Generalized Geometric Fitting Problems and Weighted Dynamic Voronoi Diagrams

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1. Introduction

Geometric fitting two similar sets of $n$ points is fundamental problem in image processing and pattern recognition. This minimax geometric fitting problem between two similar sets of points is considered in [10]. For example, it arises in an industrial robot attaching a pin-grid-array type LSI to a board by using visual sensors. The robot first takes an image of the pins of LSI by its visual sensor. Then it tries to fit the LSI package to the corresponding patterns on the board by geometric operations such as translation and rotation. The patterns are a collection of disks or squares of the same size. Using the furthest Voronoi diagram for moving points in the plane, this geometric fitting problem has been solved in $O(n^2\lambda_7(n)\log n)$ time [10].

In a more general setting, the following variants of the fitting problem should be considered. (a) In the case that there are some points which cannot be put into the corresponding disks, minimize the number of such points. (b) In the case that the radii of disk patterns are different from one another, solve this non-uniform geometric fitting problem (Figure 1).

Recently, the Voronoi diagrams for moving objects have been investigated in connection with motion planning in robotics and geometric optimization problems[2,5,8,9,10]. In this paper, to solve the generalized problem (b), we extend the concept of the weighted Voronoi diagram for $n$ points in the plane to the one for the case that the coordinates and the weights of each point are represented by polynomials or rational functions of a parameter. We show that the combinatorial complexity of this dynamic Voronoi diagram is $O(n^2\lambda_s(n))$ where $s$ is some fixed number determined by the degree of the functions and $\lambda_s(n)$ is the maximum length of $(n,s)$ Davenport-Schinzel sequence. Since a lower bound of this dynamic Voronoi diagram can be shown to be $\Omega(n^3)$, our bounds are tight within $\log^*n$ factor, which beats bounds within $n^\epsilon$ factor.

2. Generalized Geometric Fitting Problems

We here formulate the geometric fitting problems for two corresponding sets of points. The non-weighted problem has been formulated as follows [10]. Given two sets $S = \{s_j = (x_j,y_j) \mid j = 1, \cdots, n\}$ and $T = \{t_j = (u_j,v_j) \mid j = 1, \cdots, n\}$ of points in the plane such that $s_j$ is associated with $t_j$, translate, rotate (or transform in a more complicated way) and/or scale the set $S$ simultaneously so that the maximum of the $L_2$ (or $L_\infty$) distances, according to each pattern, between $t_j$ and the transformed $s_j$ is minimized.
For example, in the case that the patterns are disks, and translation and rotation are used as geometric operations, the problem is expressed as follows:
\[
\min_{z, 0 \leq \theta < 2\pi} \max_{j=1, \ldots, n} \| s_j e^{i\theta} - t_j - z \|,
\]
where \( s_j \) and \( t_j \) are identified with complex numbers \( x_i + iy_i \) and \( u_i + iv_i \), respectively.
and \( \theta \) is an angle \( 0 \leq \theta < 2\pi \) and the \( S \) is translated by making the origin to \( z = x + iy \). \( \| \cdot \| \) denotes the Euclidean norm. Using the furthest Voronoi diagram for moving points in the plane, this geometric fitting problem can be solved in \( O(n^2 \lambda(n) \log n) \) time in \( L_2 \) norm.

Defining a points \( p_j(\theta) \) by \( s_j e^{i\theta} - t_j \) in the complex plane, the problem is rewritten as
\[
\min_{0 \leq \theta < 2\pi} \left( \min_{z} \max_{j=1, \ldots, n} \| p_j(\theta) - z \| \right).
\]
Fixing \( \theta \), the problem becomes the minimum enclosing circle problem for \( n \) points \( p_j(\theta) \).
The minimum enclosing circle problem can be solved by using the furthest Voronoi diagram.

We consider the generalized case where the radii of disk patterns are different from one another and solve this non-uniform geometric fitting problem (Figure 1). The weighted Voronoi diagram for moving points can be used to solve this generalized problem.

There are various kinds of Voronoi diagram and efficient algorithms to construct such a diagram. The weighted Voronoi diagram is one of these generalizations of the Voronoi diagram.

