数理解析研究所講究録
第 834 巻 1993 年 13-21

Contact transformations, Huygens's principle, and the fourfold picture of the calculus of variations

STEFAN HILDEBRANDT

1. The transformations of Hölder and Legendre.

Let $F(x,z,p)$ be a C^2-function on $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$; the points in this space are denoted by (x,z,p). With F we associate an *adjoint function* $\Phi(x,z,p)$ defined by

$$\Phi := p \cdot F_p - F, \tag{1.1}$$

and a *momentum tensor* T defined by

$$T := p \otimes F_p - FI \tag{1.2}$$

where I is the identity, i.e. the components of T are given by

$$T^i_k = p_k F_{p_i} - F\delta^i_k.$$

Then we define E. Hölder's transformation generated by F :

$$\mathcal{H}_F : (x,z,p) \mapsto (x,z,y)$$

by

$$y = \frac{p}{F(x,z,p)}. \tag{1.3}$$

If this map is invertible, we can introduce the Hölder transform $H(x,z,y)$ of F by

$$H := 1/F \circ \mathcal{H}_F^{-1} \tag{1.4}$$

For the sake of simplicity we shall here and in similar cases assume that the inverse mappings exist locally ; thus all considerations are to be understood in a local sense.

First we can prove:

Lemma 1.1. *We have*

$$\det T = (-1)^{n-1} F^{n-1} \Phi \tag{1.5}$$

Lemma 1.2. *The Jacobi matrix of the mapping $p \mapsto y$ given by (1.3) can be computed as*

$$\frac{\partial y}{\partial p} = -F^{-2} T \tag{1.6}$$

whence

$$\det \frac{\partial y}{\partial p} = -F^{-n-1}\Phi. \tag{1.7}$$

Hence $\mathcal{H}_F^{-1}$ exists locally if we assume

$$F \neq 0 \quad \text{and} \quad \Phi \neq 0. \tag{A1}$$

Thus H is well-defined if we require (A1), and we immediately see that the transformation (1.3), (1.4) is an involution. In fact, if (x, z, p) and (x, z, y) are related by (1.3), we obtain

$$\begin{aligned} y &= \frac{p}{F(x,z,p)}, & p &= \frac{y}{H(x,z,y)}, \\ H(x,z,y) &= \frac{1}{F(x,z,p)}, & F(x,z,p) &= \frac{1}{H(x,z,p)}. \end{aligned} \tag{1.8}$$

Suppose that $\Psi(x, z, y)$ is the adjoint of $\Phi(x, z, y)$ given by

$$\Psi := y \cdot H_y - H, \tag{1.9}$$

and let P be the momentum tensor of H,

$$P := y \otimes H_y - HI. \tag{1.10}$$

Then formulas (1.8) can be complemented by the following remarkable relations where $F = F(x,z,p), F_x = F_x(x,z,p), \cdots$, and $H = H(x,z,y)$, $H_y = H_y(x,z,y), \cdots$, and (x,z,p) and (x,z,y) are related by formula (1.3) :

$$\begin{gathered} y = \frac{p}{F}, \ H = \frac{1}{F}, \ \Psi = \frac{1}{\Phi}, \\ H_x = \frac{F_x}{F\Phi}, \ H_y = \frac{F_p}{\Phi}, \ H_z = \frac{F_z}{F\Phi}, \\ \det H_{yy} = \left(\frac{F}{\Phi}\right)^{n+2} \det F_{pp}. \end{gathered} \tag{1.11}$$

A completely analogous set of formulas holds for F and its derivatives. Moreover, one can even express H_{yy} in terms of F ; this is, however, more elegantly done by the Legendre transform W of F, see formula (1.16) below.

Next we recall Legendre's transformation

$$\mathcal{L}_F : (x,z,p) \mapsto (x,z,\xi)$$

generated by F which is defined by

$$\xi = F_p(x, z, p) \tag{1.12}$$

The Legendre transform $W(x,z,\xi)$ of $F(x,z,p)$ is defined by

$$W := \Phi \circ \mathcal{L}_F^{-1}.$$

Here $\mathcal{L}_F^{-1}$ exists (locally) if we suppose

$$\det F_{pp} \neq 0. \tag{A2}$$

Let

$$M := \xi \cdot W_\xi - W \tag{1.13}$$

be the adjoint of W, and let

$$\Gamma := \xi \otimes W_\xi - WI \tag{1.14}$$

be its momentum tensor.

