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# Factorization of matrix polynomials and its applications to partial differential operators

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## 1. Introduction

We consider the quadratic form

$$\mathfrak{a}[u] = \int_\Omega \sum_{j,k=1}^{n} \left\langle A^{jk}\partial_{x_k}\boldsymbol{u}, \partial_{x_j}\boldsymbol{u}\right\rangle dx \tag{1.1}$$

on the space $\boldsymbol{H}^1(\Omega) := H^1(\Omega;\mathbb{C}^N)$ of $N$-vector $H^1$ functions $\boldsymbol{u}$ on a domain $\Omega$ of $\mathbb{R}^n$ $(n \geq 2,\ N \geq 1)$ with constant coefficients $A^{jk} \in \mathcal{M}_N$ having the symmetry relations

$$(A^{jk})^* = A^{kj}, \qquad 1 \leq j,k \leq n, \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the hermitian inner product on $\mathbb{C}^N$, $\mathcal{M}_N = \mathcal{M}_{N,N}$ with $\mathcal{M}_{\ell,m}$ the set of complex $\ell\times m$ matrices, and $(A^{jk})^*$ the adjoint of $A^{jk}$. Note that by (1.2) $\mathfrak{a}[\cdot]$ is real-valued. It is fundamental in PDE theory to examine the coercivity or the positivity of quadratic forms such as $\mathfrak{a}[\cdot]$ on $\boldsymbol{H}^1(\Omega)$ or its certain subspaces. Our first aim is to describe, in terms of the coefficients $A^{jk}$, conditions for the following basic inequalities to hold:

$$\mathfrak{a}[\boldsymbol{u}] \geq c_K \|\nabla\boldsymbol{u}\|^2 \qquad \forall\boldsymbol{u} \in \boldsymbol{H}^1(\mathbb{R}^n) \text{ or } \boldsymbol{H}^1(\mathbb{R}^n_+), \tag{1.3}$$

$$\mathfrak{a}[\boldsymbol{u}] \geq c_P \|\boldsymbol{u}\|^2 \qquad \forall\boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega_1) \text{ or } {}^0\boldsymbol{H}^1(\Omega_1) \tag{1.4}$$

$$\mathfrak{a}[\boldsymbol{u}] \geq c_S [\![\boldsymbol{u}]\!]^2 \qquad \forall\boldsymbol{u} \in {}^0\boldsymbol{H}^1(\Omega_1) \tag{1.5}$$

with some *positive* constants $c_K, c_P, c_S$, where $\|\cdot\|$ stands for the $L^2$ norm on the domain considered, $\mathbb{R}^n_+ = \{(x', x_n);\ x_n > 0\}$ the upper half space, $\Omega_1 = \{(x', x_n);\ 0 < x_n < 1\}$ a slab in $\mathbb{R}^n$, and $[\![\boldsymbol{u}]\!]^2 = \int_{\mathbb{R}^{n-1}} |\boldsymbol{u}(x',0)|^2 dx'$ for $\boldsymbol{u} \in {}^0\boldsymbol{H}^1(\Omega_1) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega_1); \boldsymbol{v}|_{x_n=1} = \boldsymbol{0}\}$.

Our strategy for attacking the problem mentioned above is *symmetric factorization of positive-semidefinite matrix-polynomials*: it is known (see Jakubovič [5], Gohberg, Lancaster & Rodmann [3], [4]) that, if a matrix polynomial $H(\tau)$ in the form

$$(1.6)\qquad H(\tau) = H_2\tau^2 + H_1\tau + H_0 \quad \text{with } H_2, H_1, H_0 \in \mathcal{M}_N \text{ hermitian and } H_2 > O$$

satisfies $H(\alpha) \geq O$ for all $\alpha \in \mathbb{R}$, then there exists a matrix $\Lambda \in \mathcal{M}_N$ such that

$$(1.7)\qquad H(\tau) = (I\tau - \Lambda^*)\, H_2\, (I\tau - \Lambda) \qquad \text{as polynomial in } \tau.$$

For the purpose of application to the inequalities above (and to more general PDE's of second order) we investigate some properties, suitable for that use, of such a $\Lambda$ as satisfies (1.7). This is our second aim in the present paper.

The following §2 will be devoted to factorization of matrix polynomials of form (1.6). In §3, using the technique developed in §2, we shall reveal to some extent the relationship between the coefficients $A^{jk}$ of (1.1) and the validity of inequlities (1.3)–(1.5). If $n = 2$ or if $N = 1$ in particular, our method will be so powerful that we can obtain satisfactory results. The results in §3 will be roughly proved in §4. Finally, in §5 we shall apply results obtained in §3 to the following two practical cases:

Example 1: the scalar case.

$$(1.8)\qquad \mathfrak{a}[u] = \int_{\Omega_1} \sum_{j,k=1}^{n} A^{jk}\partial_{x_k} u\, \overline{\partial_{x_j} u}\, dx \quad \text{for } i \in H^1(\Omega_1) \quad (N = 1).$$

Example 2: the case of linear isotropic elasticity:

$$(1.9)\qquad \mathfrak{a}[\boldsymbol{u}] = \lambda\, \|\mathrm{div}\, \boldsymbol{u}\|^2 + 2\mu\, \|\varepsilon(\boldsymbol{u})\|^2 \quad \text{for } \boldsymbol{u} \in H^1(\Omega_1; \mathbb{C}^n) \quad (n = N \geq 2),$$

where $\lambda, \mu \in \mathbb{R}$ are the Lamé moduli and $\varepsilon(\boldsymbol{u})$ denotes the symmetric part of $\nabla \boldsymbol{u}$: $\varepsilon(\boldsymbol{u}) = 2^{-1}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right)$.

## 2. Factorization of matrix polynomials

Let $H(\tau)$ be as in (1.6). It is known (for details see [2, Chap. VI], [4, Chap. S1], etc.) that, by applying a sequence of *elementary transformations*, $H(\tau)$ can be reduced to a diagonal polynomial matrix

$$D(\tau) = \begin{pmatrix} d_1(\tau) & & & \\ & d_2(\tau) & & \\ & & \ddots & \\ & & & d_N(\tau) \end{pmatrix} \tag{2.1}$$

with monic (i.e., the leading coefficient is 1) scalar polynomials $d_j(\tau)$ such that $d_j(\tau)$ is divisible by $d_{j-1}(\tau)$; in other words there are matrix polynomials $P(\tau)$ and $Q(\tau)$ with *constant nonzero determinants* such that

$$D(\tau) = P(\tau)\,H(\tau)\,Q(\tau). \tag{2.2}$$

While the invertible matrix polynomials $P(\tau)$ and $Q(\tau)$ in (2.2) are not uniquely determined, the scalar polynomials $d_1(\tau), d_2(\tau), \dots, d_N(\tau)$ are uniquely defined. Indeed, denoting by $g_j(\tau)$ the G.C.D. of the minors of $H(\tau)$ of order $j$, we have

$$d_1(\tau) = g_1(\tau) \quad \text{and} \quad d_j(\tau) = g_j(\tau)/g_{j-1}(\tau) \quad \text{for } 2 \le j \le N.$$

We call $D(\tau)$ the *Smith form* of $H(\tau)$ and $d_1(\tau), d_2(\tau), \dots, d_N(\tau)$ the *invariant polynomials* of $H(\tau)$. Note that the invariant polynomials of $H(\tau)$ considered here are with real coefficients. Now we represent each of them as a product of irreducible factors over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$:

$$d_j(\tau) = \phi_{j1}(\tau)^{p_{j1}}\phi_{j2}(\tau)^{p_{j2}} \cdots \phi_{j,k_j}(\tau)^{p_{j,k_j}}, \quad 1 \le j \le N,$$

where $\phi_{j1}(\tau), \dots, \phi_{j,k_j}(\tau)$ are all the distinct factors of $d_j(\tau)$ over $\mathbb{K}$, and $p_{j1}, \dots, p_{j,k_j} \in \mathbb{N}$ (positive integers). The factors $\phi_{jk}(\tau)^{p_{jk}}$ $(1 \le k \le k_j, 1 \le j \le N)$ are called the *elementary divisors* of $H(\tau)$ if $\mathbb{K} = \mathbb{C}$, and the *real elementary divisors* if $\mathbb{K} = \mathbb{R}$.

**Theorem 2.1.** (cf. Gohberg et al. [3], [4]) *For a matrix polynomial $H(\tau)$ of form* (1.6), *the following three conditions are equivalent:*

(i) $H(\tau) \geq O$ *for all* $\tau \in \mathbb{R}$.

(ii) *There exists a matrix* $\Lambda \in \mathcal{M}_N$ *which satisfies* (1.7).

(iii) *The real elementary divisors of* $H(\tau)$ *have the forms*

$$\{(\tau - \overline{\alpha_\ell})(\tau - \alpha_\ell)\}^{p_\ell}, \qquad 1 \leq \ell \leq L \tag{2.3}$$

*with* $\alpha_\ell \in \mathbb{C}$ *and* $p_\ell \in \mathbb{N}$ *satisfying* $\sum_{\ell=1}^{L} p_\ell = N$.

