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The Complexity of Selecting Maximal Solutions

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1 Introduction

Intuitively, a maximization problem is to select a maximal solution for a given input according to some selection criterion. The maximal independent set problem (MIS) [5] and the minimal unsatisfiability problem (MinUnsat) [11] are two standard examples of such problems. Much work has been devoted to the study of maximization problems [1,2,3,4,5,7,9,11,12]. Most of the previous work has involved studying *specific* maximization problems and either finding an efficient algorithm (e.g., [5]) or proving the problem is hard to solve (e.g., [11]). An attractive alternative approach is to study maximization problems in a general framework and to prove general results.

In this paper, we formalize a maximization problem (MAXP) Q as a pair (D, R), where D is the set of *instances* and $R: D \times \{0,1\}^* \rightarrow \{true, false\}$ is the *instance-solution relation.* The objective in solving Q is to select, given an instance $x \in D$, a *maximal solution*, i.e., a binary string y such that $R(x,y)$ is true but changing one or more arbitrary 0-bits of y to 1-bits will change the value of $R(x,y)$ to false. As an example, consider MIS in our framework. For it, D is the set of all undirected graphs, and $R(G, b_1b_2\cdots b_n)$ is true if and only if G has n vertices (say, 1, 2, $\cdots$, n) and $\{i : b_i = 1\}$ is an independent set in G. Our goal is to demonstrate what factors make Q easy or hard to solve and how the factors influence the complexity of solving Q. We are able to find two such factors. One obvious factor is the complexity of R. This can be seen by comparing MIS with MinUnsat. The instance-solution relation of MIS is decidable in NC while that of MinUnsat is coNP-complete. Because of this gap, solving MinUnsat is much harder than solving MIS. In fact, MIS is solvable in NC [5,7] while solving MinUnsat is $\mathrm{D^P}$-hard [11]. The other factor is whether R is *hereditary* or not, where R is said to be hereditary if and only if for every x and w, whenever $R(x,w)$ is true, $R(x,w)$ remains true even one or more arbitrary 1-bits of w are changed to 0-bits. The instance-solution relation of MIS (also MinUnsat) is hereditary. In [9], Papadimitriou considered the following problem (MinModel): Given a CNF boolean formula ϕ, find a satisfying truth assignment $\vec{a}$ to ϕ such that changing one or more arbitrary 1-bits of $\vec{a}$ to 0-bits will make $\vec{a}$ no longer satisfy ϕ. The instance-solution relation of MinModel is not hereditary but is decidable in NC. Unlike MIS, solving MinModel is obviously NP-hard.

In this paper, we restrict to consider only those MAXP's whose instance-solution relation is decidable in NP or coNP. We first consider upper bounds on the complexity of solving such MAXP's. Let $Q = (D, R)$ be a MAXP. The following give trivial upper bounds: (i) Q is solvable in FP if R is decidable in P and hereditary; (ii) Q is solvable in NPMV//OptP$[O(\log n)]$ if R is decidable in NP; (iii) Q is solvable in $\mathrm{FP^{NP}}$ if R is decidable in coNP and hereditary; (iv) Q is solvable in $\mathrm{FP^{\Sigma_2^P}}$ if R is decidable in coNP.

Our main results concerning upper bounds are the following:

(v) Suppose Q is a MAXP whose instance-solution relation is NP decidable. Let e be an arbitrary polynomial. Then, there exist a function $F \in \mathrm{FP}_{\|}^{\mathbf{NP}}$ and a polynomial p such that for every x, $\Pr[F(x,w)$ is a maximal solution of x in $Q] \geq 1 - 2^{-e(|x|)}$, where $w \in \{0,1\}^{p(|x|)}$ is randomly chosen under uniform distribution.

(vi) Suppose Q is a MAXP whose instance-solution relation is coNP decidable. Let e be an arbitrary polynomial. Then, there exist a function $F \in \mathrm{FP}_{\|}^{\Sigma_2^{\mathbf{P}}}$ and a polynomial p such that for every x, $\Pr[F(x,w)$ is a maximal solution of x in $Q] \geq 1 - 2^{-e(|x|)}$, where $w \in \{0,1\}^{p(|x|)}$ is randomly chosen under uniform distribution.

(v) and (vi) are shown by extending the technique used in [3].