It is well-known that the 2-step procedure $(x,z,p), F(x,z,p) \mapsto (x,z,\xi), W(x,z,\xi)$ of first assigning ξ to p and then defining W is an involution, satisfying

$$\begin{aligned} &F + W = p \cdot \xi, \quad \xi = F_p, \quad p = W_\xi, \\ &F_x + W_x = 0, \quad F_z + W_z = 0, \end{aligned} \tag{1.15}$$

where again $F = F(x,z,p), F_x = F_x(x,z,p), \cdots, W = W(x,z,\xi), W_x = W_x(x,z,\xi)$, $\cdots$, and $(x,z,p) \leftrightarrow (x,z,\xi)$.
By using the corresponding sloppy, but instructive notation if $(x,z,p) \leftrightarrow (x,z,\xi) \leftrightarrow (x,z,y)$, we obtain

$$\begin{aligned} &M = F = \frac{1}{H}, \quad W = \Phi = \frac{1}{\Psi}, \quad F_{pp} = W_{\xi\xi}^{-1}, \\ &H_{yy} = (W^{-2}\Gamma) \cdot F_{pp} \cdot (H^{-2}P), \end{aligned} \tag{1.16}$$

and from the second line follows the last equation of (1.11). Thus we infer

Proposition 1.1. *If* $F \neq 0, \Phi \neq 0$, *and* $\det F_{pp} \neq 0$, *then also* $H \neq 0, \Psi \neq 0$, $\det H_{yy} \neq 0$, *and also* $W \neq 0, M \neq 0$, $\det W_{\xi\xi} \neq 0$.

Thus we have under the assumptions (A1) and (A2) in past-center the following formulas

$$\begin{aligned} &\mathcal{L}_F^{-1} = \mathcal{L}_W, \qquad \mathcal{H}_F^{-1} = \mathcal{H}_H, \\ &W = \Phi \circ \mathcal{L}_F^{-1}, \quad F = M \circ \mathcal{L}_W^{-1}, \\ &H = F \circ \mathcal{H}_F^{-1}, \quad F = H \circ \mathcal{H}_H^{-1}. \end{aligned} \tag{1.17}$$

Furthermore, assumptions (A1), (A2) carry over from F to its Hölder transform H and its Legendre transform W.

From the above formulas we infer the following result :

Proposition 1.2. *We have*

$$\mathcal{L}_H \circ \mathcal{H}_F = \mathcal{H}_W \circ \mathcal{L}_F \tag{1.18}$$

provided that (A1) *and* (A2) *are satisfied, i.e. the following diagram is commuting* :

$$\begin{tikzcd} x,z,p,F & \boxed{\mathrm{I}} \arrow[r, "\mathcal{H}_F"] \arrow[d, "\mathcal{L}_F"] & \boxed{\mathrm{III}} \arrow[d, "\mathcal{L}_H"'] & x,z,y,H \\ x,z,\xi,W & \boxed{\mathrm{II}} \arrow[r, "\mathcal{H}_W"'] & \boxed{\mathrm{IV}} & x,z,\nu,L \end{tikzcd} \tag{1.19}$$

Here $L(x,z,v)$ is the Legendre transform of H obtained by

$$L = H \circ \mathcal{L}_H^{-1} \tag{1.20}$$

and $\mathcal{L}_H : (x,z,y) \mapsto (x,z,v)$ is the Legendre transformation generated by H, i.e.

$$v = H_y(x,z,y). \tag{1.21}$$

Note that the transformation $\mathcal{R}_F : (x,z,p) \mapsto (x,z,v)$ defined by

$$\mathcal{R}_F := \mathcal{L}_H \circ \mathcal{H}_F \tag{1.22}$$

is an involution, which in a different context was introduced by A. Haar. However, this transformation appears (implicitely) at many places in the calculus of variations as it is closely related to a classical contact transformation, the mapping by reciprocal polars. Proposition 2 implies that

$$\mathcal{R}_F = \mathcal{R}_L^{-1}, \tag{1.23}$$

and this relation shows that Haar's transformation is an involution.