*Furthermore, under condition* (iii), *we can construct a* $\Lambda \in \mathcal{M}_N$ *which satisfies* (1.7) *and whose Jordan form is*

$$\bigoplus_{\ell=1}^{L} J(\alpha_\ell, p_\ell) = \begin{pmatrix} J(\alpha_1, p_1) & & & \\ & J(\alpha_2, p_2) & & \\ & & \ddots & \\ & & & J(\alpha_L, p_L) \end{pmatrix},$$

*where* $J(\alpha, p)$ *denotes a Jordan block of size* $p$ *and eigenvalue* $\alpha$.

For convenience of explanation we define three sets $\mathfrak{X}, \mathfrak{X}_0, \mathfrak{X}_1$ of matrix polynomials by

$$\mathfrak{X} = \{H(\tau) = H_2\tau^2 + H_1\tau + H_0;\ H_2, H_1, H_0 \in \mathcal{M}_N \text{ and hermitian}\},$$

$$\mathfrak{X}_0 = \{H(\tau) \in \mathfrak{X};\ H(\alpha) \geq O\ \forall \alpha \in \mathbb{R} \text{ and } H_2 > O\},$$

$$\mathfrak{X}_1 = \{H(\tau) \in \mathfrak{X};\ H(\alpha) > O\ \forall \alpha \in \mathbb{R}\} \subset \mathfrak{X}_0 \subset \mathfrak{X}.$$

We can identify $\mathfrak{X}$ with the set of triplets $(H_2, H_1, H_0)$ of hermitian matrices of order $N$, so with $\left(\mathbb{R}^N \times \mathbb{C}^{N(N-1)/2}\right)^3 \cong \mathbb{R}^{3N^2}$. Given $H(\tau) \in \mathfrak{X}_0$ we are interested in a matrix $\Lambda$ which satisfies (1.7) and all whose eigenvalues are included in the closed upper half $\overline{\mathbb{C}_+} =$

$\{z \in \mathbb{C}; \operatorname{Im} z \geq 0\}$ of the compex plane $\mathbb{C}$. The last assertion of Theorem 2.1 guarantees the existence of such a $\Lambda$, and the following proposition shows an explicit construction of it. Later, this $\Lambda$ will prove to be unique (Theorem 2.4).

**Proposition 2.2.** *Let $H(\tau) \in \mathfrak{X}_0$. First, by means of elementary transformations, represent $H(\tau)$ in form (2.2). Second, express the real elementary divisors of $H(\tau)$ as the form (2.3) but with $\alpha_\ell \in \overline{\mathbb{C}_+}$ for all $1 \leq \ell \leq L$. For each $\ell$ let $e^{(\ell)}$ be the unit vector in $\mathbb{C}^N$ such that the $j_\ell$-th element is 1 and the others are 0, where $j_\ell$ is the index such that the elementary divisor $\{(\tau - \overline{\alpha_\ell})(\tau - \alpha_\ell)\}^{p_\ell}$ is chosen out of $d_{j_\ell}(\tau)$. Finally, using $Q(\tau)$ and $e^{(\ell)}$ given above, define a matrix $R$ by*

$$\text{(2.4)} \qquad \begin{cases} \boldsymbol{r}_k^{(\ell)} = \frac{1}{(k-1)!} Q^{(k-1)}(\alpha_\ell)\, e^{(\ell)} \in \mathbb{C}^N, & 1 \leq k \leq p_\ell, \\ R^{(\ell)} = \left(\boldsymbol{r}_1^{(\ell)}\ \boldsymbol{r}_2^{(\ell)}\ \cdots\ \boldsymbol{r}_{p_\ell}^{(\ell)}\right) \in \mathcal{M}_{N,p_\ell}, & 1 \leq \ell \leq L, \\ R = \left(R^{(1)}\ R^{(2)}\ \cdots\ R^{(L)}\right) \in \mathcal{M}_N, \end{cases}$$

*where $Q^{(k-1)}(\alpha_\ell) = \left(\frac{d}{d\tau}\right)^{k-1} Q(\tau)|_{\tau=\alpha_\ell}$. Then $R$ is nonsingular. Furthermore the matrix*

$$\Lambda = R\left(\bigoplus_{\ell=1}^{L} J(\alpha_\ell, p_\ell)\right) R^{-1}.$$

*satisfies (1.7) and its spectrum $\sigma(\Lambda)$ is included in $\overline{\mathbb{C}_+}$.*

**Proposition 2.3.** *Let $H(\tau) \in \mathfrak{X}_0$ and let $\Lambda$ be a matrix such that (1.7) is valid and $\sigma(\Lambda) \subset \overline{\mathbb{C}_+}$. Define a subspace $\boldsymbol{V}_1$ of $\mathbb{C}^N$ by*

$$\text{(2.5)} \qquad \boldsymbol{V}_1 = \bigoplus_{\alpha \in \sigma(\Lambda) \cap \mathbb{C}_+} \ker (\alpha I - \Lambda)^N.$$

*Then we have the following:*

(i) *Given $\boldsymbol{d} \in \mathbb{C}^N$ the boundary-value problem*

$$\begin{cases} H(D_t)\boldsymbol{u}(t) = \boldsymbol{0} & \text{for } 0 < t < 1, \\ \boldsymbol{u}(0) = \boldsymbol{d}, \quad \boldsymbol{u}(1) = \boldsymbol{0} & \left(D_t = \frac{1}{i}\frac{d}{dt}\right) \end{cases}$$

*admits a unique solution* $\boldsymbol{u}$*, which is given by*

$$\boldsymbol{u}(t) = e^{it\Lambda}\left\{I - \left(\int_0^t e^{-is\Lambda}H_2^{-1}e^{is\Lambda^*}ds\right)\left(\int_0^1 e^{-is\Lambda}H_2^{-1}e^{is\Lambda^*}ds\right)^{-1}\right\}\boldsymbol{d}.$$

(ii) *Given* $\boldsymbol{d} \in \boldsymbol{V}_1$ *the boundary-value problem*

$$\begin{cases} H(D_t)\boldsymbol{u}(t) = \boldsymbol{0} & \text{for } t > 0, \\ \boldsymbol{u}(0) = \boldsymbol{d}, \quad \boldsymbol{u}(t) \to \boldsymbol{0} & \text{as } t \to \infty \end{cases}$$

*admits a unique solution* $\boldsymbol{u}$*, which is given by*

$$\boldsymbol{u}(t) = e^{it\Lambda}\boldsymbol{d}.$$

*Conversely if the problem has a solution* $\boldsymbol{u}$*, then there must be* $\boldsymbol{d} \in \boldsymbol{V}_1$.

**Theorem 2.4.** *Let* $H = H(\tau) \in \mathfrak{X}_0$*. Then there exists a unique* $\Lambda \in \mathcal{M}_N$ *such that* (1.7) *holds and the spectrum* $\sigma(\Lambda)$ *of* $\Lambda$ *is included in* $\overline{\mathbb{C}_+} = \{\operatorname{Im} z \geq 0\}$*. Furthermore, the mapping* $\lambda : \mathfrak{X}_0 \to \mathcal{M}_N$ *defined by* $\lambda(H) = \Lambda$ *is continuous on* $\mathfrak{X}_0$ *and real-analytic in* $\mathfrak{X}_1$*. If* $H \in \mathfrak{X}_1$ *in particular,* $\lambda(H)$ *is represented as*

$$\lambda(H) = \left(\int_\Gamma H(z)^{-1}dz\right)^{-1}\left(\int_\Gamma zH(z)^{-1}dz\right),$$

*where* $\Gamma$ *is a contour in* $\mathbb{C}_+ = \{\operatorname{Im} z > 0\}$ *with* $\sigma(\Lambda)$ *inside.*

Given $H = H(\tau) \in \mathfrak{X}_0$ let $\Lambda = \lambda(H)$. We define two subspaces $\boldsymbol{V}_0(H), \boldsymbol{V}_1(H)$ of $\mathbb{C}^N$ by

$$\boldsymbol{V}_0(H) = \bigoplus_{\alpha\in\sigma(\Lambda)\cap\mathbb{R}} \ker(\alpha I - \Lambda)^N, \quad \boldsymbol{V}_1(H) = \bigoplus_{\alpha\in\sigma(\Lambda)\cap\mathbb{C}_+} \ker(\alpha I - \Lambda)^N; \tag{2.6}$$

$\ker(\alpha I - \Lambda)^N$ is so-called the *generalized eigenspace* of $\Lambda$ corresponding to $\alpha \in \sigma(\Lambda)$. With the notation (2.4) used in Proposition 2.2, these spaces are expressed more explicitly as

$$\boldsymbol{V}_0(H) = \bigoplus_{\ell;\,\alpha_\ell\in\mathbb{R}} S[\boldsymbol{r}_1^{(\ell)}, \boldsymbol{r}_2^{(\ell)}, \ldots, \boldsymbol{r}_{p_\ell}^{(\ell)}], \quad \boldsymbol{V}_1(H) = \bigoplus_{\ell;\,\alpha_\ell\in\mathbb{C}_+} S[\boldsymbol{r}_1^{(\ell)}, \boldsymbol{r}_2^{(\ell)}, \ldots, \boldsymbol{r}_{p_\ell}^{(\ell)}], \tag{2.7}$$

where $S[\boldsymbol{r}_1^{(\ell)}, \dots, \boldsymbol{r}_{p_\ell}^{(\ell)}]$ denotes the linear subspace of $\mathbb{C}^N$ generated by $\boldsymbol{r}_1^{(\ell)}, \dots, \boldsymbol{r}_{p_\ell}^{(\ell)}$. We note that $\boldsymbol{V}_0(H) \oplus \boldsymbol{V}_1(H) = \mathbb{C}^N$, and that if $H \in \mathfrak{X}_1$ then $\boldsymbol{V}_0(H) = \{\boldsymbol{0}\}$ and $\boldsymbol{V}_1(H) = \mathbb{C}^N$. Further note that, by Theorem 2.3, given $H \in \mathfrak{X}_0$ the $\boldsymbol{V}_1$ in (2.5) and the $\boldsymbol{V}_1(H)$ in (2.6) are the same.