Let \( S \) denote a set of \( n \) weighted points \( p_i \) \( (i = 1, \ldots, n) \) in the Euclidean plane. Each point \( p_i \) is associated with a non-negative real weight \( w_i \). The weighted distance \( d_i(p) \) between \( p_i \) and an arbitrary point \( p \) in \( \mathbb{E}^2 \) is defined by \( d_i(p) = w_i \sqrt{(x - x_i)^2 + (y - y_i)^2} \), where the coordinates of \( p_i \), \( p \) are \( (x_i, y_i) \), \( (x, y) \), respectively. The weighted Voronoi region of \( p_i \) is given by
\[
V(p_i) = \bigcap_{j \neq i} \{ p \mid d_i(p) < d_j(p) \}.
\]
and the subdivision of the Euclidean plane defined by the weighted Voronoi regions is called the weighted Voronoi diagram of $S$ (Figure 2). The weighted Voronoi diagram for fixed $n$ points in the plane can be constructed in $O(n^2)$ time [4]. The algorithm in [4] is optimal as the diagram can consist of $\Theta(n^2)$ faces, edges and vertices.

The weighted Voronoi diagram for moving points can be used to solve the problem (b), since the radii of the disks may be considered as weights of their centers. Suppose that point $s_j$ should be put into disk with center $t_j$ and radius $r_j$ in the pattern. By rotating the set $S$ of points by $\theta$ and further translating it by $z$, $s_j$ is the corresponding disk iff $\|s_je^{i\theta} - t_j - z\| \leq r_j$. Then, all the points in $S$ can be put into the corresponding disks iff the following value

$$\min_{z, 0 \leq \theta < 2\pi} \max_{j=1, \ldots, n} \frac{1}{r_j} \|s_je^{i\theta} - t_j - z\|$$

is less than or equal to 1. $\frac{1}{r_j}$ can be considered as the weight of $p_j(\theta) = s_je^{i\theta} - t_j$. Hence, the weighted Voronoi diagram for moving points can be applied to its generalized geometric fitting problems.

To solve the geometric fitting problems, we need the weighted furthest Voronoi diagram. Although the furthest Voronoi diagram is different from the nearest one, the below arguments hold with a slight modification. Hence, we consider the dynamic nearest Voronoi diagram in the below arguments.

3. Definitions and Some Properties of the Weighted Dynamic Voronoi Diagram

In the weighted Voronoi Diagram, if the regions of two point $p_i$ and $p_j$ intersect each other, then the intersection is a subset of the curve defined by the equation $d_i(p) = d_j(p)$. The equidistant curve from $p_i$ and $p_j$ is a circle, and one of the points $p_i$, $p_j$ whose weight is greater than that of the other is enclosed in the circle, which is called the
Apollonius' circle for $p_i$ and $p_j$. Let $p_{ij}$ be the center of the circle $C_{ij}$ and $r_{ij}$ be its radius:

$$p_{ij} = \left( \frac{w_i^2 x_i - w_j^2 x_j}{w_i^2 - w_j^2}, \frac{w_i^2 y_i - w_j^2 y_j}{w_i^2 - w_j^2} \right), \quad r_{ij} = \frac{w_i w_j}{|w_i^2 - w_j^2|} \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}.$$ 

Note here that the equidistant curve from the two points with same weights is a line.

In the weighted Voronoi diagram, the set of the equidistants from three points of $S$ consists of at most two points. See Figure 2. For each $i$, 

$$z = f_i(x, y) = w_i \sqrt{(x - x_i)^2 + (y - y_i)^2}$$ 

is a cone with the singular point at $(x_i, y_i, 0)$. When the cone $z = f_i(x, y)$ intersects another cone $z = f_j(x, y)$, the intersection of two cones is a quadratic curve in $E^3$. The projection of the quadratic curve is nothing but the Apollonius' circle for $p_i$, $p_j$. Consider the lower envelope of the graphs of $n$ functions $f_i(x, y)$. Projecting down this lower envelope to the $(x, y)$-plane yields the weighted Voronoi diagram for $n$ points $p_i$.