Note also that $\mathcal{R}_F : (x,z,p) \mapsto (x,z,v)$ is given by

$$p \mapsto v = \frac{F_p}{p \cdot F_p - F} \tag{1.24}$$

i.e.

$$v = \frac{F_p(x,z,p)}{\Phi(x,z,p)} \tag{1.24'}$$

and the Haar transform L of F is just

$$L(x,z,v) = \frac{1}{\Phi(x,z,p)}. \tag{1.25}$$

From the second line of (1.16) one can derive that

$$H_{yy} = (F^3/\Phi) P^T F_{pp} P, \tag{1.26}$$

and this result can be used to show global invertibility of $\mathcal{L}_F, \mathcal{H}_F$, and $\mathcal{R}_F$ assuming also that F_{pp} is definite.

2. Contact transformations and Lie's equations.

Let us consider the configuration space $M = \mathbf{R}^n \times \mathbf{R}$ with the points (x, z) and the corresponding *contact space* $\hat{M}$ with the points (x, z, p). The contact form ω on $\hat{M}$ is defined by

$$\omega = dz - p \cdot dx. \tag{2.1}$$

Let $\mathcal{U}$ be an n-dimensional parameter domain. An immersion $\mathcal{E} : \mathcal{U} \to \hat{M}$ is called a *strip* (more precisely, an n-dimensional strip) if it annulles ω, that is, if

$$\mathcal{E}^*\omega = 0 \tag{2.2}$$

where $\mathcal{E}^*\omega$ denotes the pull-back of ω with respect to $\mathcal{E}$. A *contact transformation* is a mapping $\mathcal{T} : \Omega \to \hat{M}$ of some domain $\Omega \subset \hat{M}$ into $\hat{M}$ such that

$$\mathcal{T}^*\omega = \rho\omega \tag{2.3}$$

for some nonvanishing function ρ. Contact transformations map strips into strips. Lie has proved that every one-parameter group of contact transformations

$$\bar{x} = X(\theta, x, z, p), \ \ \bar{z} = Z(\theta, x, z, p), \ \ \bar{p} = P(\theta, x, z, p) \tag{2.4}$$

or

$$\bar{e} = \sigma(\theta, e) = \mathcal{T}_\theta e, \tag{2.5}$$

$e := (x, z, p)$, $\bar{e} := (\bar{x}, \bar{z}, \bar{p})$, is generated by a vector field $\mathcal{H}_F$ of the form

$$\mathcal{H}_F = \Pi^i \frac{\partial}{\partial x^i} + \Phi \frac{\partial}{\partial z} + A_i \frac{\partial}{\partial \phi_i} \tag{2.6}$$

where

$$\Pi^i = F_{pi}, \ \Phi = p \cdot F_p - F, \ A_i = -F_{xi} - p_i F_z, \tag{2.7}$$

i.e. $\mathcal{H}_F$ is derived from a single scalar function $F(x, z, p)$ which will be called *Lie's function.* In other words, $\sigma(\theta, e) := \mathcal{T}_\theta e$ is a flow of *Lie's system of differential equations*

$$\dot{x} = F_p, \quad \dot{z} = p \cdot F_p - F, \quad \dot{p} = -F_x - pF_z \tag{2.8}$$

where $\dot{x} = \dfrac{dx}{d\theta}$, etc. Conversely, every flow generated in the form (2.8) yields a (local) one-parameter group of contact transformations.