Now we write $\check{H}(\tau) = H(-\tau)$ for $H \in \mathfrak{X}_0$; note that $H \in \mathfrak{X}_0$ if and only if $\check{H} \in \mathfrak{X}_0$. The following theorem explains how the positive semidefiniteness of $H(\tau)$ is reflected in $\lambda(H)$ (and $\lambda(\check{H})$).

**Theorem 2.5.** *Let $H \in \mathfrak{X}_0$. Then the matrix*

$$K = (2i)^{-1} H_2 \left(\lambda(H) + \lambda(\check{H})\right)$$

*is hermitian and satisfies*

$$\text{(i)} \quad K \geq O, \qquad\qquad \text{(ii)} \quad \ker K = \boldsymbol{V}_0(H).$$

*If $H_2, H_1, H_0$ are real matrices in addition, then $\lambda(\check{H}) = -\overline{\lambda(H)}$, so that $K = H_2 \operatorname{Im} \lambda(H)$, where $\operatorname{Im} \Lambda$ denotes the imaginary part of $\Lambda$ : $\operatorname{Im} \Lambda = (\Lambda - \bar{\Lambda})/(2i)$.*

We give a proof only to Theorem 2.5 because it is the most important in this section. For convenience of explanation we write $\Lambda_\pm = \lambda(H(\pm\tau))$ and set for $\varepsilon > 0$

$$\begin{aligned} H_0^\varepsilon &= H_0 + \varepsilon I, \qquad & H^\varepsilon(\tau) &= H(\tau) + \varepsilon I, \\ \Lambda_\pm^\varepsilon &= \lambda(H^\varepsilon(\pm\tau)), \qquad & K^\varepsilon &= (2i)^{-1} H_2 (\Lambda_+^\varepsilon + \Lambda_-^\varepsilon). \end{aligned}$$

Since $H^\varepsilon(\pm\tau) \in \mathfrak{X}_1$, we see from Theorem 2.4 that, given $\boldsymbol{d} \in \mathbb{C}^N$, $\boldsymbol{u}_\pm^\varepsilon(t) := e^{it\Lambda_\pm^\varepsilon} \boldsymbol{d}$ are the unique solutions of the problems

$$\begin{cases} H^\varepsilon(\pm D_t) \boldsymbol{u}_\pm^\varepsilon(t) = \boldsymbol{0} \quad \text{for } t > 0, \\ \boldsymbol{u}_\pm^\varepsilon(0) = \boldsymbol{d}, \quad \boldsymbol{u}_\pm^\varepsilon(1) = \boldsymbol{0}. \end{cases}$$

Now define quadratic forms $\mathfrak{h}^\varepsilon_\pm[\boldsymbol{u}]_I$ on $\boldsymbol{H}^1(I)$, $I$ being an open interval in $\mathbb{R}$, by

$$\mathfrak{h}^\varepsilon_\pm[\boldsymbol{u}]_I = (H_2\boldsymbol{u}', \boldsymbol{u}')_{L^2(I)} \pm \mathrm{Im}\,(H_1\boldsymbol{u}', \boldsymbol{u})_{L^2(I)} + (H^\varepsilon_0\boldsymbol{u}, \boldsymbol{u})_{L^2(I)} \quad \text{for } \boldsymbol{u} \in \boldsymbol{H}^1(I),$$

where $\boldsymbol{u}' = d\boldsymbol{u}/dt$. If we set further

$$G(\tau) = -i(H_2\tau + 2^{-1}H_1),$$

integration by parts gives

$$\mathfrak{h}^\varepsilon_\pm[\boldsymbol{u}^\varepsilon_\pm]_{\mathbb{R}_+} = \big\langle \big(\pm G(\pm D_t)\boldsymbol{u}^\varepsilon_\pm\big)(0), \boldsymbol{u}^\varepsilon_\pm(0)\big\rangle = \big\langle -i(H_2\Lambda^\varepsilon_\pm \pm 2^{-1}H_1)\boldsymbol{d}, \boldsymbol{d}\big\rangle \tag{2.8}$$

The function $\boldsymbol{v}^\varepsilon(t)$ defined by

$$\boldsymbol{v}^\varepsilon(t) = \boldsymbol{u}^\varepsilon_+(t) \quad \text{for } t \geq 0, \quad = \boldsymbol{u}^\varepsilon_-(-t) \quad \text{for } t < 0$$

is in $\boldsymbol{H}^1(\mathbb{R})$ and satisfies $\mathfrak{h}^\varepsilon_+[\boldsymbol{v}^\varepsilon]_{\mathbb{R}} = \mathfrak{h}^\varepsilon_+[\boldsymbol{u}^\varepsilon_+]_{\mathbb{R}_+} + \mathfrak{h}^\varepsilon_-[\boldsymbol{u}^\varepsilon_-]_{\mathbb{R}_+}$. Furthermore we have by the Fourier transformation

$$\mathfrak{h}^\varepsilon_+[\boldsymbol{v}^\varepsilon]_{\mathbb{R}} = \int_{\mathbb{R}} \langle H^\varepsilon(\tau)\hat{\boldsymbol{v}}^\varepsilon(\tau), \hat{\boldsymbol{v}}^\varepsilon(\tau)\rangle\, d\tau \geq 0 \tag{2.9}$$

Thus it follows from (2.8) and (2.9) that

$$2\,\langle K^\varepsilon\boldsymbol{d}, \boldsymbol{d}\rangle = \mathfrak{h}^\varepsilon_+[\boldsymbol{u}^\varepsilon_+]_{\mathbb{R}_+} + \mathfrak{h}^\varepsilon_-[\boldsymbol{u}^\varepsilon_-]_{\mathbb{R}_+} = \mathfrak{h}^\varepsilon_+[\boldsymbol{v}^\varepsilon]_{\mathbb{R}} \geq 0 \quad \forall \boldsymbol{d} \in \mathbb{C}^N,$$

where we tend $\varepsilon \to 0$ to obtain $\langle K\boldsymbol{d}, \boldsymbol{d}\rangle \geq 0$ for all $\boldsymbol{d} \in \mathbb{C}^N$. Since this implies that $K$ satisfies (i) above, our remaining task is to show $\ker K = \boldsymbol{V}_0(H)$. For simplicity we set

$$\boldsymbol{V}^\pm_0 = \boldsymbol{V}_0(H(\pm\tau)), \quad \boldsymbol{V}^\pm_1 = \boldsymbol{V}_1(H(\pm\tau)).$$

We begin with showing $\boldsymbol{V}^\pm_0 \subset \ker K$. In order to find the explicit forms, such as in (2.6), of $\boldsymbol{V}^\pm_0$, choose invertible matrix polynomials $P(\tau)$ and $Q(\tau)$ such that $P(\tau)H(\tau)Q(\tau)$

is the Smith form of $H(\tau)$. Then $\check{P}(\tau)\check{H}(\tau)\check{Q}(\tau)$ is the Smith form of $\check{H}(\tau) = H(-\tau)$. If the real elementary divisors of $H(\tau)$ are given by

$$\{(\tau - \overline{\alpha_\ell})(\tau - \alpha_\ell)\}^{p_\ell}, \qquad 1 \le \ell \le L, \tag{2.10}$$

then those of $\check{H}(\tau)$ are given by

$$\{(\tau + \alpha_\ell)(\tau + \overline{\alpha_\ell})\}^{p_\ell}, \qquad 1 \le \ell \le L. \tag{2.11}$$

