nowhere dense closed P-set of ω^*. Let $O(U)$ be a basic open set of βR and $O(U) \cap M(\mathcal{I}, B) \neq \phi$. Let $A = \{n \in \omega : \iota(n) \cap U \neq \phi\}$. Then $A \in [\omega]^\omega$. Since B is nowhere dense, there is $A_1 \in [\omega]^\omega$ such that $A_1 \subset A$ and $A_1^* \cap B = \phi$. Therefore, $M(\mathcal{I}, A_1^*) \cap M(\mathcal{I}, B) = \phi$. But $O(U) \cap M(\mathcal{I}, A_1^*) \neq \phi$. So $O(U) \cap R^* \setminus M(\mathcal{I}, B) \neq \phi$. It follows that $M(\mathcal{I}, B)$ is nowhere dense. Suppose that $\{O(U_n) : n \in \omega\}$ is a family of neighbourhoods of $M(\mathcal{I}, B)$. Let $A_n = \{\iota(I) : I \in \mathcal{I}$ and $I \subset U_n\}$ for $n \in \omega$. As we showed in the last paragraph, A_n^* is a neighbourhood of B for all $n \in \omega$. Since B is a P-set, there is $A \in [\omega]^\omega$ such that $B \subset A^*$ and $A^* \subset A_n$ for all $n \in \omega$. We choose a strictly increasing sequence $\{m_n : n \in \omega\}$ of integers so that for each $n \in \omega$ $A \setminus m_n \subset A_n$ and $[m_n, m_{n+1}) \cap A \neq \phi$, where $[m_n, m_{n+1}) = \{i \in \omega : m_n \leq i < m_{n+1}\}$. For each $i \in [m_n, m_{n+1}) \cap A$, let J_i be an open interval of R such that $\iota(i) \subset J_i \subset U_n$. Let $V = \bigcup \{J_n : n \in A \setminus m_0\}$. Then $M(\mathcal{I}, B) \subset M(\mathcal{I}, A^*) \subset O(V)$ and $O(V) \subset O(U_n)$ for $n \in \omega$. This completes the proof of Lemma 3.

Since we can easily choose $\mathcal{I}, \mathcal{I}' \in \Omega$ such that $\cup(\mathcal{I} \cup \mathcal{I}') = [0, \infty)$ and R^* is the topological sum of $\beta(-\infty, 0] - (-\infty, 0]$ and $\beta[0, \infty) - [0, \infty)$, we have

Corollary 4. *If* ω^* *can be covered by nowhere dense closed P-sets, then so can* R^*.

Blass proved in [2] that, under NCF, for any $u\in\omega^*$ there is a finite-to-one non-decreasing function $f:\omega\longrightarrow\omega$ such that $v=\beta f(u)$ is a P-point. It is easily seen that $\beta f^{-1}(v)$ is a nowhere dense closed P-set of ω^* and $u\in\beta f^{-1}(v)$. Therefore, NCF implies that ω^* can be covered by nowhere dense closed P-sets. Our purpose is to sharpen the conclusion so that ω^* can be covered by an increasing sequence of nowhere dense closed P-sets under NCF.

We regard ω^* as a subspace of $\beta[0,\infty)-[0,\infty)$. The following lemma is an easy observation.

Lemma 5. *If* $u\in\omega^*$ *is a* P-*point, then* $M(\mathcal{I},u)\cap\omega^*$ *is a nowhere dense closed* P-*set of* ω^* *for* $\mathcal{I}\in\Omega$.

Proof. Let $X=\omega\cap(\cup\mathcal{I})$ and $Y=\{i^{-1}(I):I\cap\omega\neq\phi\}$. If $Y\notin u$, then, $M(\mathcal{I},u)\cap\omega^*=\phi$. So we assume that $Y\in u$. We define a finite to one function $f:X\longrightarrow Y$ from X onto Y by $f(n)=m$ if and only if $n\in i(m)$. Then $M(\mathcal{I},u)\cap\omega^*=\beta f^{-1}(u)$. Since $\beta f^{-1}(u)$ is a nowhere dense closed P-set in X^*, $M(\mathcal{I},u)\cap\omega^*$ is a nowhere dense closed P-set in ω^*.

By Lemma 3 and 5, our Theorem 1 and 2 follows easily from the following theorem.

Theorem 2'. NCF *is equivalent to that there is a family* $\{(\mathcal{I}_\alpha, u_\alpha):\alpha<\lambda\}$ *such that*

(1) $\mathcal{I}_\alpha \in \Omega$ *and* $u_\alpha \in \omega^*$ *is a P-point for all* $\alpha<\lambda$;

(2) $M(\mathcal{I}_\alpha, u_\alpha) \subset M(\mathcal{I}_\beta, u_\beta)$ *for all* $\alpha<\beta<\lambda$;

(3) $\beta[0,\infty)-[0,\infty) = \bigcup\{M(\mathcal{I}_\alpha, u_\alpha):\alpha<\lambda\}$.

Theorem 2' will be proved along the line of the proof of Corollary 5.7 in [5]. We first recall some properties of NCF and standard continua. We refer to [2] and [5] for details.

A subset C of a continuum K is a composant if, for some point $p\in C$, C is the set of all points x such that there is a proper subcontinuum of K containing both p and x. It is well-known that NCF is equivalent to that $\beta[0,\infty)-[0,\infty)$ is a composant of itself (See [3]). Therefore, our conditions in Theorem 2' implies NCF.