The weighted Voronoi diagram may be extended to that for moving points in the following way. Consider $n$ points $p_i(t) = (x_i(t), y_i(t))$ with weights $w_i(t)$ parametrized by $t$ in the plane, where $x_i(t), y_i(t)$ and $w_i(t)$ are functions of $t$, which are polynomials or rational functions of $t$. The degrees of these functions are assumed to be independent of $n$. To simplify our discussion, it is also assumed that these functions are different from one another. As we mentioned before, for fixed $t$, the weighted Voronoi diagram for $p_i(t)$ is the orthogonal projection of the lower envelope of functions $z = f_i(x, y) = w_i \sqrt{(x - x_i)^2 + (y - y_i)^2}$ of two variables $x$ and $y$. In a similar way, the problem of constructing the weighted dynamic diagram may be regarded as computing the lower envelope of $n$ functions

$$f_i(x, y, t) = w_i(t) \sqrt{(x - x_i(t))^2 + (y - y_i(t))^2}$$

of three variables $x$, $y$ and $t$. Define $f(x, y, t)$ by

$$f(x, y, t) = \min_{i=1,\ldots,n} f_i(x, y, t).$$

Now the problem is how to compute $f(x, y, t)$.

For this function $f(x, y, t)$, the minimum diagram is a subdivision of $(x, y, t)$-space such that, with each region, a function $f_i$ attaining the minimum in the definition of $f$ for any point in the region is associated. This is nothing but the projection of the pointwise minimum of these functions onto the $(x, y, t)$-space. The intersection of this diagram with the plane $t = t_0$ is a weighted Voronoi diagram for $t = t_0$, and the whole diagram is called the weighted dynamic Voronoi diagram.

By the assumptions on $p_i(t)$, each region of the minimum diagram of $f$ consists of a maximal connected 3-dimensional set of points at which the minimum is attained by a
function $f_i$. The faces, edges and vertices of the subdivision consist of points at which the minimum is attained simultaneously by two, three and four, respectively, functions. It is easy to see the following properties of the intersections of the trivariate functions $f_i(x,y,t)$ by solving the simultaneous equations each of which defines a function $f_i(x,y,t)$ in the 4-dimensional Euclidean space $\mathbb{E}^4$. In the sequel, variables $x, y, t$ of $f_i$ will be often omitted.

**Lemma 1:** (1) For each $i \neq j$, $f_i(t,x,y) = f_j(t,x,y)$ is a connected surface in $\mathbb{E}^4$.

(2) For each triple $i, j, k$ of distinct indices, $f_i(t) = f_j(t) = f_k(t)$ consists of at most two points for fixed $t$ and $f_i = f_j = f_k$ is a curve with parameter $t$. The curve has at most two components and is discontinuous at most constant times.

(3) For four distinct indices $i, j, k$ and $l$, $f_i = f_j = f_k = f_l$ consists of at most $s$ points, where $s$ is a constant. \qed

4. Combinatorial Complexity of the Minimum Diagram and Constructing the Diagram

A trivial bound on the number of vertices on the minimum diagram is $O(n^4)$, since the number of vertices is bounded by $s(n^4)$ by Lemma 1. However, this is loose as shown below.

Let $i$ and $j$ be distinct two indices, and fix them. Let $l_{ij}$ be the oriented line passing $p_i(t)$ and $p_j(t)$ and the orientation is defined as follows. When weight of $p_i$ is less than that of $p_j$, $l_{ij}$ is oriented from $p_i$ to $p_j$. There are two points on the oriented line which are at equal distances from $p_i(t)$ and $p_j(t)$. One of the two points is between $p_i(t)$ and $p_j(t)$ and the point is denoted by $m_{ij}$. The other is denoted by $\tilde{m}_{ij}$. See Figure 3(a). Draw the circle $C_{jk}$ for each $k \neq i,j$ and consider the intersection points of $C_{ij}$, $C_{jk}$. Those intersection points are nothing but the equidistants from $p_i$, $p_j$, $p_k$. Using the intersection points of $C_{ij}$, $C_{jk}$, we define four points $q^+_k(t)$, $q^-_k(t)$, $\tilde{q}^+_k(t)$ and $\tilde{q}^-_k(t)$ as follows. $d_E(p,q)$ is the Euclidean distance between $p$ and $q$.

**Case 1:** $d_E(p_{ij},p_{jk}) < r_{ij} + r_{jk}$ and $d_E(p_{ij},p_{jk}) > |r_{ij} - r_{jk}|$.

In this case, $C_{ij}$ intersects $C_{jk}$ at two points $q$ and $q'$. Let $s_k$ be the line segment connecting $q$ and $q'$.