For the elementary divisors (2.10) of $H(\tau)$ and (2.11) of $\check{H}(\tau)$ we set respectively

$$\boldsymbol{r}_k^{(\ell)} = \frac{1}{(k-1)!} Q^{(k-1)}(\alpha_\ell) \boldsymbol{e}^{(\ell)}, \qquad \check{\boldsymbol{r}}_k^{(\ell)} = \frac{1}{(k-1)!} \check{Q}^{(k-1)}(-\overline{\alpha_\ell}) \boldsymbol{e}^{(\ell)}$$

for $1 \le k \le p_\ell$ and $1 \le \ell \le L$. If $\alpha_\ell \in \mathbb{R}$ in particular, then $\check{\boldsymbol{r}}_k^{(\ell)} = (-1)^{k-1} \boldsymbol{r}_k^{(\ell)}$, so that

$$\boldsymbol{V}_0^+ = \boldsymbol{V}_0^- = \bigoplus_{\ell;\, \alpha_\ell \in \mathbb{R}} S[\boldsymbol{r}_1^{(\ell)}, \boldsymbol{r}_2^{(\ell)}, \dots, \boldsymbol{r}_{p_\ell}^{(\ell)}]$$

.Moreover, for such $\ell$,

$$\begin{aligned} 2\left\langle K\boldsymbol{r}_k^{(\ell)}, \boldsymbol{r}_k^{(\ell)} \right\rangle &= \operatorname{Im}\left( \left\langle H_2\Lambda_+ \boldsymbol{r}_k^{(\ell)}, \boldsymbol{r}_k^{(\ell)} \right\rangle + \left\langle H_2 \Lambda_- \boldsymbol{r}_k^{(\ell)}, \boldsymbol{r}_k^{(\ell)} \right\rangle \right) \\ &= \operatorname{Im}\left( \left\langle H_2(\lambda_\ell \boldsymbol{r}_k^{(\ell)} + \boldsymbol{r}_{k-1}^{(\ell)}), \boldsymbol{r}_k^{(\ell)} \right\rangle + \left\langle H_2(-\lambda_\ell \check{\boldsymbol{r}}_k^{(\ell)} + \check{\boldsymbol{r}}_{k-1}^{(\ell)}), \check{\boldsymbol{r}}_k^{(\ell)} \right\rangle \right) = 0, \end{aligned}$$

where we have used the notation $\boldsymbol{r}_0^{(\ell)} = \boldsymbol{0}$. Hence by (i) we obtain $K\boldsymbol{r}_k^{(\ell)} = \boldsymbol{0}$ for all $\ell$ such that $\alpha_\ell \in \mathbb{R}$.

Finally, suppose that $\boldsymbol{V}_0^+ \; (= \boldsymbol{V}_0^-) \ne \ker K$. Let $\beta_1, \beta_2, \dots, \beta_M \in \mathbb{C}_+$ be the distinct imaginary eigenvalues of $\Lambda_-$. Then we have

$$\boldsymbol{V}_1^- = \bigoplus_{1 \le m \le M} \boldsymbol{V}^-(\beta_m) \quad \text{with } \boldsymbol{V}^-(\beta_m) = \ker(\beta_m I - \Lambda_-)^N. \tag{2.12}$$

Since $\mathbb{C}^N = \boldsymbol{V}_0^- \oplus \boldsymbol{V}_1^-$ and $\boldsymbol{V}_0^- \subset \ker K$, the supposition above implies that there is a nontrivial $\boldsymbol{r} \in \ker K \cap \boldsymbol{V}_1^-$. We decompose this $\boldsymbol{r}$ to the sum corresponding to (2.12):

$$\boldsymbol{r} = \sum_{1 \le m \le M} \boldsymbol{r}_m \quad \text{with } \boldsymbol{r}_m \in \boldsymbol{V}^-(\beta_m).$$

Choose $m_1$ such that $\boldsymbol{r}_{m_1} \neq \mathbf{0}$ and $p_1$ such that

$$(\beta_{m_1} I - \Lambda_-)^{p_1} \boldsymbol{r}_{m_1} \neq \mathbf{0}, \qquad (\beta_{m_1} I - \Lambda_-)^{p_1+1} \boldsymbol{r}_{m_1} = \mathbf{0},$$

and define $\tilde{\boldsymbol{r}} \neq \mathbf{0}$ by

$$\begin{aligned} \tilde{\boldsymbol{r}} &= \left( \prod_{1 \le m \le M, m \neq m_1} (\beta_k I - \Lambda_-)^N \right) (\beta_{m_1} I - \Lambda_-)^{p_1} \boldsymbol{r} \\ &= \left( \prod_{1 \le m \le M, m \neq m_1} (\beta_k I - \Lambda_-)^N \right) (\beta_{m_1} I - \Lambda_-)^{p_1} \boldsymbol{r}_{m_1}. \end{aligned}$$

Then it follows that $\Lambda_- \tilde{\boldsymbol{r}} = \beta_{m_1} \tilde{\boldsymbol{r}}$. On the other hand, since $\boldsymbol{r} \in \ker K = \ker(\Lambda_+ + \Lambda_-)$, we have $\tilde{\boldsymbol{r}} \in \ker(\Lambda_+ + \Lambda_-)$. Hence we arrive at a contradiction that $\Lambda_+ \tilde{\boldsymbol{r}} = -\Lambda_- \tilde{\boldsymbol{r}} = -\beta_{m_1} \tilde{\boldsymbol{r}}$. Therefore $\boldsymbol{V}_0^+ \ (= \boldsymbol{V}_0^-) = \ker K$.

## 3. Positivity of the quadratic form $\mathfrak{a}[\cdot]$

The quadratic form $\mathfrak{a}[\cdot]$ of (1.1) with $\Omega = \mathbb{R}^n_+$ determines differential operaters $\mathcal{A} = A(D)$ in $\mathbb{R}^n_+$ and $\mathcal{B} = B(D)$ on $\partial \mathbb{R}^n_+$ whose symbols are respectively

$$A(\xi) = \sum_{j,k=1}^{n} A^{jk} \xi_j \xi_k, \quad B(\xi) = -i \sum_{k=1}^{n} A^{nk} \xi_k \quad \text{for } \xi \in \mathbb{R}^n.$$

Indeed, integration by parts gives

$$\mathfrak{a}[\boldsymbol{u}] = \int_{\mathbb{R}^n_+} \langle \mathcal{A}\boldsymbol{u}, \boldsymbol{u} \rangle \, dx + \int_{\partial \mathbb{R}^n_+} \langle \mathcal{B}\boldsymbol{u}, \boldsymbol{u} \rangle \, dx' \qquad \forall \boldsymbol{u} \in \boldsymbol{H}^2(\mathbb{R}^n_+).$$

Let $A(\xi) \geq O$ for all $\xi$ and $A^{nn} = A(0, \dots, 0, 1) > O$. Then, by Theorem 2.2, for each $\eta \in \mathbb{R}^{n-1}$ there exists a $\Lambda(\eta) \in \mathcal{M}_N$ such that $\sigma(\Lambda(\eta)) \subset \overline{\mathbb{C}_+}$ and

$$A(\eta, \tau) = \big(I\tau - \Lambda(\eta)^*\big) A^{nn} \big(I\tau - \Lambda(\eta)\big) \quad \text{as polynomial of } \tau.$$

By means of this $\Lambda(\eta)$ we define for each $\eta \in \mathbb{R}^{n-1}$

$$T(\eta) = -i \Big( \sum_{k=1}^{n-1} A^{nk} \eta_k + A^{nn} \Lambda(\eta) \Big), \tag{3.1}$$

$$\boldsymbol{V}_0(\eta) = \bigoplus_{\alpha \in \sigma(\Lambda(\eta)) \cap \mathbb{R}} \ker \big(\alpha I - \Lambda(\eta)\big)^N \quad (= \boldsymbol{V}_0(A(\eta, \tau))).$$

Note that $T(\eta)$ is a hermitian matrix-valued continous function of $\eta \in \mathbb{R}^{n-1}$ which is positively homogeneous in $\eta$ of degree 1 (i.e., $T(r\eta) = rT(\eta)\ \forall r > 0$).

Let $\mathcal{A}$ be a strongly elliptic system, that is, let $A(\xi) > O$ for all $\xi \in \mathbb{S}^{n-1} := \{|\xi| = 1\}$. Then, given $\boldsymbol{\phi} \in \boldsymbol{H}^{1/2}(\partial\mathbb{R}^n_+)$, the Dirichlet problem

$$(3.2) \qquad \mathcal{A}\boldsymbol{u} = \boldsymbol{0} \quad \text{in } \mathbb{R}^n_+, \qquad \boldsymbol{u} = \boldsymbol{\phi} \quad \text{on } \partial\mathbb{R}^n_+$$

has a unique $H^2$ solution $\boldsymbol{u}$. We define a mapping $\mathcal{T}\colon \boldsymbol{H}^{1/2}(\partial\mathbb{R}^n_+) \to \boldsymbol{H}^{3/2}(\partial\mathbb{R}^n_+)$ by

$$\mathcal{T}\boldsymbol{\phi} = \mathcal{B}\boldsymbol{u}\big|_{x_n=0}$$

using the unique $H^2$ solution $\boldsymbol{u}$ of (3.2); $\mathcal{T}$ is called the *Dirichlet-to-Neumann map* for $\{\mathcal{A}, \mathcal{B}\}$.