Recall that there is a natural partial order $<^{\mathcal{I}}_u$ on $M(\mathcal{I},u)$ for $\mathcal{I}\in\Omega$ and $u\in\omega^*$, defined as follows: For any $x,y\in M(\mathcal{I},u)$,

$$x<^{\mathcal{I}}_u y \text{ if there are } F\in x \text{ and } H\in y \text{ such that}$$
$$\{i^{-1}(I):I\in\mathcal{I} \text{ and } F\cap I<H\cap I\}\in u,$$

For $x\in M(\mathcal{I},u)$, we let

$$[x]_u^{\mathcal{I}}=\{y\in M(\mathcal{I},u):y \text{ is } <_u^{\mathcal{I}}\text{-incomparable with } x \text{ or } y=x\}.$$

$[x]_u^{\mathcal{I}}$ is called a layer of $M(\mathcal{I},u)$. It is well-known that layers are indecomposable subcontinua of $M(\mathcal{I},u)$ and every indecomposable subcontinuum of $M(\mathcal{I},u)$ is contained in a layer.

Lemma 6 (Corollary 2.11 in [5]). *Let* C *and* D *be subcontinua of* R^*. *If one of them is indecomposable, then* $C\subset D$, $D\subset C$ *or* $C\cap D=\phi$.

A point $u\in\omega^*$ is a Q-point if every finite-to-one function from ω to ω is one-to-one on a set in u. By Proposition 5.1 in [5], it is equivalent to require the functions in the definition of Q-points to be non-decreasing. Blass proved in [2] that NCF implies that there is no Q-points.

Lemma 7. *Under* NCF, *for every proper subcontinuum* C *of* $\beta[0,\infty)-[0,\infty)$, *there is a standard continuum* $M(\mathcal{I},u)$ *and a layer* T *of* $M(\mathcal{I},u)$ *such that* $C\subset T$ *and* $M(\mathcal{I},u)$ *is a nowhere dense P-set of* $\beta[0,\infty)-[0,\infty)$.

Proof. Since every proper subcontinuum of $\beta[0,\infty)-[0,\infty)$ is contained in a standard subcontinuum, we assume that $C\subset M(\mathcal{I}_1,u_1)$ for some $\mathcal{I}_1\in\Omega$ and $u\in\omega^*$. Since NCF implies that there is no

Q-points, there is a finite-to-one non-decreasing function $f:\omega\longrightarrow\omega$ which witnesses that u_1 is not a Q-point. We define $\mathcal{I}_2=\{I_n:n\in\omega\}$ as follows: I_n is the convex hull of the set $\bigcup\{i_1(m):m\in f^{-1}(n)\}$. Let $u_2=f(u_1)$. Then, $M(\mathcal{I}_1,u_1)\subset M(\mathcal{I}_2,u_2)$. Moreover, for any $x,y\in M(\mathcal{I}_1,u_1)$, x and y are $<^{\mathcal{I}_2}_{u_2}$-incomparable or $x=y$. Therefore, $M(\mathcal{I}_1,u_1)$ is contained in a layer T' of $M(\mathcal{I}_2,u_2)$. By NCF, there is a finite-to-one non-decreasing function $g:\omega\longrightarrow\omega$ such that $u=g(u_2)$ is a P-point. By the same method as above, we can find $\mathcal{I}\in\Omega$ such that $M(\mathcal{I}_2,u_2)\subset M(\mathcal{I},u)$. Since T' is an indecomposable subcontinuum of $M(\mathcal{I},u)$, there is a layer T of $M(\mathcal{I}.u)$ such that $C\subset T'\subset T$. By Lemma 3, $M(\mathcal{I},u)$ is a nowhere dense P-set of $\beta[0,\infty)-[0,\infty)$.

Now we are in a position to complete the proof of Theorem 2'. We assume NCF. We define, inductively, $\mathcal{I}_\alpha\in\Omega$, $u_\alpha\in\omega^*$ and a layer T_α of $M(\mathcal{I}_\alpha,u_\alpha)$ for $\alpha\geq 0$ satisfying that

(a) u_α is a P-point for all $\alpha\geq 0$;

(b) $M(\mathcal{I}_\alpha,u_\alpha)\subset T_\beta$ for $\alpha<\beta$.

Our induction process will stop at some λ if $\beta[0,\infty)-[0,\infty)=\bigcup\{M(\mathcal{I}_\alpha,u_\alpha):\alpha<\lambda\}$. Suppose that we have defined $\mathcal{I}_\beta$, u_β and T_β for all $\beta<\alpha$ satisfying (a) and (b). If $\alpha=0$ or $\gamma+1$, then, by Lemma 7, we can easily define $\mathcal{I}_\alpha$, u_α and T_α satisfying (a) and (b). Assume that $\alpha\neq 0$ is a limit and $\beta[0,\infty)-[0,\infty)$ is not covered by $\{M(\mathcal{I}_\beta,u_\beta):\beta<\alpha\}$. Note that by (b) $\bigcup\{M(\mathcal{I}_\beta,u_\beta):\beta<\alpha\}=$

$\bigcup\{T_\beta : \beta<\alpha\}$ since α is a limit. Take $x,y\in\beta[0,\infty)-[0,\infty)$ such that $x\in T_0$ and $y\notin T_\beta$ for all $\beta<\alpha$. By NCF, there is a proper subcontinuum C of $\beta[0,\infty)-[0,\infty)$ containing both x and y. By Lemma 6, $T_\beta\subset C$ for all $\beta<\alpha$. By Lemma 7, there is $\mathcal{I}_\alpha\in\Omega$, $u_\alpha\in\omega^*$ and a layer T_α of $M(\mathcal{I}_\alpha,u_\alpha)$ such that $C\subset T_\alpha$ and u_α is a P-point. This completes our inductive construction. Since $\{M(\mathcal{I}_\alpha,u_\alpha):\alpha\geq 0\}$ is a strictly increasing sequence, our induction process can not go over $|R^*|$ steps. This completes the proof of Theorem 2'.

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