As was observed by Lie, Vessiot, and E. Hölder, equations (2.8) describe the motion $\mathcal{T}_\theta \mathcal{E}$ of strips $\mathcal{E}$ in time by means of *Huygens's principle.* More precisely, if $\zeta = W(x, z, \xi)$ is the *indicatrix* of an optical medium at the point (x, z) described in running coordinates ξ, ζ, and if F is the Legendre transform of W, then *Huygens's envelope construction* applied to surfaces $\mathcal{S}$ in M and to the strips $\mathcal{E}$ supported by $\mathcal{S}$ leads to a motion $\mathcal{S}_\theta$ and $\mathcal{E}_\theta$ of $\mathcal{S}$ and $\mathcal{E}$ given by $\mathcal{E}_\theta = \mathcal{T}_\theta\mathcal{E}$, i.e. by (2.8), if each point (x,z) on $\mathcal{S}$ is viewed as source of an elementary wave $\{(\xi,\zeta) : \zeta = W(x,z,\xi)\}$. We call an n-parameter solution $\sigma(\theta,c)$, $c = (c_1,\dots,c_n)$ of (2.8) a *Lie flow* provided that σ_c has rank n. The pull-back $\sigma^*\omega$ of the contact form ω with respect to a Lie flow is of the form

$$\sigma^*\omega = -\varphi\, d\theta + \lambda_\alpha\, dc_\alpha \tag{2.9}$$

where $\varphi := F(\sigma)$ and $\lambda_1,\dots,\lambda_n$ are solutions of the same homogeneous linear differential equation, namely

$$\dot\varphi + F_z(\sigma)\varphi = 0, \quad \dot\lambda_\alpha + F_z(\sigma)\lambda_\alpha = 0. \tag{2.10}$$

A Lie flow is said to be a *Huygens flow* if it satisfies

$$\sigma^*\omega = -F(\sigma)d\theta. \tag{2.11}$$

It turns out that a Lie flow is Huygens if and only if its initial values $\mathcal{E}(c) = \sigma(0,c)$ *satisfy the strip condition*

$$\mathcal{E}^{*'}\omega = 0.$$

The *ray map* $r(\theta,c) = (X(\theta,c), Z(\theta,c))$ of a Huygens flow

$$(\theta,c) \mapsto \sigma(\theta,c)$$

given by (2.4) is said to be a *Huygens field* (on G), if r defines a diffeomorphism (onto some domain G in M). Let $P(\theta,c)$ be the p-component of $\sigma(\theta,c)$ and set

$$\mathcal{N} := P \circ r^{-1} \tag{2.12}$$

and

$$\nu(x,z) = (x,z,\mathcal{N}(x,z)). \tag{2.13}$$

Moreover, we set

$$s(x,z) = (S(x,z), T(x,z)), \tag{2.14}$$

that is, $s = r^{-1}$ is given by

$$\theta = S(x,z), \quad c = T(x,z). \tag{2.15}$$

Then we have

$$\nu(x,z) = (\sigma \circ s)(x,z) = (r \circ s, P \circ s)(x,z),$$

and (2.11) implies

$$\nu^*\omega = -F(\nu)dS \tag{2.16}$$

i.e.

(2.16') $$d_z - \mathcal{N} \cdot dx = -F(\nu)dS.$$

This implies

(2.17) $$\mathcal{N} = -S_x/S_z \quad \text{and} \quad F(\cdot,\cdot,\mathcal{N})S_z + 1 = 0$$

whence we obtain *Vessiot's differential equation*

(2.18) $$F(x, z, -S_x/S_z)S_z + 1 = 0.$$

The motion of wave fronts of a Huygens field with the direction field $\nu(x,z) = (x, z, \mathcal{N}(x,z)$ is then obtained by forming the level surfaces

(2.19) $$\mathcal{S}_\theta = \{(x,z) : S(x,z) = \theta\}$$

of the function $S(x,z)$ which is called the *eikonal* of the Huygens field r.