**Proposition 3.1.** *Let $\mathcal{A}$ be a strongly elliptic system. Then the symbol of the Dirichlet-to-Neumann map $\mathcal{T}$ for $\{\mathcal{A}, \mathcal{B}\}$ is equal to the $T(\eta)$ defined in* (3.1). *Moreover $T(\eta)$ is real-analytic in $\eta \in \mathbb{S}^{n-2}$.*

INEQUALITY (1.3).

It is well-known under what condition on the coefficients $A^{jk}$ inequality (1.3) holds.

**Theorem 3.2.** *The following is a necessary and sufficient condition for inequality* (1.3) *to hold for some constant $c_K > 0$ :*

*Case* 1 : *on the space $\boldsymbol{H}^1(\mathbb{R}^n)$.* $\quad A(\xi) > O\ \forall \xi \in \mathbb{S}^{n-1}$.

*Case* 2 : *on the space $\boldsymbol{H}^1(\mathbb{R}^n_+)$.* $\quad A(\xi) > O\ \forall \xi \in \mathbb{S}^{n-1}$ *and* $T(\eta) > O\ \forall \eta \in \mathbb{S}^{n-2}$.

The symbol $A(\xi)$ depends only on the hermitian parts $\mathfrak{H}A^{jk} := (A^{jk} + (A^{jk})^*)/2$ of the coefficient matrices $A^{jk}$; note that any $A \in \mathcal{M}_N$ can be decomposed as $A = \mathfrak{H}A + i\,\mathfrak{S}A$ with $\mathfrak{S}A := (A - A^*)/(2i)$ the skew-hermitian part of $A$. Thus the following question is naturally raised: Can we change the skew-hermitian parts $\mathfrak{S}A^{jk}$ of the coefficients $A^{jk}$ so that $T(\eta) > O$ for all $\eta \in \mathbb{S}^{n-2}$ if $\mathcal{A}$ is strongly elliptic?

**Theorem 3.3.** *Let $\mathcal{A}$ be strongly elliptic. Then we can change the coefficients $A^{jk}$ with $A(\xi)$ left unchanged so that $T(\eta) > O$ for all $\eta \in \mathbb{S}^{n-2}$ if $n = 2$ or if $N = 1$.*

INEQUALITY (1.4) ON $\boldsymbol{H}_0^1(\Omega_1)$.

**Theorem 3.4.** *For inequality (1.4) to hold on $\boldsymbol{H}_0^1(\Omega_1)$ with some constant $c_P > 0$, it is necessary that $A(\xi)$ satisfies the following conditions:*

(i) $A(\xi) \geq O \ \forall \xi \in \mathbb{S}^{n-1}$; (ii) $A^{nn} > O$;

(iii) *For every $\eta \in \mathbb{S}^{n-2}$ the matrix polynomial $A(\eta, \tau)$ in $\tau$ has no elementary divisors in the form $(\tau - \alpha)^{2p}$ with $\alpha \in \mathbb{R}$ and $p \geq 2$.*

*Furthermore, if $n = 2$ or if $N \leq 2$, then it is also sufficient.*

**Corollary 3.5.** *Assume $n = 2$ or $N = 1$. Let $\Omega$ be an arbitrary bounded domain in $\mathbb{R}^n$. Then, inequality (1.4) holds on $\boldsymbol{H}_0^1(\Omega)$ with some constant $c_P > 0$ if and only if the following conditions are satisfied:*

(i) $A(\xi) \geq O \ \forall \xi \in \mathbb{S}^{n-1}$;

(ii)′ *There is a $\xi_0 \in \mathbb{S}^{n-1}$ such that $A(\xi_0) > O$;*

(iii)′ *For some (or any) $\xi_0$ satisfying (ii)′ above, if we rotate the coordinate axes so that $\xi_0$ is on the $x_n$-axis, the transformed $A(\xi)$ satisfies condition (iii) in Theorem 3.4.*

INEQUALITIES (1.4) AND (1.5) ON ${}^0\boldsymbol{H}^1(\Omega_1)$.

**Theorem 3.6.** *For inequality* (1.4) *to hold on* ${}^0\boldsymbol{H}^1(\Omega_1)$ *with some constant* $c_P > 0$, *it is necessary that* $A(\xi)$ *and* $T(\eta)$ *satisfy the following conditions*:

(i) – (iii) : *as in* Theorem 3.4;

(iv) $T(\eta) \geq O \ \forall \eta \in \mathbb{S}^{n-2}$; (v) $\operatorname{Ker} T(\eta) = \boldsymbol{V}_0(\eta) \ \forall \eta \in \mathbb{S}^{n-2}$.

*Furthermore, if* $n = 2$ *or if* $N = 1$, *then it is also sufficient.*

**Theorem 3.7.** *For inequality* (1.5) *to hold on* ${}^0\boldsymbol{H}^1(\Omega_1)$ *with some constant* $c_S > 0$, *it is necessary that* $A(\xi)$ *and* $T(\eta)$ *satisfy conditions* (i)–(iv) *as in Theorem 3.6. Furthermore, if* $n = 2$ *or if* $N = 1$, *then it is also sufficient.*

**Corollary 3.8.** *Assume that* $\mathcal{A}$ *is strongly elliptic system. Then the following four conditions are equivalent:*

(1) *Inequality* (1.3) *holds on the space* $\boldsymbol{H}^1(\mathbb{R}^n_+)$ *for some constant* $c_K > 0$.

(2) *Inequality* (1.4) *holds on the space* ${}^0\boldsymbol{H}^1(\Omega_1)$ *for some constant* $c_P > 0$.

(3) *Inequality* (1.5) *holds on the space* ${}^0\boldsymbol{H}^1(\Omega_1)$ *for some constant* $c_S > 0$.

(4) $T(\eta) > O$ *for all* $\eta \in \mathbb{S}^{n-2}$.

# 4. Sketchy proofs of Theorems in the case $n = 2$

As mentioned in Introduction, our method is fit for the case $n = 2$ or $N = 1$. Proofs of Theorems of the preceding section in the case $N = 1$ are found in Ito [1]. Here we give those in the case $n = 2$. Now let $n = 2$. Then $\Omega_1 = \mathbb{R} \times (0, 1)$ and

$$A(\eta, \tau) = A^{22}\tau^2 + (A^{21} + A^{12})\eta\tau + A^{11}\eta^2 \qquad \text{for } (\eta, \tau) \in \mathbb{R} \times \mathbb{R}.$$

If $A(\xi)$ satisfies conditions (i) and (ii) in Theorem 3.4, the $\Lambda(\eta)$ defined at the beginning of §3 is given by

$$\Lambda(\eta) = \Lambda\eta \quad \text{if} \quad \eta \geq 0, \qquad = -\check{\Lambda}\eta \quad \text{if} \quad \eta < 0, \tag{4.1}$$

where $\Lambda = \lambda(A(1,\tau))$ and $\check{\Lambda} = \lambda(A(-1,\tau))$. We further set, for simplicity,

$$V_0 = V_0(A(1,\tau)) \ (= V_0(A(-1,\tau))), \quad V_1 = V_1(A(1,\tau)), \quad \check{V}_1 = V_1(A(-1,\tau)).$$

In what follows we shall use the notation above.

PROOF OF THEOREM 3.3.

Using (4.1) we have

$$\Lambda(\eta) + \Lambda(-\eta) = (\Lambda + \check{\Lambda})|\eta|, \quad \Lambda(\eta) - \Lambda(-\eta) = (\Lambda - \check{\Lambda})\eta.$$

Thus the $T(\eta)$ defined by (3.1) is calculated as follows:

$$\begin{aligned} T(\eta) &= -i\left(A^{21}\eta + A^{22}\Lambda(\eta)\right) \\ &= (2i)^{-1}A^{22}(\Lambda + \check{\Lambda})|\eta| - i\left(A^{21} + 2^{-1}A^{22}(\Lambda - \check{\Lambda})\right)\eta \\ &= (2i)^{-1}A^{22}(\Lambda + \check{\Lambda})|\eta| + \left(-\mathfrak{S}A^{12} + 2^{-1}\mathfrak{S}(A^{22}(\Lambda - \check{\Lambda}))\right)\eta. \end{aligned} \tag{4.2}$$

Therefore, if $\mathfrak{S}A^{12} \, (= -\mathfrak{S}A^{21}) = 2^{-1}\mathfrak{S}(A^{22}(\Lambda - \check{\Lambda}))$, then $T(\pm 1) > O$ by Theorem 2.5. Note that the quadratic form

$$\mathfrak{a}[u] = \frac{1}{2}\left\{\left(A^{22}(\partial_{x_2} - \Lambda\partial_{x_1})u, (\partial_{x_2} - \Lambda\partial_{x_1})u\right) + \left(A^{22}(\partial_{x_2} + \check{\Lambda}\partial_{x_1})u, (\partial_{x_2} + \check{\Lambda}\partial_{x_1})u\right)\right\}$$

has the property just stated.

PROOF OF THEOREM 3.4.