We then show that NPMV//OptP$[O(\log n)]$ is also a lower bound for solving those MAXP's whose instance-solution relation is decidable in NP or is decidable in P but not hereditary, and that $\mathrm{FP}_{\|}^{\Sigma_2^{\mathbf{P}}}$ is also a lower bound for solving those MAXP's whose instance-solution relation is decidable in coNP but not hereditary. Combining the upper and lower bounds, we obtain characterizations of NPMV//OptP$[O(\log n)]$ and $\mathrm{FP}_{\|}^{\Sigma_2^{\mathbf{P}}}$ via MAXP's. As an important consequence of the characterization of NPMV//OptP$[O(\log n)]$, we obtain the first natural complete problem for NPMV//OptP$[O(\log n)]$. The problem (called X-MinModel) is defined as follows: Given a CNF boolean formula ϕ and a subset X of the set of variables in ϕ, find a satisfying truth assignment $\vec{a}$ to ϕ such that changing one or more arbitrary 1-bits of $\vec{a}$ corresponding to variables in X to 0-bits will make $\vec{a}$ no longer satisfy ϕ. X-MinModel was first considered by Papadimitriou in [9], and was claimed *without a precise proof* to be $\Delta_2^{\mathbf{P}}$-complete there. However, Papadimitriou later withdrew his claim and thus left the complexity of X-MinModel open [10]. In [3], we proved that the complexity of X-MinModel is roughly captured by $\mathrm{FP}_{\|}^{\mathbf{NP}}$. Now, the results in this paper give, for the first time, the exact complexity of solving X-MinModel.

We also characterize complexity classes of sets via MAXP's. The following are shown:

(a) coNP is the class of all sets L that can be expressed as $L = \{x : f(x)$ is a maximal solution of x in $Q\}$ for some $f \in$ FP and some MAXP Q whose instance-solution relation is P-decidable.

(b) $\mathrm{D}^{\mathbf{P}}$ is the class of all sets L that can be expressed as $L = \{x : f(x)$ is a maximal solution of x in $Q\}$ for some $f \in$ FP and some MAXP Q whose instance-solution relation is NP-decidable.

(c) $\mathrm{D}^{\mathbf{P}}$ is the class of all sets L that can be expressed as $L = \{x : f(x)$ is a maximal solution of x in $Q\}$ for some $f \in$ FP and some MAXP Q whose instance-solution relation is coNP-decidable and hereditary.

(d) $\Pi_2^{\mathbf{P}}$ is the class of all sets L that can be expressed as $L = \{x : f(x)$ is a maximal solution of x in $Q\}$ for some $f \in$ FP and some MAXP Q whose instance-solution relation is coNP-decidable.

As consequences, we obtain several new natural problems that are $\leq_m^{\mathbf{P}}$-complete for coNP or $\mathrm{D}^{\mathbf{P}}$.

2 Preliminaries

We use $\Sigma = \{0,1\}$ as our alphabet. By a *set*, we mean a subset of Σ^*. Similarly, by a *string*, we mean a string in Σ^*. We denote by $|x|$ the length of a finite string x. The bits of a finite string with length n are indexed from left to right as the 1st, 2nd, $\cdots$, nth bits, respectively. For a finite string x, we usually identify x with the set of all indices i such that the ith bit of x is 1. Thus we will often use some set-theoretical notations for finite strings. A finite string x is *smaller* than another finite string y if either $|x| < |y|$ or $|x| = |y|$ and $x \subset y$. A *maximal string* in a set S of finite strings is a string in S that is not smaller than any other string in S.

We assume a one-to-one pairing function from $\Sigma^* \times \Sigma^*$ to Σ^* that is polynomial-time computable and polynomial-time invertible. For strings x and y, we denote the output of the pairing function by $\langle x, y\rangle$; this notation is extended to denote any k-tuples for $k > 2$ in a usual manner. W.l.o.g., we assume that $|\langle x, y\rangle|$ depends only on $|x|$ and $|y|$.

For any finite set A, $||A||$ denotes the number of strings in A. For a set L, $\bar{L}$ denotes its complement (i.e., $\Sigma^* - L$), and χ_L denotes the characteristic function of L. For a class $\mathbf{C}$ of sets, co$\mathbf{C}$ denotes the class of all sets whose complement is in $\mathbf{C}$. Let Σ^n denote the set of all strings with length n. For two sets L_1 and L_2, $L_1 \oplus L_2$ denotes the set $\{0x \ : \ x \in L_1\} \cup \{1y \ : \ y \in L_2\}$.

All functions considered here are ones from Σ^* to $\Sigma^* \cup \{\#\}$. The symbol $\#$ is assumed to be not in Σ^*. We consider both single-valued functions and multi-valued functions, but by a *function* we mean a (partial) single-valued function. For a multi-valued function G, $G(x)$ denotes the set of all possible values of G at x. Thus, when $G(x) = \emptyset$, the multi-valued function G is *undefined* at the argument x.

We assume that the reader is familiar with the basic concepts from the theory of computational complexity. Our computational models are variations of standard Turing machines. A machine is either an acceptor or a transducer, and may be deterministic or nondeterministic. An acceptor is denoted by M or M_i while a transducer is denoted by T or T_i. A deterministic (resp., nondeterministic) Turing machine is abbreviated as DTM (resp., NTM). On a given input, a branch of a (nondeterministic) machine may halt by entering either a rejecting state or an accepting state. For simplicity, we say that a branch of a machine *halts* if the branch halts by entering an accepting state. Let $L(M)$ denote the set of all strings accepted by M. A transducer T *computes a string* y *on input* x if some branch of T on input x halts with y on the output tape. $T(x)$ denotes the set of all strings computed by T on input x. A DTM T *computes a function* f if for all $x \in \Sigma^*$, $T(x) = \emptyset$ if $f(x)$ is undefined, and the unique element of $T(x)$ is $f(x)$ otherwise.