**Case 1-1:** $s_k \cap l_{ij} = \emptyset$.

**Case 1-1-1:** Consider the case where $C_{jk}$ intersects $l_{ij}$ and these two intersection points are between $m_{ij}$ and $\tilde{m}_{ij}$. When the segment is on the right side of $l_{ij}$, choose the point which is counterclockwisely nearer along the circle from $m_{ij}$ between $q$ and $q'$ and let $q^-_k$ be the point and $q^+_k$ be the other. See Figure 3(a). When the
Figure 3. Definitions of $q_k^+$, $q_k^-$, $\tilde{q}_k^+$ and $\tilde{q}_k^-$.

Segment is on the left side of $l_{ij}$, choose the point which is clockwise nearer along the circle from $m_{ij}$ and let $q_k^+$ be the point and $q_k^-$ be the other. Moreover, we define $\tilde{q}_k^+ = q_k^+$, $\tilde{q}_k^- = q_k^-$. 

Case 1-1-2: Consider the case where $C_{jk}$ does not intersect $l_{ij}$, or $C_{jk}$ intersects $l_{ij}$ at two points and $m_{ij}$, $\tilde{m}_{ij}$ are between those two points. When the segment is on the right side of $l_{ij}$, choose the point which is counterclockwisely nearer along the circle from $m_{ij}$ between $q$ and $q'$ and let $q_k^+$ be the point and $\tilde{q}_k^-$ be the other. Furthermore, we define $\tilde{q}_k^+ = +\infty$, $\tilde{q}_k^- = -\infty$. If $\tilde{q}_k^- = \tilde{m}_{ij}$, then we consider $\tilde{q}_k^+ = -\tilde{m}_{ij}$. When the segment is on the left side of $l_{ij}$, choose the point which is clockwise nearer along the circle from $m_{ij}$ between $q$ and $q'$ and let $q_k^+$ be the point and $\tilde{q}_k^+$ be the other. Furthermore, we define $q_k^+ = +\infty$, $\tilde{q}_k^- = -\infty$. If $\tilde{q}_k^- = \tilde{m}_{ij}$, then we consider $\tilde{q}_k^+ = -\tilde{m}_{ij}$.

Case 1-2: $s_k \cap l_{ij} \neq \emptyset$.

$l_{ij}$ intersects $C_{ik}$ at two points.

Case 1-2-1: Consider the case where $w_j > w_k$ and $m_{ij}$ is between these intersection points of $l_{ij}$ and $C_{jk}$, or $w_j < w_k$ and $\tilde{m}_{ij}$ is between these intersection points of $l_{ij}$ and $C_{jk}$. Define the point lying on the right side of $l_{ij}$ among $q$ and $q'$ to be $q_k^+$. The other point, lying on the left side of $l_{ij}$, is called $q_k^-$. We define $\tilde{q}_k^+$ and $\tilde{q}_k^-$ as $\tilde{q}_k^+ = q_k^+$, $\tilde{q}_k^- = q_k^-$. See Figure 3(c).

Case 1-2-2: Consider the case where $w_j > w_k$ and $\tilde{m}_{ij}$ is between these intersection points of $l_{ij}$ and $C_{jk}$, or $w_j < w_k$ and $m_{ij}$ is between these intersection points of $l_{ij}$ and $C_{jk}$.
of \( l_{ij} \) and \( C_{jk} \). Define the point lying on the right side of \( l_{ij} \) among \( q \) and \( q' \) to be \( \tilde{q}_k^- \). The other point, lying on the left side of \( l_{ij} \), is called \( q_k^+ \). Set \( \tilde{q}_k^- = +\infty \), \( q_k^+ = -\infty \). See Figure 3(d). If \( \tilde{q}_k^- = \tilde{m}_{ij} \) (resp. \( q_k^+ = \tilde{m}_{ij} \) ), then we consider \( \tilde{q}_k^- = +m_{ij} \) (resp. \( q_k^+ = -m_{ij} \) ).

Case 2: \( d_E(p_{ij},p_{jk}) = r_{ij} + r_{jk} \).

In this case, there is only one intersection point of \( C_{ij} \) and \( C_{jk} \). We call the intersection point \( q \) and define four points by \( q_k^+ = q_k^- = \tilde{q}_k^+ = \tilde{q}_k^- = q \).

Case 3: \( d_E(p_{ij},p_{jk}) > r_{ij} + r_{jk} \) or \( d_E(p_{ij},p_{jk}) < |r_{ij} - r_{jk}| \).

There is no intersection point of \( C_{ij} \) and \( C_{ik} \).