It is easy to see that conditions (i) and (ii) are necessary. So, under (i) and (ii), we show that (iii) is a necessary and sufficient condition. Since

$$A(\eta,\tau) = (I\tau - \Lambda^*\eta)A^{22}(I\tau - \Lambda\eta) \qquad \text{for} \ (\eta,\tau) \in \mathbb{R} \times \mathbb{R},$$

we have for every $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega_1)$, which is regarded as an element of $\boldsymbol{H}^1(\mathbb{R}^2)$ by 0-extension,

$$\begin{aligned}
\mathfrak{a}[\boldsymbol{u}] &= (A(\eta,\tau)\hat{\boldsymbol{u}}(\eta,\tau), \hat{\boldsymbol{u}}(\eta,\tau))_{L^2(\mathbb{R}^2_{\eta,\tau})} \\
&= \Big(A^{22}(I\tau - \Lambda\eta)\hat{\boldsymbol{u}}(\eta,\tau), (I\tau - \Lambda\eta)\hat{\boldsymbol{u}}(\eta,\tau)\Big)_{L^2(\mathbb{R}^2_{\eta,\tau})} \\
&= \Big(A^{22}(\partial_{x_2} - \Lambda\partial_{x_1})\boldsymbol{u}, (\partial_{x_2} - \Lambda\partial_{x_1})\boldsymbol{u}\Big)_{L^2(\mathbb{R}^2_x)} \\
&= \Big(R^*A^{22}R(\partial_{x_2} - J\partial_{x_1})R^{-1}\boldsymbol{u}, (\partial_{x_2} - J\partial_{x_1})R^{-1}\boldsymbol{u}\Big)_{L^2(\mathbb{R}^2_x)},
\end{aligned}$$

where $J$ and $R$ denote a Jordan form of $\Lambda$ and a corresponding transforming matrix: $J = R^{-1}\Lambda R$, and $\hat{\boldsymbol{u}}(\eta,\tau)$ represents the Fourier transform of $\boldsymbol{u}(x)$. Thus the validity of inequality (1.4) on $\boldsymbol{H}_0^1(\Omega_1)$ is equivalent to that of the inequality

$$\|(\partial_{x_2} - J\partial_{x_1})\boldsymbol{u}\|^2 \geq \tilde{c}_P\|\boldsymbol{u}\|^2 \qquad \forall \boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega_1)$$

for some $\tilde{c}_P > 0$. Moreover the inequality just above is due to the following rough estimates for $c_P(\alpha,p) := \inf\{\|(\partial_{x_2} - J(\alpha,p)\partial_{x_1})\boldsymbol{v}\|^2/\|\boldsymbol{v}\|^2;\ \boldsymbol{v} \in H_0^1(\Omega_1;\mathbb{C}^p)\}$ with $\alpha \in \mathbb{C}$ and $p \in \mathbb{N}$:

$$c_P(\alpha,1) = \pi^2,$$

$$c_1(p)\min\{1, |\mathrm{Im}\,\alpha|^{2(p-1)}\} \leq c_P(\alpha,p) \leq c_2(p)\min\{1, |\mathrm{Im}\,\alpha|^{2(1-\frac{1}{p})}\} \quad \text{if } p \geq 2,$$

where $c_1(p)$ and $c_2(p)$ are positive constants depending only on $p \geq 2$. We here note that, if $\alpha \in \mathbb{R}$ and $p \geq 2$, for arbitrary nontrivial functions $\phi \in C_0^2(\mathbb{R})$ and $\psi \in C_0^2(0,1)$ the family $\{\boldsymbol{w}_\varepsilon\}_{\varepsilon>0}$ in $H_0^1(\Omega_1;\mathbb{C}^p)$ defined by

$$\boldsymbol{w}_\varepsilon(x_1,x_2) = \Big(\phi_\varepsilon'(x_1+\alpha x_2)\,\psi(x_2),\ \phi_\varepsilon(x_1+\alpha x_2)\,\psi'(x_2),\ 0,\ldots,0\Big)^T$$

with $\phi_\varepsilon(t) = \varepsilon^{-1/2}\phi(\varepsilon^{-1}t)$ satisfies

$$\frac{\|(\partial_{x_2} - J(\alpha,p)\partial_{x_1})\boldsymbol{w}_\varepsilon\|^2}{\|\boldsymbol{w}_\varepsilon\|^2} \leq \frac{\|\phi_\varepsilon\|^2_{\mathbb{R}}}{\|\phi_\varepsilon'\|^2_{\mathbb{R}}} \cdot \frac{\|\psi''\|^2_{(0,1)}}{\|\psi\|^2_{(0,1)}} \to 0 \qquad \text{as } \varepsilon \to 0.$$

PROOF OF THEOREM 3.7.

Since conditions (i) and (ii) are easily checked to be necessary, we show under (i) and (ii) that for inequality (1.5) to hold it is necessary and sufficient that conditions (iii)–(v) are satisfied together.

First, assume that inequality (1.5) is valid. By Proposition 2.3 the boundary-value problem

$$\begin{cases} A(\eta, D_t)U(t) = O \quad \text{for } 0 < t < 1, \\ U(0) = I, \quad U(1) = O \end{cases}$$

with parameter $\eta \in \mathbb{R}$ admits a unique solution $U(t) = U(\eta, t)$, which is given by

(4.3)

$$U(\eta,t) = e^{it\Lambda(\eta)}\left\{I - \left(\int_0^t e^{-is\Lambda(\eta)}(A^{22})^{-1}e^{is\Lambda(\eta)^*}ds\right)\left(\int_0^1 e^{-is\Lambda(\eta)}(A^{22})^{-1}e^{is\Lambda(\eta)^*}ds\right)^{-1}\right\}.$$

Note that $U(\eta,t)$ satisfies

$$\begin{cases} c_1(1+|\eta|)^{-1}|\boldsymbol{r}|^2 \le \int_0^1 |U(\eta,t)\boldsymbol{r}|^2 dt \le c_2|\boldsymbol{r}|^2, \\ \int_0^1 |\partial_t U(\eta,t)\boldsymbol{r}|^2 dt \le c_2(1+|\eta|)|\boldsymbol{r}|^2 \qquad \forall \boldsymbol{r} \in \mathbb{C}^N, \forall \eta \in \mathbb{R} \end{cases} \tag{4.4}$$

for some positive constants $c_1$ and $c_2$ independent of $\boldsymbol{r}$ and $\eta$. Introducing $T_0(\eta) := B(\eta, D_t)U(\eta,t)|_{t=0}$, we have

$$T_0(\eta) = T(\eta) + T_{00}(\eta) \quad \text{with } T_{00}(\eta) = \left(\int_0^1 e^{-is\Lambda(\eta)}(A^{22})^{-1}e^{is\Lambda(\eta)^*}ds\right)^{-1}.$$

Moreover, for any $\boldsymbol{u} \in {}^0\boldsymbol{H}^1(\Omega_1) \cap \boldsymbol{H}^2(\Omega_1)$, if we define $\boldsymbol{v} \in {}^0\boldsymbol{H}^1(\Omega_1)$ by $\hat{\boldsymbol{v}}(\eta, x_2) = U(\eta, x_2)\hat{\boldsymbol{u}}(\eta, 0)$ (see (4.4)), then $\boldsymbol{u} - \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega_1)$ and

$$\mathfrak{a}[\boldsymbol{v}] = \int_{\mathbb{R}} \langle T_0(\eta)\hat{\boldsymbol{u}}(\eta,0), \hat{\boldsymbol{u}}(\eta,0)\rangle d\eta, \tag{4.5}$$

where $\hat{\boldsymbol{v}}(\eta, x_2)$ denotes the partial Fourier transform of $\boldsymbol{v}(x_1, x_2)$. Simple observation using (4.5) yields that the best constant, denoted by $c_S$ again, in inequality (1.5) is given by

$$c_S = \inf_{\eta \in \mathbb{R}} c_S(\eta) \quad \text{with } c_S(\eta) = \text{minimum eigenvalue of } T_0(\eta).$$

Thus, by assumption, condition (iv) follows from

$$T(\pm 1) = \lim_{\rho\to\infty} \rho^{-1} T_0(\pm\rho) \geq O,$$

where we have used the fact that $T_{00}(\eta)$ is uniformly bounded in $\eta$ (see (4.6)). Hence, on the subspaces $\ker T(\pm 1)$ of $\mathbb{C}^N$, the behaviors of $T_{00}(\eta)$ as $\eta \to \pm\infty$ determine whether $c_S > 0$ or $c_S = 0$.