Classes in the first three levels of the polynomial-time hierarchy are denoted in the usual way: P, NP, coNP, Σ_2^{P}, $\Pi_2^{\mathrm{P}} = \mathrm{co}\Sigma_2^{\mathrm{P}}$. Let $\mathrm{D^P} = \{L_1 \cap L_2 : L_1 \in \mathrm{NP} \text{ and } L_2 \in \mathrm{coNP}\}$.

FP denotes the class of all functions computed by polynomial-time bounded DTM's. Let A be a set. FP^A denotes the class of all functions computed by polynomial-time bounded deterministic oracle Turing machines (DOTM) with oracle A. $\mathrm{FP}_{||}^A$ denotes the class of all functions F for which there exists a polynomial-time bounded DOTM T such that T, while computing $F(x)$ for a given x, prepares all its query strings before asking them to the oracle A. More precisely, a function F is in $\mathrm{FP}_{||}^A$ if there exist two functions f and g in FP such that for all strings x, $F(x) = g(x, \chi_A(y_1)\cdots\chi_A(y_m))$, where $f(x) = \langle y_1, \cdots, y_m\rangle$. For a class $\mathbf{C}$ of sets, $\mathrm{FP}^{\mathbf{C}} = \cup_{A\in\mathbf{C}}\mathrm{FP}^A$ and $\mathrm{FP}_{||}^{\mathbf{C}} = \cup_{A\in\mathbf{C}}\mathrm{FP}_{||}^A$.

An NP *metric Turing machine* is a polynomial-time bounded NTM T such that on every input, every branch of T outputs a binary number and halts [6]. $\mathrm{OptP}[O(\log n)]$ denotes the class of all (total) integer-valued functions H for which there exist a polynomial p and an NP metric Turing machine T such that for every x, $H(x) \leq p(|x|)$ and

$H(x)$ equals to the maximum number in $T(x)$. NPMV//OptP$[O(\log n)]$ denotes the class of all (partial) multi-valued functions G for which there exist an NTM T and a function $H \in$ OptP$[O(\log n)]$ such that for every x, $G(x) = T(\langle x, H(x)\rangle)$.

A *maximization problem* (MAXP) Q is a pair (D, R), where (i) D is the set of *instances* and (ii) $R : D \times \Sigma^* \to \{true, false\}$ is the *instance-solution relation.*

R is said to be *hereditary* if for every $x \in D$ and every $w \in \Sigma^*$, whenever $R(x, w)$ is true, $R(x, w')$ is also true for every w' with $|w'| = |w|$ and $w' \subset w$. Let $x \in D$. A string w is called a *solution* of x if $R(x, w)$ is true. A *maximal solution* of x is a maximal string in the set of all solutions of x. The objective in solving Q is to compute, given an instance $x \in D$, a maximal solution of x.

Each MAXP $Q = (D, R)$ considered in this paper is required to satisfy the following: (1) D is P-decidable (i.e., decidable in polynomial time), (2) there is a polynomial p such that for every $x \in D$ and every string w, whenever $R(x, w)$ is true, $|w| \leq p(|x|)$, and (3) R is NP-decidable or coNP-decidable.

Definition 2.1 A function F *solves* Q if for every $x \in D$, (a) $F(x)$ is undefined if x has no solution in Q and (b) $F(x)$ is a maximal solution of x in Q otherwise. A multi-valued function G *solves* Q if for every $x \in D$, (a) $G(x) = \emptyset$ if x has no solution in Q and (b) $G(x)$ is nonempty and each element of $G(x)$ is a maximal solution of x in Q otherwise. Q is *solvable* in a class $\mathbf{H}$ of (single-valued or multi-valued) functions if some $H \in \mathbf{H}$ solves Q.

Definition 2.2 Let F be a function, and let G be a multi-valued function. Then, F (resp., G) is *reducible* to Q if there exist two functions f, g in FP such that for every x, $f(x) \in D$ and $g(x, w) = F(x)$ (resp., $g(x, w) \in G(x)$) for every maximal solution w of $f(x)$ in Q. Q is *hard* for a class $\mathbf{H}$ of (single-valued or multi-valued) functions if every $H \in \mathbf{H}$ is reducible to Q. Q is *complete* for a class $\mathbf{H}$ of (single-valued or multi-valued) functions if Q is solvable in and hard for $\mathbf{H}$. Q is *hard* for a class $\mathbf{C}$ of sets if Q is hard for the class $\{\chi_L : L \in \mathbf{C}\}$.

Definition 2.3 The set $L_Q = \{\langle x, w\rangle : w$ is a maximal solution of x in $Q\}$ is called the *decision problem associated with* Q.