Case 3-1: In the cases that \( d_E(p_{ij} - p_{jk}) > r_{ij} + r_{jk} \) or \( d_E(p_{ij} - p_{jk}) < -r_{ij} + r_{jk} \), the four points are defined by \( q_k^+ = q_k^- = +\infty \), \( q_k^- = q_k^+ = -\infty \).

Case 3-2: If \( d_E(p_{ij} - p_{jk}) < r_{ij} - r_{jk} \), we set \( q_k^+ = q_k^- = -\infty \), \( q_k^- = q_k^+ = +\infty \).

Let \( d_k^- \left( d_k^+, \tilde{d}_k^-, \tilde{d}_k^+ \right) \) be the Euclid distance between \( p_{ij} \) and \( q_k^- \left( q_k^+, \tilde{q}_k^-, \tilde{q}_k^+ \right) \).

We consider the distance between a point and \( \pm\infty \) is infinity and the point \( +\infty \) (resp. \( -\infty \)) is on the right (resp. left) side of \( l_{ij}(t) \). We define four functions \( g_k^+ \), \( g_k^- \), \( \tilde{g}_k^+ \), and \( \tilde{g}_k^- \) by using four points \( q_k^+ \), \( q_k^- \), \( \tilde{q}_k^+ \), and \( \tilde{q}_k^- \) as follows:

\[
g_k^+(t) = \begin{cases} +d_k^+(q_k^+ \text{ is on the right side of } l_{ij}(t)) \\ -d_k^-(q_k^- \text{ is on the left side of } l_{ij}(t)) \end{cases}
\]

\[
g_k^-(t) = \begin{cases} -d_k^-(q_k^- \text{ is on the right side of } l_{ij}(t)) \\ +d_k^+(q_k^+ \text{ is on the left side of } l_{ij}(t)) \end{cases}
\]

\[
\tilde{g}_k^+(t) = \begin{cases} +\tilde{d}_k^+(\tilde{q}_k^+ \text{ is on the right side of } l_{ij}(t)) \\ -\tilde{d}_k^-(\tilde{q}_k^- \text{ is on the left side of } l_{ij}(t)) \end{cases}
\]

\[
\tilde{g}_k^-(t) = \begin{cases} -\tilde{d}_k^-(\tilde{q}_k^- \text{ is on the right side of } l_{ij}(t)) \\ +\tilde{d}_k^+(\tilde{q}_k^+ \text{ is on the left side of } l_{ij}(t)) \end{cases}
\]

For \( g_k^+(t) \), \( g_k^-(t) \), \( \tilde{g}_k^+(t) \), \( \tilde{g}_k^-(t) \), further define \( g^+(t) \), \( g^-(t) \), \( g'(t) \), \( \tilde{g}^+(t) \), \( \tilde{g}^-(t) \) and \( \tilde{g}'(t) \) as follows:

\[
g^+(t) = \min_{k \neq i,j} g_k^+(t), \quad \tilde{g}^+(t) = \min_{k \neq i,j} \tilde{g}_k^+(t),
\]

\[
g^-(t) = \min_{k \neq i,j} g_k^-(t), \quad \tilde{g}^-(t) = \min_{k \neq i,j} \tilde{g}_k^-(t),
\]

\[
g'(t) = \max\{g^-(t) + g^+(t), 0\}, \quad \tilde{g}'(t) = \max\{\tilde{g}^-(t) + \tilde{g}^+(t), 0\}.
\]

\[
h(t) = \max\{g^+(t) - \tilde{g}^-(t), 0\}
\]

Geometric implications of these definitions are given by the following lemma. Note that Voronoi edge \( E_{ij} \), which is on the boundary of Voronoi regions \( V_i \), \( V_j \), may be disconnected and may consist of some connected components \( \{e_{ij}\} \).

**Lemma 2:** (1) For \( t \), suppose \( g'(t) > 0 \), \( g^+(t) \neq \pm \infty \) (resp. \( g^-(t) \neq \pm \infty \)) is attained by a function \( g_k^+(t) \) (resp. \( g_l^-(t) \)). Then, \( q_k^+(t) \) and \( q_l^-(t) \) are Voronoi points which are
on the boundaries of Voronoi regions $V_i$, $V_j$, $V_k$ ($V_l$), and equidistant from $p_i$, $p_j$, $p_k$ ($p_l$). $q^+_i(t)$ (resp. $q^-_i(t)$) is the right (resp. left) end point of the leftest component $e_{ij}$ of $E_{ij}$ regarding the orientation defined the oriented line $l_{ij}$ and the point $m_{ij}$.