Now remind Proposition 2.2 in order to find explicit forms of $T_{00}(\pm\infty) := \lim_{\eta\to\pm\infty} T_{00}(\eta)$, which do exist. Firstly, we write the real elementary divisors of $A(1,\tau)$ in the form

$$\{(\tau - \overline{\beta_\ell})(\tau - \beta_\ell)\}^{q_{\ell m}}, \qquad 1 \leq m \leq M_\ell,\ 1 \leq \ell \leq L_1 + L_2,$$

where $\beta_1, \beta_2, \ldots, \beta_{L_1+L_2}$ are distinct complex numbers such that

$$\beta_1, \ldots, \beta_{L_1} \in \mathbb{R}, \qquad \beta_{L_1+1}, \ldots, \beta_{L_1+L_2} \in \mathbb{C}_+,$$

and $q_{\ell 1}, q_{\ell 2}, \ldots, q_{\ell M_\ell}$ are positive integers satisfying $\sum_{\ell=1}^{L_1+L_2} \sum_{m=1}^{M_\ell} q_{\ell m} = N$ and

$$q_{\ell 1} \geq q_{\ell 2} \geq \cdots \geq q_{\ell M_\ell}, \qquad 1 \leq \ell \leq L_1 + L_2.$$

Secondly, using the invertible matrix polynomial $Q(\tau)$ in the representation (2.2) with $H(\tau)$ replaced by $A(1,\tau)$, we define

$$\begin{cases} \boldsymbol{r}_k^{(\ell,m)} = \frac{1}{(k-1)!} Q^{(k-1)}(\beta_\ell) e^{(\ell,m)} \in \mathbb{C}^N \quad 1 \leq k \leq q_{\ell m}, \\ R^{(\ell,m)} = \left(\boldsymbol{r}_1^{(\ell,m)}\ \boldsymbol{r}_2^{(\ell,m)}\ \cdots\ \boldsymbol{r}_{q_{\ell m}}^{(\ell,m)}\right) \in \mathcal{M}_{N,q_{\ell m}}, \quad 1 \leq m \leq M_\ell,\ 1 \leq \ell \leq L_1 + L_2, \\ R = \left(R^{(1,1)} \ldots R^{(1,M_1)}\ R^{(2,1)} \ldots R^{(2,M_2)} \ldots R^{(L_1+L_2,1)} \ldots R^{(L_1+L_2,M_{L_1+L_2})}\right) \in \mathcal{M}_N, \end{cases}$$

where the unit vectors $e^{(\ell,m)}$ are chosen in the same way as we chose $e^{(\ell)}$ in Proposition 2.2. The $R$ just obtained transforms $\Lambda$ to its Jordan form:

$$R^{-1}\Lambda R = \bigoplus_{\ell=1}^{L_1+L_2} \bigoplus_{m=1}^{M_\ell} J(\beta_\ell, q_{\ell m}).$$

Thirdly, corresponding to the form $R$, we write $(R^{-1})^*$ in the form

$$\begin{cases} (R^{-1})^* = \left(S^{(1,1)} \ \ldots \ S^{(1,M_1)} \ S^{(2,1)} \ \ldots \ S^{(2,M_2)} \ \ldots \ S^{(L_1+L_2,1)} \ \ldots \ S^{(L_1+L_2,M_{L_1+L_2})}\right) \in \mathcal{M}_N, \\ S^{(\ell,m)} = \left(\boldsymbol{s}_1^{(\ell,m)} \ \boldsymbol{s}_2^{(\ell,m)} \ \cdots \ \boldsymbol{s}_{q_{\ell m}}^{(\ell,m)}\right) \in \mathcal{M}_{N,q_{\ell m}}, \quad 1 \le m \le M_\ell, \ 1 \le \ell \le L_1+L_2. \end{cases}$$

Note here that

$$\sum_{\ell=1}^{L_1+L_2} \sum_{m=1}^{M_\ell} \sum_{k=1}^{q_{\ell m}} \boldsymbol{r}_k^{(\ell,m)} \otimes \overline{\boldsymbol{s}_k^{(\ell,m)}} = I,$$

$$\left\langle \boldsymbol{r}_{k_1}^{(\ell_1,m_1)}, \boldsymbol{s}_{k_2}^{(\ell_2,m_2)} \right\rangle = \begin{cases} 1 & \text{if } \ell_1 = \ell_2, m_1 = m_2 \text{ and } k_1 = k_2, \\ 0 & \text{otherwise}, \end{cases}$$

where $\boldsymbol{a} \otimes \boldsymbol{b} \in \mathcal{M}_N$ denotes the tensor product of $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^N$. In these circumstances we obtain

$$(4.6) \qquad \begin{aligned} H_{00}(+\infty) &= \lim_{\rho\to\infty} \left( \frac{1}{\rho} \int_0^\rho e^{-it\Lambda} (A^{22})^{-1} e^{it\Lambda^*} dt \right)^{-1} \\ &= \sum_{\ell=1}^{L_1} \sum_{j,k=1}^{M_\ell} \left( \sum_{h=1}^{\min\{q_{\ell j}, q_{\ell k}\}} (2h-1) c_{h,jk}^{(\ell)} \right) \boldsymbol{s}_{q_{\ell j}}^{(\ell,j)} \otimes \overline{\boldsymbol{s}_{q_{\ell k}}^{(\ell,k)}}, \end{aligned}$$

where $c_{h,jk}^{(\ell)}$ is the $(j,k)$-th element of the matrix $\left((S_h^{(\ell)})^*(A^{22})^{-1}S_h^{(\ell)}\right)^{-1} \in \mathcal{M}_{m(\ell,h)}$ with

$$m(\ell,h) = \max\{m; q_{\ell m} \ge h\}, \quad S_h^{(\ell)} = \left(\boldsymbol{s}_{q_{\ell 1}}^{(\ell,1)} \ \boldsymbol{s}_{q_{\ell 2}}^{(\ell,2)} \ \cdots \ \boldsymbol{s}_{q_{\ell,m(\ell,h)}}^{(\ell,m(\ell,h))}\right) \in \mathcal{M}_{N,m(\ell,h)}.$$

In addition, for any $\boldsymbol{r}_1, \boldsymbol{r}_2 \in V_1$, the function $\langle H_{00}(\eta)\boldsymbol{r}_1, \boldsymbol{r}_2 \rangle$ of $\eta$ decays exponentially as $\eta \to +\infty$.

It follows from (4.6) that

$$(4.7) \qquad \ker T_{00}(+\infty) = \left( \bigoplus_{\ell=1}^{L_1} \bigoplus_{m=1}^{M_\ell} S[\boldsymbol{r}_1^{(\ell,m)}, \boldsymbol{r}_2^{(\ell,m)}, \ldots, \boldsymbol{r}_{q_{\ell m}-1}^{(\ell,m)}] \right) \oplus V_1.$$

Similarly we have

$$(4.8) \qquad \ker T_{00}(-\infty) = \left( \bigoplus_{\ell=1}^{L_1} \bigoplus_{m=1}^{M_\ell} S[\boldsymbol{r}_1^{(\ell,m)}, \boldsymbol{r}_2^{(\ell,m)}, \ldots, \boldsymbol{r}_{q_{\ell m}-1}^{(\ell,m)}] \right) \oplus \check{V}_1.$$

On the other hand, Theorem 2.5 and (4.2) show that

$$T(\eta)|_{V_0} = \mathfrak{S}\big(-A^{12} + 2^{-1}A^{22}(\Lambda - \check{\Lambda})\big)\,\eta \geq O \quad \forall\eta \in \mathbb{R},$$

from which we can conclude that $\ker T(\pm 1) \supset \boldsymbol{V}_0$. Now suppose that there is an index $\ell$ such that $1 \leq \ell \leq L_1$ and $q_{\ell 1} \geq 2$. Then we have by (4.7) and (4.8)

$$\langle T_0(\eta)\boldsymbol{r}^{(\ell,1)}_{q_{\ell 1}-1}, \boldsymbol{r}^{(\ell,1)}_{q_{\ell 1}-1}\rangle = \langle T_{00}(\eta)\boldsymbol{r}^{(\ell,1)}_{q_{\ell 1}-1}, \boldsymbol{r}^{(\ell,1)}_{q_{\ell 1}-1}\rangle \to 0 \quad \text{as } \eta \to \pm\infty,$$

which is a contradiction. Thus condition (iii) is satisfied. Suppose again that $\ker T(1) \neq \boldsymbol{V}_0$. Then there is a nontrivial $\boldsymbol{r} \in \boldsymbol{V}_1$ such that $T(1)\boldsymbol{r} = \boldsymbol{0}$, which satisfies

$$\langle T_0(\eta)\boldsymbol{r}, \boldsymbol{r}\rangle = \langle T_{00}(\eta)\boldsymbol{r}, \boldsymbol{r}\rangle \to 0 \quad \text{as } \eta \to +\infty.$$

Since this contradict the positivity of $c_S$, we have $\ker T(1) = \boldsymbol{V}_0$. Likewise we can show that $\ker T(-1) = \boldsymbol{V}_0$. Hence, we lastly obtain condition (v).

We proceed to the sufficiency part. Under condition (iii), (4.6) and the corresponding representation for $T_{00}(-\infty)$ show that $T_{00}(\pm\infty)|_{V_0} > O$. Hence, using conditions (iv) and (v) we can deduce that

$$T_0(\eta) \geq c_3 I \qquad \forall\eta \in \mathbb{R}$$

for some constant $c_3 > 0$ independent of $\eta$.

Proof of Theorem 3.6.