3 Upper bounds

In this section, we show upper bounds on the complexity of solving MAXP's. The following proposition shows trivial upper bounds.

Proposition 3.1 Let $Q = (D, R)$ be a MAXP.
(1) If R is hereditary and P-decidable, then Q is solvable in FP.
(2) If R is NP-decidable, then Q is solvable in NPMV//OptP$[O(\log n)]$.
(3) If R is hereditary and coNP-decidable, then Q is solvable in $\mathrm{FP}^{\mathrm{NP}}$.
(4) If R is coNP-decidable, then Q is solvable in $\mathrm{FP}^{\Sigma_2^{\mathrm{P}}}$.

We next proceed to show two other non-trivial upper bounds. To do this, we need several definitions and a known result.

Definition 3.1 Let $\mathbf{F}$ be a class of functions. Then we define a class RP$\cdot\mathbf{F}$ of multi-valued functions as follows: A multi-valued function G is in RP$\cdot$**F** if for every polynomial e, there exist a function $F \in \mathbf{F}$ and a polynomial p such that for every string x, (a) $F(x,w)$ is undefined for all $w \in \{0,1\}^{p(|x|)}$ if $G(x)$ is undefined and (b) $\Pr[F(x,w) \in G(x) \cup \{\#\}] = 1$ and $\Pr[F(x,w) \in G(x)] \geq 1 - 2^{-e(|x|)}$ otherwise, where w is a random string chosen from $\{0,1\}^{p(|x|)}$. Intuitively speaking, G is in RP$\cdot$**F** if for every string x, we can randomly pick up an element of $G(x)$ using a function in **F**.

Notation: For $k \geq 1$, $[1,k]$ denotes the set of all integers i with $1 \leq i \leq k$.

Definition 3.2 Let S be a finite set and let k be a positive integer. A *weight function* over S is a function from the elements of S to positive integers. A *k-weight function* over S is a weight function f over S such that for each $s \in S$, $f(s)$ is in $[1,k]$. A *random k-weight function* over S is a k-weight function f over S such that for each $s \in S$, $f(s)$ is chosen uniformly and independently from $[1,k]$. The *weight* of a subset S' of S under a weight function f is $\Sigma_{s \in S'} f(s)$. Note that for every k-weight function over S, the weight of each subset of S under f is no more than $k||S||$ and that the empty set $\emptyset$ is the unique subset of S with weight 0.

Lemma 3.1 [8]. Let **S** be a nonempty family of subsets of a finite set S. Then, for any random k-weight function f over S with $k \geq 2||S||$, Pr[There is a unique maximum weight set in **S** under f] $\geq \frac{1}{2}$.

Now we are ready to show the two non-trivial upper bounds. The idea used in the proof is a generalization of the one used in [3].

Theorem 3.1 Let $Q = (D, R)$ be a MAXP.
(1) If R is NP-decidable, then Q is solvable in RP$\cdot$FP$_{\|}^{\mathrm{NP}}$.
(2) If R is coNP-decidable, then Q is solvable in RP$\cdot$FP$_{\|}^{\Sigma_2^{\mathrm{P}}}$.
Proof. We only show a proof for (2). (1) can be shown in a similar manner.

(2) We first explain the idea behind the proof. Let p_Q be a polynomial such that for all $x \in D$, the length of each solution of x is no more than $p_Q(|x|)$. Let x be an instance of Q. Then, we consider **S**, the family of all solutions of x with maximum length. To find a maximal solution for x, we first get a random $2p_Q(|x|)$-weight function f over $[1, p_Q(|x|)]$. Then, by Lemma 3.1, with probability at least $\frac{1}{2}$, there is a unique solution in **S** of maximum weight. To find this unique solution of maximum weight, it suffices to ask only one round of parallel queries to a Σ_2^{P} oracle set. Since the weight assigned to each element of $[1, p_Q(|x|)]$ is positive, all maximum weight solutions are maximal solutions (but not necessarily solutions of maximum 1-bits). In order to get the high probability of success, we may perform several copies of this computation in parallel.

We now proceed to give the precise proof. Let p_Q be a polynomial that bounds the lengths of solutions of x from above. For convenience, let $n_x = p_Q(|x|)$ for all $x \in D$. Then we define five sets as follows:

$L_R = \{x : x \text{ has a solution}\}$,

$B_1 = \{\langle x, i\rangle : 0 \leq i \leq n_x \text{ and } x \text{ has a solution of length } i\}$,

$B_2 = \{\langle x, i, f, j\rangle : x \in D,\ 0 \leq i \leq n_x,\ f$ is a $2^{1+\lceil \log_2 n_x \rceil}$-weight function over $[1, n_x]$, $0 \leq j \leq i2^{1+\lceil \log_2 n_x \rceil}$, and x has a solution u such that $|u| = i$ and j is the weight of u under $f\}$,

$B_3 = \{\langle x, i, f, j\rangle : x \in D,\ 0 \leq i \leq n_x,\ f$ is a $2^{1+\lceil \log_2 n_x \rceil}$-weight function over $[1, n_x]$,

$0 \le j \le i2^{1+\lceil \log_2 n_x \rceil}$, and x has two or more solutions u such that $|u| = i$ and j is the weight of u under $f\}$, and

$B_4 = \{\langle x, i, f, j, k\rangle : x \in D,\ 0 \le i \le n_x,\ f$ is a $2^{1+\lceil \log_2 n_x \rceil}$-weight function over $[1, n_x]$, $0 \le j \le i2^{1+\lceil \log_2 n_x \rceil}$, $1 \le k \le i$, and x has a solution u such that $|u| = i$, j is the weight of u under f, and the kth bit of u is $1\}$.

