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Construction of Morse flows to a variational functional of harmonic map type

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In this paper we shall construct solutions of the parabolic differential equations associated to a simple variational functional, the Euler-Lagrange equations of which are linear equations.

Let Ω be a bounded domain in $\mathbf{R}^m$, $m \geq 2$, with C^2-boundary $\partial\Omega$. In the following, a map u means the one from Ω to $\mathbf{R}^M$, $M \geq 1$. For a map u belonging to Sobolev space $H^1(\Omega)$, we consider the functional

$$F(u) = \int_\Omega A_{ij}^{\alpha\beta}(x) D_\alpha u^i(x) D_\beta u^j(x)dx, \tag{1}$$

where $u = (u^i), D_\alpha u^i = \partial u^i/\partial x^\alpha, 1 \leq i \leq M, 1 \leq \alpha \leq m$. The summation convention is used. The coefficients $A_{ij}^{\alpha\beta}(x)$, $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$, are assumed to be bounded measurable in Ω and to satisfy the elliptic condition: There exists a positive λ such that

$$A_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j \geq \lambda|\xi|^2 \quad \text{for} \quad \xi = (\xi_\alpha^i) \in \mathbf{R}^{mM} \quad \text{and} \quad x \in \Omega.$$

Hereafter, we use the notation

$$A(x)(Du, Du) = A_{ij}^{\alpha\beta}(x) D_\alpha u^i D_\beta u^j.$$

'Morse flows' of variational functional F are defined as solutions of parabolic partial defferential equations

$$\frac{\partial u^i}{\partial t} = D_\beta(A_{ji}^{\alpha\beta}(x) D_\alpha u^j) \qquad (1 \leq i \leq M). \tag{2}$$

Let u_0 be a given map belonging to $H^1(\Omega)$ and T a positive number. We take a positive integer N and put

$$h = T/N \quad \text{and} \quad t_n = nh \quad (n = 0, 1, \cdots, N). \tag{3}$$

In the following, we use a function space

$$H^1_{u_0}(\Omega) = \{u \in H^1(\Omega); u - u_0 \in H^1_0(\Omega)\},$$

$H^1_0(\Omega)$ being the space obtained by taking the closure of $C^\infty_0(\Omega)$ in the space $H^1(\Omega)$. Beginning with u_0, we inductively construct two sequences of maps u_n and functionals F_n, $1 \leq n \leq N$, as follows: For each n, $1 \leq n \leq N$, we introduce the functional

$$F_n(u) = \int_\Omega \left(A(x)(Du, Du) + \frac{1}{h}|u - u_{n-1}|^2 \right) dx \tag{4}$$

and define u_n as a minimizer of F_n in $H^1_{u_0}(\Omega)$, the existence of which is assured by the lower semi-continuity of F_n with respect to weak convergence of $H^1(\Omega)$. We here remark the Euler-Lagrange equations of F_n in $H^1_{u_0}(\Omega)$ are of the form: For n, $1 \leq n \leq N$,

$$\frac{u^i_n - u^i_{n-1}}{h} = D_\beta(A^{\alpha\beta}_{ji} D_\alpha u^j_n) \qquad (1 \leq i \leq M), \tag{5}$$

which are Rothe's approximate equations of (2). Upon comparing u_{n-1} with a minimizer u_n of F_n, we infer

$$\int_\Omega A(x)(Du_n, Du_n)dx + \int_\Omega \frac{1}{h}|u_n - u_{n-1}|^2 dx \leq \int_\Omega A(x)(Du_{n-1}, Du_{n-1})dx$$

and hence have the following result.

Theorem 0 ([4]). *For $\{u_n\}(1 \leq n \leq N)$ constructed as above, there hold the estimates*

$$\int_\Omega A(x)(Du_n, Du_n)dx \leq \int_\Omega A(x)(Du_0, Du_0)dx \quad \text{for any } n \ (1 \leq n \leq N) \tag{6}$$

and

$$h \sum_{n=1}^{N} \int_\Omega \left| \frac{u_n - u_{n-1}}{h} \right|^2 dx \leq \int_\Omega A(x)(Du_0, Du_0)dx. \tag{7}$$

We define a map $u(t) \in H^1_{u_0}(\Omega), -h \le t \le T$, by means of the identities :

$$u(t) = u_n \qquad \text{for } t_{n-1} < t \le t_n \ (1 \le n \le N)$$

and (8)

$$u(t) = u_0 \qquad \text{for } -h \le t \le 0.$$

We put

$$\partial_t u(t) = \frac{1}{h}(u_n - u_{n-1}) \qquad \text{for } t_{n-1} < t \le t_n \ (1 \le n \le N)$$

and (9)

$$\widetilde{u}(t) = u(t-h) \qquad \text{for } 0 \le t \le T.$$

For the gradient of u constructed as above, we have the estimate of higher integrability. To state the result, we shall prepare the notations as follows. We set

$$Q = (0,T) \times \Omega.$$

For $z_0 = (t_{n_0}, x_0) \in Q, 1 \le n_0 \le N$ and positive s, we put

$$Q_s(z_0) = \{t \in (0,T); t_{n_0} - s^2 < t < t_{n_0}\} \times B_s(x_0),$$

where $B_s(x_0) = \{x \in \Omega; |x - x_0| < s\}$.