Since conditions (i)–(iii) are necessary by Theorem 3.4, assuming those to be satisfied we show that inequality (1.4) is valid on the space ${}^0\boldsymbol{H}^1(\Omega_1)$ if and only if conditions (iv) and (v) hold.

We begin with the necessity part. For any $\boldsymbol{\phi} \in \boldsymbol{H}^1(\mathbb{R})$, define $\boldsymbol{v} \in {}^0\boldsymbol{H}^1(\Omega_1)$ by $\hat{\boldsymbol{v}}(\eta, x_2) = U(\eta, x_2)\hat{\boldsymbol{\phi}}(\eta)$, where $U(\eta, t)$ is as in (4.3). Then inequality (1.4) on ${}^0\boldsymbol{H}^1(\Omega_1)$ yields, by (4.5) and the arbirariness of $\phi$, that

$$\langle T_0(\eta)\boldsymbol{r},\boldsymbol{r}\rangle \geq c_{P2}\int_0^1 |U(\eta,t)\boldsymbol{r}|^2 dt \quad \forall \boldsymbol{r}\in\mathbb{C}^N, \forall\eta\in\mathbb{R}. \tag{4.9}$$

This implies that $T_0(\eta)\geq O$ for all $\eta\in\mathbb{R}$, so that $T(\pm 1)\geq O$ (condition (iv)). Thus, as in Proof of Theorem 3.7, we have $\ker T(\pm 1)\supset \boldsymbol{V}_0$. Now suppose that there exists a nonzero $\boldsymbol{r}_1\in\ker T(1)\cap \boldsymbol{V}_1$. Since, from what stated just below (4.6), we have

$$\langle \eta\, T_0(\eta)\boldsymbol{r}_1,\boldsymbol{r}_1\rangle = \langle \eta\, T_{00}(\eta)\boldsymbol{r}_1,\boldsymbol{r}_1\rangle \to 0 \quad \text{as } \eta\to+\infty,$$

so by (4.9) that $\eta\int_0^1 |U(\eta,t)\boldsymbol{r}_1|^2 dt\to 0$ as $\eta\to+\infty$. This contradicts (4.4), so $\ker T(1) = \boldsymbol{V}_0$. Likewise $\ker T(1) = \boldsymbol{V}_0$ is obtained. Therefore condition (v) holds.

Conversely, let (iv) and (v) be satisfied. Given $\boldsymbol{u}\in {}^0\boldsymbol{H}^1(\Omega_1)\cap \boldsymbol{H}^2(\Omega_1)$ we write $\phi(x_1) = \boldsymbol{u}(x_1,0)\in \boldsymbol{H}^1(\mathbb{R})$ and define $\boldsymbol{v}\in {}^0\boldsymbol{H}^1(\Omega_1)$ as above. Using (4.4) we have

$$\|\boldsymbol{v}\|^2 = \int_{\mathbb{R}} d\eta \int_0^1 |U(\eta,t)\hat{\phi}(\eta)|^2 dt \leq c_2\|\phi\|^2_{\mathbb{R}}.$$

Since $\boldsymbol{u}-\boldsymbol{v}\in \boldsymbol{H}^1_0(\Omega_1)$, Theorems 3.4 and 3.7 yield that

$$\mathfrak{a}[\boldsymbol{u}] = \mathfrak{a}[\boldsymbol{u}-\boldsymbol{v}] + \mathfrak{a}[\boldsymbol{v}] \geq c_{P1}\|\boldsymbol{u}-\boldsymbol{v}\|^2 + c_S c_2^{-1}\|\boldsymbol{v}\|^2 \geq \frac{c_{P1}c_S}{c_2 c_{P1}+c_S}\|\boldsymbol{u}\|^2,$$

where $c_{P1}$ denotes the best constant in inequality (1.4) on $\boldsymbol{H}^1_0(\Omega_1)$. This completes the proof.

## 5. Examples

EXAMPLE 1. Consider the quadratic form (1.8). This determines the symbols

$$A(\xi) = \sum_{j,k=1}^{n} A^{jk}\xi_j\xi_k, \quad B(\xi) = -i\sum_{k=1}^{n} A^{nk}\xi_k \quad \text{for } \xi\in\mathbb{R}^n.$$

Conditions (i) and (ii) in §3 hold if and only if

$$A^{nn}>0 \quad\text{and}\quad A^{nn}\sum_{j,k=1}^{n-1} A^{jk}\eta_j\eta_k \geq \left(\operatorname{Re}\sum_{j=1}^{n-1} A^{nj}\eta_j\right)^2 \quad \forall\eta\in\mathbb{S}^{n-2}. \tag{5.1}$$

Let (5.1) be satisfied. Then condition (iii) always holds. The $\Lambda(\eta)$ and $T(\eta)$, defined at the beginning of §3, are given by

$$\Lambda(\eta) = (A^{nn})^{-1}\left\{ i\sqrt{A^{nn}\sum_{j,k=1}^{n-1} A^{jk}\eta_j\eta_k - \left(\operatorname{Re}\sum_{j=1}^{n-1} A^{nj}\eta_j\right)^2} - \operatorname{Re}\sum_{j=1}^{n-1} A^{nj}\eta_j \right\},$$

$$T(\eta) = \sqrt{A^{nn}\sum_{j,k=1}^{n-1} A^{jk}\eta_j\eta_k - \left(\operatorname{Re}\sum_{j=1}^{n-1} A^{nj}\eta_j\right)^2} + \operatorname{Im}\sum_{j=1}^{n-1} A^{nj}\eta_j \quad \text{for } \eta \in \mathbb{R}^{n-1}.$$

Thus conditions (iv) and (v) are satisfied if and only if

$$A^{nn}\sum_{j,k=1}^{n-1} A^{jk}\eta_j\eta_k \geq \left|\sum_{j=1}^{n-1} A^{nj}\eta_j\right|^2 \quad \forall \eta \in \mathbb{S}^{n-2}$$

with the equality attained only by $\eta$ satisfying $\operatorname{Im}\sum_{j=1}^{n-1} A^{nj}\eta_j = 0$. If all the coefficients $A^{jk}$ are real in particular, then conditions (iv) and (v) follows directly from (5.1).

EXAMPLE 2. The quadratic form (1.9) determines the symbols

$$A(\xi) = (\lambda + \mu)\,\boldsymbol{\xi} \otimes \boldsymbol{\xi} + \mu\,|\xi|^2 I,$$

$$B(\xi) = -i(\lambda\, \boldsymbol{e}_n \otimes \boldsymbol{\xi} + \mu\,\boldsymbol{\xi} \otimes \boldsymbol{e}_n + \xi_n I) \quad \text{for } \xi \in \mathbb{R}^n,$$

where $\boldsymbol{e}_n = (0, \ldots, 0, 1)^T$. Condition (i) in §3 is equivalent that

$$\mu > 0 \quad \text{and} \quad \lambda + 2\mu > 0. \tag{5.2}$$

Let (5.2) be satisfied. Then conditions (ii) and (iii) are satisfied. The real elementary divisors of the matrix polynomial $A(\eta, \tau)$ in $\tau$ with parameter $\tau \in \mathbb{R}^{n-1}$ are given by

$$(\tau^2 + |\eta|^2)^2,\ \underbrace{\tau^2 + |\eta|^2, \ldots, \tau^2 + |\eta|^2}_{n-2} \quad \text{if } \lambda + \mu \neq 0 \text{ and } \eta \neq 0;$$

$$\underbrace{\tau^2 + |\eta|^2, \ldots, \tau^2 + |\eta|^2}_{n} \quad \text{if } \lambda + \mu = 0 \text{ or } \eta = 0.$$

Moreover, the $\Lambda(\eta)$ and $T(\eta)$ are given by

$$\Lambda(\eta) = \begin{pmatrix} i\left(I_{n-1}|\eta| + \beta\dfrac{\boldsymbol{\eta}\otimes\boldsymbol{\eta}}{|\eta|}\right) & -\beta\boldsymbol{\eta} \\ -\beta\boldsymbol{\eta}^T & i(1-\beta)|\eta| \end{pmatrix},$$

$$T(\eta) = \mu\begin{pmatrix} \left(I_{n-1}|\eta| + \beta\dfrac{\boldsymbol{\eta}\otimes\boldsymbol{\eta}}{|\eta|}\right) & -i(1-\beta)\boldsymbol{\eta} \\ i(1-\beta)\boldsymbol{\eta}^T & (1+\beta)|\eta| \end{pmatrix} \qquad \text{for } \eta \in \mathbb{R}^{n-1},$$

where $\beta = (\lambda+\mu)/(\lambda+3\mu)$; the eigenvalues of $T(\eta)$ are

$$2\mu|\eta|,\ 2\beta\mu|\eta|,\ \underbrace{\mu|\eta|,\ \ldots,\ \mu|\eta|}_{n-2}.$$

Since $V_0(\eta) = \{0\}$ for all $\eta \in \mathbb{S}^{n-2}$, conditions (i)–(v) hold together if and only if

$$\mu > 0 \quad \text{and} \quad \lambda + \mu > 0.$$

We finally note that Corollary 3.8 is applicable to this case under (5.2).

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