Obviously, L_R, B_1, B_2, B_3, and B_4 are in Σ_2^{P}. Let $B = (((L_R \oplus B_1) \oplus B_2) \oplus B_3) \oplus B_4)$. Then, $B \in \Sigma_2^{\mathrm{P}}$.

Let e be an arbitrary polynomial. We define a polynomial p as follows: $p(i) = e(i) \cdot (p_Q^2(i) + p_Q(i))$. Below, we define a DOTM T which uses B as an oracle set. Given an input $\langle x, w\rangle$ with $x \in D$ and $w \in \{0,1\}^{p(|x|)}$, T operates as follows:

Step 1: T checks whether x has a solution by asking a query to L_R. If x has no solution, then T halts by entering a rejecting state.

Step 2: T finds n_1, the length of the longest solutions of x. This is done by asking the queries $\langle x, 0\rangle$, $\langle x, 1\rangle$, $\cdots$, $\langle x, n_x\rangle$ to the oracle set B_1.

Step 3: T computes $n_2 = 2^{1+\lceil \log_2 n_x \rceil}$ and constructs, from w, n_2-weight functions f_1, f_2, $\cdots$, $f_{e(|x|)}$ over the set $[1, n_x]$ as follows:

Step 3.1: T first computes $e(|x|)$ strings w_1, $\cdots$, $w_{e(|x|)}$ from w such that $|w_1| = \cdots = |w_{e(|x|)}| = n_x \log_2 n_2$ and the string $w_1 w_2 \cdots w_{e(|x|)}$ is a prefix of w (the remaining part of w is ignored), and then for each $1 \le k \le e(|x|)$, it partitions w_k into n_x substrings $w_{k,1}$, $\cdots$, w_{k,n_x} each of length $\log_2 n_2$. (Note: T can do this because $n_x^2 + n_x \ge n_x \log_2 n_2$.)

Step 3.2: For each $1 \le k \le e(|x|)$ and each l in $[1, n_x]$, T sets $f_k(l) = d_{k,l} + 1$, where $d_{k,l}$ is the integer whose binary representation is $w_{k,l}$.

Step 4: For each $1 \le k \le e(|x|)$, T computes the maximum number m_k with $\langle x, n_1, f_k, m_k\rangle \in B_2$. This is done by asking the queries $\langle x, i, f_k, j\rangle$ with $0 \le i \le n_x$, $1 \le k \le e(|x|)$, and $0 \le j \le in_2$ to the oracle set B_2. (Note: In this step, T asks the queries of the form $\langle x, i, f_k, j\rangle$ *for all possible values* of i, k, and j because the machine needs to prepare all queries independently of each other.)

Step 5: For $1 \le k \le e(|x|)$ and $1 \le l \le n_1$, T computes $a_{k,l} = \chi_{B_4}(\langle x, n_1, f_k, m_k, l\rangle)$. This is done by asking the queries $\langle x, i, f_k, j, l\rangle$ with $0 \le i \le n_x$, $1 \le k \le e(|x|)$, $0 \le j \le in_2$, and $1 \le l \le i$ to the oracle set B_4. (Note: In this step, T asks the queries of the form $\langle x, i, f_k, j, l\rangle$ *for all possible values* i, k, j, and l because the machine needs to prepare all queries independently of each other.)

Step 6: For each $1 \le k \le e(|x|)$, T checks whether $\langle x, n_1, f_k, m_k\rangle \in B_3$. This is done by asking the queries $\langle x, i, f_k, j\rangle$ with $0 \le i \le n_x$, $1 \le k \le e(|x|)$, and $0 \le j \le in_2$ to the oracle set B_3. If for some k, $\langle x, n_1, f_k, m_k\rangle \notin B_3$, then T outputs $a_{k,1}a_{k,2} \cdots a_{k,n_1}$ and halts; otherwise, T outputs the special symbol # and halts.

Let F denote the function computed by T with oracle B. We can easily see that T is polynomial-time bounded and all query strings are prepared independently of each other; this means that the query strings made by T on input $\langle x, w\rangle$ can be realized as parallel queries to the oracle set B. Thus, F is in $\mathrm{FP}_{\parallel}^{\Sigma_2^{\mathrm{P}}}$.