(2) For $t$, suppose $\tilde{g}'(t) > 0$, $\tilde{g}^+(t) \neq \pm \infty$ (resp. $\tilde{g}^-(t) \neq \pm \infty$) is attained by a function $\tilde{g}^+_k(t)$ (resp. $\tilde{g}^-_k(t)$). Then, $q^+_k(t)$ and $q^-_k(t)$ are Voronoi points which are on the boundaries of Voronoi regions $V_i$, $V_j$, $V_k$ ($V_l$), and equidistant from $p_i$, $p_j$, $p_k$ ($p_l$). $q^+_k(t)$ (resp. $q^-_k(t)$) is the right (resp. left) end point of the rightest component $e_{ij}$ of $E_{ij}$ regarding the orientation defined the oriented line $l_{ij}$ and the point $m_{ij}$.

Graphs of $g^+$, $g^-$ and $g'$ are composed of maximal connected portions of graphs of $g^+_k$, $g^-_k$, $g^+_k + g^-_l$ and 0. As usual, define the combinatorial complexity of these functions to be the maximum number of such maximal connected portions. Also, call $t'$ an intersecting value of a function $(g^+, g^-, g')$ if the functions $(g^+_k$, $g^-_k$, $g^+_k + g^-_l$, 0) attaining its minimum at $t' - t_e$ and $t' + t_e$ are different for sufficiently small $t_e$. The number of intersecting values is nearly equal to the combinatorial complexity. This complexity can be evaluated as follows, where $\lambda_s(n)$ is the maximum length of $(n, s)$ Davenport-Schinzel sequence (e.g., see [1,3,6,11,12]), and is almost linear in $n$. For $\tilde{g}^+$, $\tilde{g}^-$, $\tilde{g}'$ and $h$, similar.

**Lemma 3:** The combinatorial complexity of $g^+$, $g^-$ and $g'$ ($\tilde{g}^+$, $\tilde{g}^-$ and $\tilde{g}'$) are $O(\lambda_{s+2}(n))$. These functions can be computed in $O(\lambda_{s+1}(n) \log n)$ time and $O(n)$ space.

**Proof:** Each $g^+_k$ may be discontinuous at most a constant number of times from Lemma 1(2). Any two functions among $g^+_k$ intersect at most $s$ points from Lemma 1(3). Hence, the combinatorial complexity of $g^+$ is $O(\lambda_{s+2}(n))$ [3]. For $g^-$, similar. Any two functions among $g^+_k$, $g^-_k$, $g^+_k + g^-_l$ and 0 intersect at most constant times. Then, the combinatorial complexity of $g'$ is within that of $g^+$ and $g^-$ by a constant factor. The time complexity follows from [7]. For $\tilde{g}^+$, $\tilde{g}^-$ and $\tilde{g}'$, similar. For $h$, the graph of $h$ are composed of maximal connected portions of graphs of $g^+_k$, $g^-_l$, $g^+_k - g^-_l$ and 0. Any two functions among these functions also intersect at most constant times. Hence, the combinatorial complexity of $h$ is within that of $g^+$ and $\tilde{g}^-$ by a constant factor.

Fixing $i$ and $j$, the end points of the leftest and rightest components of Voronoi edge $E_{ij}$ regarding the orientation defined by the oriented line $l_{ij}$ and the middle point $m_{ij}$, are the points $q$, $q'$ associated with the function $g^+$, $g^-$, $\tilde{g}^+$, $\tilde{g}^-$. However, some of the Voronoi points on $E_{ij}$ may be incident to the components of $E_{ij}$ which are not the leftest or rightest ones. Such Voronoi points cannot be listed by using the functions $g^+$, $g^-$, $\tilde{g}^+$ and $\tilde{g}^-$ for the indices $i$, $j$. The remaining problem is how to
maintain such Voronoi points and count the topological changes. We assume that $q$ is the Voronoi point which is the end points of neither of the leftest and rightest components and $w_i < w_j < w_k$. In this case, we choose the indices $j$ and $k$ instead of $i$ and $j$, and define the functions $g^+, g^-, \tilde{g}^+, \tilde{g}^-$ for new indices $j$ and $k$. It is easily to see that the point is the end points of the leftest and rightest components of Voronoi edge $E_{jk}$ regarding the orientation defined by the oriented line $l_{jk}$ and the middle point $m_{jk}$. Therefore, any end point of Voronoi edge is the end point of the leftest and rightest component regarding the orientation defined by the oriented line $l_{i'j'}$ and the middle point $m_{i'j'}$ for a pair of indices $i', j'$.