Theorem 1. *For the map u defined as in* (8), *there exist positive C and ε not depending on h such that*

$$\begin{aligned}\left(\fint_{Q_{r/2}(z_0)} |Du|^{2+\varepsilon} dz \right)^{1/(2+\varepsilon)} &\le C \left(\fint_{Q_{r/2}(z_0)} |Du|^2 dz \right)^{1/2} \\ &+ ch^{(\overline{p}-1)(m+2)/2m} \left(\fint_{Q_r(z_0)} |\partial_t u|^{(1+\varepsilon/2)\overline{p}} |u - \widetilde{u}|^{(1+\varepsilon/2)(2-\overline{p})} dz \right)^{1/(2+\varepsilon)}\end{aligned} \tag{10}$$

holds for any $Q_r(z_0) \subset Q$ and any $\overline{p}$, $1 < \overline{p} < 2$.

Noting the estimates in Theorem 0 and 1 are vaild uniformly in h, there holds the existence theorem of a weak solution to (2) with the gradient of higher integrability.

By a weak solution to parabolic system(2), we mean a map $u \in L^\infty$ ($(0,\infty)$, $H^1(\Omega)$) $\cap H^1$ ($(0,T)$, $L^2(\Omega)$) such that

$$\int_Q \frac{\partial u^i}{\partial t}\varphi^i dz + \int_Q A_{ij}^{\alpha\beta}(x) D_\alpha u^i D_\beta \varphi^j dz = 0$$

for any $\varphi \in C_0^\infty(Q)$.

Theorem 2. *There exists a weak solution u to (2) satisfying the initial and boundary conditions*:

$$u(t) \in H^1_{u_0}(\Omega) \qquad \text{for almost every } t \in (0,T)$$

and

$$\lim_{t\downarrow 0} u(t) = u_0 \qquad \text{in } L^2(\Omega).$$

The solution u satisfies the estimate :

$$\left(⨍_{Q_{r/2}(z_0)} |Du|^{2+\varepsilon} dz \right)^{1/(2+\varepsilon)} \le C \left(⨍_{Q_r(z_0)} |Du|^2 dz \right)^{1/2}$$

for $Q_r(z_0) \subset Q$, where C and ε are positive numbers as in Theorem 1. *Furthemore, if $A_{ij}^{\alpha\beta}(x)$ are continuous in Ω, u is Hölder continuous in Q with any component $\alpha, 0 < \alpha < 1$.*

The existence proof of solutions follows from the estimates in Theorem 0. Noting the higher integrability (10) of Du and the estimate (7) and paralleling the method developed in [2], it follows from Campanato's fundamental result [1] that Hölder continuity of u is derived. The estimate (10) in Theorem 1 is derived from the following estimate of Caccioppoli type, for the verification of which we have only to follow the method due to Giaquinta-Stuwe([3]).

For positive s satisfying $B_s(x_0) \subset \Omega$ and $u \in L^1(Q)$, we put ([6])

$$u_s = u_s(t) = \int_{B_s(x_0)} \eta(x) u(t,x) dx \qquad \text{for } 0 < t < T, \tag{11}$$

where $\eta(x) = 1$ on $B_{s/2}(x_0)$ and $|D\eta| \le 4/s$.

Lemma (Caccioppoli type estimate). *For the map u defined as in* (8), *there exists a positive C not depending on h such that*

$$\int_{Q_r(z_0)} |Du|^2 dz \leq Cr^{-2} \int_{Q_{2r}(z_0)} |u - u_{2r}|^2 dz$$

$$+Ch^{\overline{p}-1} \int_{Q_{2r}(z_0)} |\partial_t u|^{\overline{p}} |u - \widetilde{u}|^{2-\overline{p}} dz$$

holds for any $Q_{2r}(z_0) \subset Q, z_0 = (t_{n_0}, x_0), 1 \leq n_0 \leq N$ *and for any* $\overline{p}, 1 < \overline{p} < 2$, *where* $|\partial_t u|^{\overline{p}} |u - \widetilde{u}|^{2-\overline{p}}, 1 < \overline{p} < 2$, *belongs to* $L^p(Q)$ *with some* $p, p > 1$, *satisfying* $p \leq m/(m-2+\overline{p})$.

We shall only sketch our proof. Let k and l be positive numbers satisfying $r < k < l < 2r$. As a comparison map in functional F_n, we adopt $v_n, 1 \leq n \leq N$, defined by

$$v_n = u_n - h\eta(u_n - u_{n,l}),$$

where u_n is a minimizer of F_n in $H^1_{u_0}(\Omega)$ and $u_{n,l}$ is defined as in (11).

We make the classification between $(l-k)^2$ and h ([5]):

$$(l-k)^2 \leq 4h, \tag{12}$$

$$(l-k)^2 > 4h. \tag{13}$$

We treat each case of (12) and (13) and follow the iteration procedure ([2]) to obtain each estimate. By adding both the estimates, we arrive at the estimate in Lemma, which is available under no restriction of (12) and (13).

The term $|\partial_t u|^{\overline{p}} |u - \widetilde{u}|^{2-\overline{p}}$ is assured to belong to $L^p(Q)$ with some $p, p > 1$, satisfying $p \leq m/(m-2+\overline{p})$, which is verified to hold from the global estimates (6) and (7).

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