Let G be a multi-valued function defined by $G(x) = \{F(x, w) : w \in \{0,1\}^{p(|x|)}$ and $F(x, w)$ is defined$\} - \{\#\}$. We show that G solves Q and is in $\mathrm{RP}\cdot\mathrm{FP}_{\parallel}^{\Sigma_2^{\mathrm{P}}}$. To this end, we first prove two claims.

Claim 1 Suppose that for some k with $1 \le k \le e(|x|)$, x has a unique solution with length n_1 and of weight m_k under f_k. Then, the string $a_{k,1}a_{k,2}\cdots a_{k,n_1}$ output by T is a maximal solution of x.

Claim 2 Suppose that x has solutions and w is randomly chosen from $\{0,1\}^{p(|x|)}$. Then, $\Pr[F(x,w)$ is a maximal solution of $x] \geq 1 - 2^{-e(|x|)}$.

Proof. Since w is randomly chosen from $\{0,1\}^{p(|x|)}$, the functions f_1, f_2, $\cdots$, $f_{e(|x|)}$ constructed in Step 3 must be random n_2-weight functions over $[1, n_x]$. Note that $n_2 = 2^{1+\lceil \log_2 n_x \rceil} \geq 2n_x$. Thus, from Claim 1 and Lemma 3.1, we have that

$\Pr[F(x,w)$ is a maximal solution of $x]$

$= \Pr[(\exists k, 1 \leq k \leq e(|x|))\ a_{k,1}a_{k,2}\cdots a_{k,n_1}$ is a maximal solution of $x]$

$\geq \Pr[(\exists k, 1 \leq k \leq e(|x|))\ x$ has a unique solution with maximum length and of maximum weight under $f_k]$

$\geq 1 - \Pi_{k=1}^{e(|x|)} \Pr[x$ has two or more solutions with maximum length and of maximum weight under $f_k]$

$\geq 1 - (\frac{1}{2})^{e(|x|)} = 1 - 2^{-e(|x|)}$. ∎

In the case when x has no solution, $F(x,w)$ is undefined for all $w \in \{0,1\}^{p(|x|)}$ by Step 1 and hence $G(x) = \emptyset$. On the other hand, when x has solutions, $G(x) \neq \emptyset$ by Claim 2 and the definition of G, and each element of $G(x)$ is a maximal solution of x by Claim 1. Therefore, G solves Q.

If $G(x) = \emptyset$, we know that x has no solution by the discussions in the last paragraph and thus that $F(x,w)$ is undefined for all $w \in \{0,1\}^{p(|x|)}$ by Step 1. On the other hand, if $G(x) \neq \emptyset$, then x has solutions by the discussions in the last paragraph, $\Pr[F(x,w) \in G(x) \cup \{\#\}] = 1$ by Step 6 and the definition of G, and $\Pr[F(x,w) \in G(x)] \geq 1 - 2^{-e(|x|)}$ by Claim 2. Therefore, G is in $\mathrm{RP}\cdot\mathrm{FP}_{||}^{\Sigma_2^P}$. ∎

4 Hardness of solving MAXP's

In the light of Proposition 3.1(1), the following proposition shows that FP is a tight lower bound on the complexity of solving MAXP's whose instance-solution relation is P-decidable and hereditary.

Proposition 4.1 There is MAXP $Q = (D, R)$ such that R is P-decidable and hereditary and Q is hard for FP.

By Proposition 3.1(2), the following theorem shows that NPMV//OptP$[O(\log n)]$ is a tight lower bound on the complexity of solving MAXP's whose instance-solution relation is P-decidable but not hereditary.

Theorem 4.1 There is a MAXP $Q = (D, R)$ such that R is P-decidable (but not hereditary) and Q is hard for NPMV//OptP$[O(\log n)]$.

The following corollary is immediate from the proof of Theorem 4.1.

Corollary 4.1 The following problem (called X-MaxModel hereafter) is complete for NPMV//OptP$[O(\log n)]$:

Instance: A CNF boolean formula ϕ and a subset X of the set of variables in ϕ.

Output: A truth assignment $\vec{a}$ to the variables in X such that $\vec{a}$ can be extended to a satisfying truth assignment to ϕ but no $\vec{b}$ with $\vec{a} \subset \vec{b}$ and $|\vec{a}| = |\vec{b}|$ can be extended to a satisfying truth assignment to ϕ.

X-MaxModel is essentially the same problem as considered by Papadimitriou in Section 3 of [9]. In [9], Papadimitriou claimed *without a precise proof* that the problem is complete for $\mathrm{FP}^{\mathrm{NP}}$. However, he later withdrew his claim and thus left the complexity of the problem open [10]. In [3], we proved that the complexity of the problem is roughly captured by $\mathrm{FP}_{\|}^{\mathrm{NP}}$. Now, Corollary 4.1 gives, for the first time, the exact complexity of the problem. Corollary 4.1 is also of special interest in the sense that no natural problem complete for NPMV//OptP$[O(\log n)]$ has been shown before.