**Theorem 1:** The weighted dynamic Voronoi diagram has the combinatorial complexity of $O(n^2\lambda_{s+2}(n))$, and can be computed in $O(n^2\lambda_{s+1}(n)\log n)$ time and $O(n)$ space.

**Proof:** Suppose that one of the intersections, whose number is at most $s$, of $f_i$, $f_j$, $f_k$ and $f_l$ in $E^4$ corresponds to a vertex on the minimum diagram in $E^3$, and the $t$-coordinate of this point is $t'$. There exists a point $q$ which is equidistant from $p_i(t')$, $p_j(t')$, $p_k(t')$, $p_l(t')$, and, for the other indices $h$, $p_h(t')$ is farther from $q$ than these four points. For $t' - t_e$ (for sufficiently small $t_e$), $q(t' - t_e)$ is the end points of the leftest or rightest component of one of the sets $E_{ij}$, $E_{ik}$, $E_{il}$, $E_{jk}$, $E_{jl}$ and $E_{kl}$. Suppose $q(t' - t_e)$ is the leftest or rightest edge in $E_{ij}$, then $t'$ is an intersecting value of one of the functions $g^+$, $g^-$, $g'$, $\tilde{g}^+$, $\tilde{g}^-$, $\tilde{g}'$ and $h$ defined for the indices $i$ and $j$. Therefore, any vertex on the minimum diagram is associated with a intersecting value of one of the functions $g^+$, $g^-$, $\tilde{g}^+$ and $\tilde{g}^-$, or with a solution of one of the equations $g^+ + g^- = 0$, $\tilde{g}^+ + \tilde{g}^- = 0$ and $g^+ - \tilde{g}^- = 0$. When $q = m_{ij}$, then it is associated with a solution of $g^+ - \tilde{g}^- = 0$. Conversely, each of such intersecting values corresponds to a unique vertex in the minimum diagram. A solution of $g^+ - \tilde{g}^- = 0$ corresponds to a vertex in the minimum diagram in the case that $g^- = +\infty$, $\tilde{g}^+ = +\infty$ and $g^+ = \tilde{g}^- = 2r_{ij}$.

Hence, the combinatorial complexity of the weighted dynamic Voronoi diagram is $O(n^2\lambda_{s+2}(n))$. By computing $g^+$, $g^-$, $g'$, $\tilde{g}^+$, $\tilde{g}^-$, $\tilde{g}'$, $h$ for any pair of $p_i(t)$ and $p_j(t)$, all the vertices on the weighted Voronoi diagram can be listed in $O(n^2\lambda_{s+1}(n)\log n)$ time and $O(n)$ space. □

### 5. Conclusion

In this paper, we extend the concept of the weighted Voronoi diagram for $n$ points to the one for the case that the coordinates of each point and the weights are represented by polynomials or rational functions of a parameter. We show that combinatorial complexity of this dynamic Voronoi diagram is $O(n^2\lambda_s(n))$ where $s$ is some fixed number determined
by the degree of the functions and $\lambda_s(n)$ is the maximum length of $(n, s)$ Davenport-Schinzel sequence. Since a lower bound of this dynamic Voronoi diagram can be shown to be $\Omega(n^3)$, our bounds are tight within $\log^* n$ factor, which beats bounds within $n^\epsilon$ factor.

Applications of these dynamic Voronoi diagrams to some generalized geometric fitting problems are also presented. As related problems, constructing the higher-order Voronoi diagram for moving points in the plane and the dynamic Voronoi diagram for $n$ circles in the Euclidean geometry and in the Laguerre geometry can be considered. For these problems, using the same technique presented in this paper, we obtain an $O(n^2\lambda_{s+1}(n)\log n)$ algorithm for moving circles in Euclidean and Laguerre geometry, and an $O(n^2m\lambda_{s+m+2}(n)\log n)$ algorithm for the dynamic $m$-th Voronoi diagram.

References