Applying the Legendre transformation $\mathcal{L}_F$ to Lie's equations (2.8), these are transformed into the *Herglotz equations*

(2.20) $$\dot{x} = \xi, \quad \dot{z} = W, \quad \frac{d}{d\theta}W_\xi - W_x - W_z W_\xi = 0$$

while Vessiot's equation for S is transformed into the *characteristic system*

(2.21) $$\begin{aligned} S_x &= W_\xi(\cdot,\cdot,\mathcal{D}/M(\cdot,\cdot,\mathcal{D})) \\ S_z &= -1/M(\cdot,\cdot,\mathcal{D}) \end{aligned}$$

for the pair $\{\mathcal{S}, \mathcal{D}\}$, where S is the eikonal of the Huygens field r, and $\mathcal{D} = F_p(\cdot,\cdot,-S_x/S_z)$ is its *slope field*, i.e. we can reconstruct the rays $r(\cdot,c)$ of the Huygens field r by solving suitable initial value problems for the system

(2.22) $$\dot{x} = \mathcal{D}(x,z). \quad \dot{z} = W(x, z, \mathcal{D}(x,z)).$$

Thus we have two equivalent pictures of geometrical optics based on Huygens's principle, one described by Lie's system (2.8) and Vessiot's equation (2.18), the other by Herglotz's equations (2.20) and by the characteristic equations (2.21). Both pictures are related by the Legendre transformation $\mathcal{L}_F$ or by its inverse $\mathcal{L}_W$. We call the description based on (2.8) and (2.18) the *Huygens-Lie picture*, while the description based on (2.20) and (2.21) is denoted as *Herglotz picture*.

3. The fourfold picture of the calculus of variations.

From either one of these to pictures we can pass to another description of geometrical optics, that of Hamilton and of Euler-Lagrange respectively. The commuting diagram (1.19) tells us how we have to proceed. For instance, let us start from picture I, the Huygens-Lie description. We subject any Lie- or Huygens flow to Hölder's transformation $\mathcal{H}_F$, and replace θ by the new variable z using the equation

$$\dot{z} = \Phi(x, z, p)$$

of (2.8); assumption (A1) yields $\dot{z} \neq 0$ on every flow line whence the substitution can be performed without any difficulty. Then Lie's equations are transformed into Hamilton's equations

$$x' = H_y(x, z, y), \quad y' = -H_x(x, z, y), \tag{3.1}$$

$' = \dfrac{d}{dz}$, i.e. *Lie flows are transformed into Hamilton flows.* Moreover, it is easy to see that Huygens flows and Huygens fields correspond to Mayer flows and Mayer fields respectively, and the eikonal S of a Huygens field is just the eikonal of the corresponding Mayer field, and vice versa. We also see that Vessiot's equation (2.18) is just transformed into the Hamilton-Jacobi equation

$$S_z + H(x, z, S_x) = 0. \tag{3.2}$$

Finally, applying $\mathcal{L}_H$ to (3.1) and (3.2), or $\mathcal{H}_W$ to (2.20) and (2.21), we obtain Euler's equations

$$x' = v, \quad \frac{d}{dz} L_v - L_x = 0 \tag{3.3}$$

and the Carathéodory equations

$$S_x = L_v(x, z, \mathcal{P}), \quad S_z = -\Lambda(x, z, \mathcal{P}) \tag{3.4}$$

for the Mayer field with the eikonal S and the slope $\mathcal{P} := H_y(\cdot, \cdot, S_x)$. The field $f(z, c) = (\mathcal{X}(z, c), z)$ is then obtained from $\mathcal{P}$ by means of the equations

$$\mathcal{X}' = \mathcal{P}(\mathcal{X}, z). \tag{3.5}$$

This leads to four equivalent descriptions of the variational problem

$$\int L(x(z), z, x'(z))dz \to \min \quad (\text{or : stationary}), \tag{3.6}$$

and in particular to the equivalence of the *principles of Fermat and Huygens.*

A detailed account of the material of this lecture will appear in the papers [2], [3] and in the forthcoming treatise [1].

Bibliography

[1] M. Giaquinta and S. Hildebrandt, Calculus of Variations, vol.1. To appear.
[2] S. Hildebrandt, On Hölder's transformation. To appear.
[3] S. Hildebrandt, Contact transformations, Huygens's principle, and calculus of variations. To appear.
[4] E. Hölder, Die infinitesimalen Berührungstransformationen der Variationsrechnung, Jahresberichte der Deutsch. Math.-Ver. 49, 799–819 (1939).