By modifying the proof of Theorem 4.1, we can show that NPMV//OptP$[O(\log n)]$ is a tight lower bound on the complexity of solving MAXP's whose instance-solution relation is NP-decidable and hereditary.

Theorem 4.2 There is a MAXP $Q = (D, R)$ such that R is NP-decidable and hereditary and Q is hard for NPMV//OptP$[O(\log n)]$.

The instance-solution relation of X-MaxModel is NP-decidable but not hereditary. A natural question arises: Are there natural MAXP's Q such that Q is hard for NPMV//OptP$[O(\log n)]$ and the instance-solution relation of Q is either NP-decidable and hereditary or P-decidable (but not hereditary)? Unfortunately, we are unable to settle this question. However, we below show that the question will have a positive answer if NPMV//OptP$[O(\log n)]$ in it is replaced by $\mathrm{FP}_{\|}^{\mathrm{NP}}$.

Definition 4.1 A MAXP $Q = (D, R)$ is *paddable* if there are two functions f and g in FP such that for every list $I = \langle x_1, x_2, \cdots, x_m \rangle$ of instances of Q, $f(I) \in D$ and for every maximal solution w of $f(I)$, $g(I, w)$ gives a maximal solution for each x_i.

Lemma 4.1 If a MAXP is paddable and hard for NP, then it is hard for $\mathrm{FP}_{\|}^{\mathrm{NP}}$.

Theorem 4.3 The following MAXP's are hard for $\mathrm{FP}_{\|}^{\mathrm{NP}}$.

(1) MAXIMAL MODEL (MaxModel)

Instance: A CNF boolean formula ϕ.

Output: A *maximal satisfying truth assignment* to ϕ, i.e., a satisfying truth assignment $\vec{a}$ to ϕ such that there is no other satisfying truth assignment $\vec{b}$ to ϕ with $\vec{a} \subset \vec{b}$.

(2) MAXIMAL CUBIC SUBGRAPH (MaxCubSubgraph)

Instance: An undirected graph G.

Output: A maximal subset F of $E(G)$ such that every vertex in the graph $(V(G), F)$ has either degree 3 or degree 0. (Note: $V(G)$ and $E(G)$ denote hereafter the sets of vertices and edges of G, respectively.)

(3) MAXIMAL SATISFIABILITY (MaxSat)

Instance: A CNF boolean formula $\phi = \{C_1, C_2, \cdots, C_m\}$.

Output: A maximal subset ϕ' of ϕ that is satisfiable.

(4) MAXIMAL k-COLORABILITY ($k \geq 3$) (Max-k-Colorability)

Instance: An undirected graph G.

Output: A maximal subset of $V(G)$ whose induced subgraph is k-colorable.

(5) MAXIMAL HAMILTONIAN SUBGRAPH (MaxHamSubgraph)

Instance: A pair $\langle G, w \rangle$ of a connected undirected graph and a vertex in G.

Output: A maximal subset U of $V(G)$ such that $w \in U$ and the subgraph induced by U has a Hamiltonian circuit.

We here note that a different proof for the $\mathrm{FP}_{\|}^{\mathrm{NP}}$-hardness of MaxModel has been given in [3]. Note that the instance-solution relations of the first two problems in Theorem 4.3 are P-decidable but not hereditary, while the instance-solution relations of the third and fourth problems in Theorem 4.3 are NP-decidable and hereditary. The last problem in Theorem 4.3 is a concrete MAXP whose instance-solution relation is NP-decidable but not hereditary.

For those MAXP's Q whose instance-solution relation is coNP-decidable and hereditary, we are only able to show a loose lower bound.

Proposition 4.2 There is a MAXP $Q = (D, R)$ such that R is a coNP-decidable hereditary relation and Q is hard for $\mathrm{FP}_{\|}^{\mathrm{NP}}$.

In the light of Theorem 3.1(2), the following theorem shows that $\mathrm{FP}_{\|}^{\Sigma_2^{\mathrm{P}}}$ is a nearly optimal lower bound on the complexity of solving MAXP's whose instance-solution relation is coNP-decidable but not hereditary.

Theorem 4.4 There is a MAXP $Q = (D, R)$ such that R is coNP-decidable and Q is hard for $\mathrm{FP}_{\|}^{\Sigma_2^{\mathrm{P}}}$.

5 Characterizations of coNP, D^{P} and Π_2^{P}

The following proposition can be easily proved.

Proposition 5.1 Let $Q = (D, R)$ be a MAXP.
(1) If R is P-decidable and hereditary, then L_Q is in P.
(2) If R is P-decidable, then L_Q is in coNP.
(3) If R is NP-decidable, then L_Q is in D^{P}.
(4) If R is coNP-decidable and hereditary, then L_Q is in D^{P}.
(5) If R is coNP-decidable, then L_Q is in Π_2^{P}.

Similar to Proposition 4.1, we can simply show that P is a tight lower bound on the complexity of L_Q for MAXP's Q whose instance-solution relation is P-decidable and hereditary.

The following theorem gives us characterizations of coNP, D^{P}, and Π_2^{P} via MAXP's.

Theorem 5.1 The following hold:
(1) A set L is in coNP if and only if it can be expressed as $L = \{x : f(x)$ is a maximal solution of x in $Q\}$ for some $f \in \mathrm{FP}$ and some MAXP Q whose instance-solution relation is P-decidable.
(2) A set L is in D^{P} if and only if it can be expressed as $L = \{x : f(x)$ is a maximal solution of x in $Q\}$ for some $f \in \mathrm{FP}$ and some MAXP Q whose instance-solution relation is NP-decidable (and hereditary).
(3) A set L is in D^{P} if and only if it can be expressed as $L = \{x : f(x)$ is a maximal solution of x in $Q\}$ for some $f \in \mathrm{FP}$ and some MAXP Q whose instance-solution relation is coNP-decidable and hereditary.

(4) A set L is in $\Pi_2^{\mathbf{P}}$ if and only if it can be expressed as $L = \{x : f(x)$ is a maximal solution of x in $Q\}$ for some $f \in$ FP and some MAXP Q whose instance-solution relation is coNP-decidable.

From the proof of Theorem 5.1(1), we easily see that there is a MAXP whose instance-solution relation is P-decidable (but not hereditary) and whose associated decision problem is $\leq_m^{\mathbf{P}}$-complete for coNP. However, the following proposition gives us two concrete such MAXP's.

Proposition 5.2 The decision problems associated with MaxModel and MaxCubSubgraph are $\leq_m^{\mathbf{P}}$-complete for coNP:

The following corollary follows immediately from the proof of Theorem 5.1(2) and Cook's theorem.

Corollary 5.1 The decision problem associated with X-MaxModel is $\leq_m^{\mathbf{P}}$-complete for $\mathrm{D}^{\mathbf{P}}$.

We next show three natural MAXP's whose instance-solution relations are in NP and whose associated decision problems are $\leq_m^{\mathbf{P}}$-complete for $\mathrm{D}^{\mathbf{P}}$.

Proposition 5.3 The decision problems associated with MaxSat, Max-k-Colorability and MaxHamSubgraph are $\leq_m^{\mathbf{P}}$-complete for $\mathrm{D}^{\mathbf{P}}$:

We next show a natural MAXP whose instance-solution relation is coNP-decidable and hereditary and whose associated decision problem is $\leq_m^{\mathbf{P}}$-complete for $\mathrm{D}^{\mathbf{P}}$. Other such natural MAXP's may be found in [2,11,12].

Proposition 5.4 The following problem is $\leq_m^{\mathbf{P}}$-complete for $\mathrm{D}^{\mathbf{P}}$:
Instance: A triple $\langle \phi, X, \vec{a} \rangle$, where ϕ is a CNF boolean formula, X is a set of variables appearing only positively in ϕ, and $\vec{a}$ is a truth assignment to the variables in X.
Question: Is it the case that $\vec{a}$ has no extension satisfying ϕ but each $\vec{b} \in \Sigma^{||X||}$ with $\vec{a} \subset \vec{b}$ has an extension satisfying ϕ?

6 Conclusion

In this paper, we have suggested a general framework for studying the complexity of solving maximization problems. Our results are summarized in Table 1 and Table 2. The results give, systematically, characterizations of several important complexity classes via MAXP's. An important consequence of the results is that the complexity of the problem X-MinModel is exactly captured by NPMV//OptP$[O(\log n)]$, giving an answer to an open question of Papadimitriou [9].

As seen from Table 1, the complexity of solving those MAXP's whose instance-solution relation is coNP-decidable and hereditary is unclear. Two obvious open questions are to ask whether the trivial upper bound $\mathrm{FP}^{\mathrm{NP}}$ can be lowered and to ask whether the trivial lower bound $\mathrm{FP}_{||}^{\mathrm{NP}}$ can be raised. As a step toward the investigation of the two questions, we may first consider what is the complexity of solving MinUnsat (or other natural such problems). Although $\mathrm{FP}_{||}^{\mathrm{NP}}$ is a loose lower bound, proving the $\mathrm{FP}_{||}^{\mathrm{NP}}$-hardness of solving MinUnsat seems to be a hard task in the sense that at least the ideas used in proving the

D^P-hardness of the decision problem associated with MinUnsat do not work [11]. Also, showing that MinUnsat is solvable in a class below FP^{NP} needs new ideas; at least, our ideas used in the proof of Theorem 3.1 do not seem to be applicable.

It would be also interesting to consider the complexity of MAXP's whose instance-solution relation is **C**-decidable and hereditary for some complexity class **C** below P. These MAXP's are obviously solvable in FP. Are they solvable in a class below FP or is there such a MAXP Q that solving Q is complete for FP (say, under $\leq^{NC}_{1-T}$ reductions)? The two questions are important in parallel computation in the case when $\mathbf{C} \subseteq \mathrm{NC}$.

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