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Pohozaev の恒等式とその応用
Pohozaev identity and its applications

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§1. Introduction

This is a joint work with Eiji Yanagida (Tokyo Institute of Technology).

Recently, in [YY1] and [YY2], we obtained classification theorems of the structure of positive radial solutions to the equation

$$\Delta u + K(|x|)u^p = 0, \qquad x \in \mathbf{R}^n,$$

where $p > 1$ and $n > 2$. (See, also, [DN1], [KL], [KN], [KNY], [KYY1], [KYY2], [LN1, LN2, LN3, LN4], [Na], [Y1], [Y2], [Y3], [Y4] and [YY3].)

We will explain the main results of them. Since we are interested in positive radial solutions (i.e. solutions with $u = u(|x|) > 0$), we study the initial value problem

$$\text{(E)} \qquad \begin{cases} (r^{n-1}u_r)_r + r^{n-1}K(r)(u^+)^p = 0, & r > 0, \\ u(0) = \alpha > 0, \end{cases}$$

where $r = |x|$ and $u^+ = \max\{u, 0\}$. We impose the following conditions on $K(r)$:

$$\text{(K)} \qquad \begin{cases} K(r) \text{ is continuous on } (0, \infty); \\ K(r) \geqq 0 \text{ and } K(r) \not\equiv 0 \text{ on } (0, \infty); \\ rK(r) \in L^1(0, 1); \\ r^{n-1-(n-2)p}K(r) \in L^1(1, \infty). \end{cases}$$

It is known that, under the first, second and third conditions in (K), the initial value problem (E) has a unique solution $u(r) \in C([0, \infty)) \cap C^2((0, \infty))$ (see, e.g., Propositions 4.1 and 4.2 of [NY]). We will denote the unique solution by $u(r;\alpha)$. We note that, if the last condition in (K) is not satisfied, then $u(r;\alpha)$ has a zero in $(0, \infty)$ for every $\alpha > 0$ (see, e.g., [A] or [N]).

We classify each solution of (E) according to its behavior as $r \to \infty$. We say that

(i) $u(r;\alpha)$ is a <u>zero-hit solution</u> if $u(r;\alpha)$ has a zero in $(0, \infty)$,

(ii) $u(r;\alpha)$ is a <u>slow-decay solution</u> if $u(r;\alpha) > 0$ on $[0, \infty)$ and $\lim_{r\to\infty} r^{n-2}u(r;\alpha) = \infty$,

(iii) $u(r;\alpha)$ is a <u>fast-decay solution</u> if $u(r;\alpha) > 0$ on $[0, \infty)$, and $\lim_{r\to\infty} r^{n-2}u(r;\alpha)$ exists and is finite and positive.

It can be shown that, if $u(r;\alpha) > 0$ on $[0, \infty)$, then $r^{n-2}u(r;\alpha)$ is non-decreasing in r. This implies that any solution of (E) is classified into one of the above three types.

Let $G(r)$ and $H(r)$ be functions defined by

$$G(r) := \frac{2}{p+1} r^n K(r) - (n-2)\int_0^r s^{n-1}K(s)ds,$$

$$H(r) := \frac{2}{p+1} r^{2-(n-2)p} K(r) - (n-2)\int_r^\infty s^{1-(n-2)p}K(s)ds,$$

By $n > 2$ and (K), the integrals in the definitions of $G(r)$ and $H(r)$ are well-defined. We note that, if $K(r)$ is differentiable, then

$$G_r(r) \equiv r^{(n-2)(p+1)}H_r(r) = \frac{2}{p+1} r^{n-1}\{rK_r(r) - \lambda K(r)\},$$

where λ is given by

$$\lambda := \frac{(n-2)p - (n+2)}{2}.$$

Finally we define

$$r_G := \inf\{\, r \in (0, \infty) \; ; \; G(r) < 0 \,\},$$

$$r_H := \sup\{\, r \in (0, \infty) \; ; \; H(r) < 0 \,\}.$$

Here we put $r_G = \infty$ if $G(r) \geqq 0$ on $(0, \infty)$, and $r_H = 0$ if $H(r) \geqq 0$ on $(0, \infty)$.

Under the condition $r_H \leqq r_G$, we can completely classify the structure of solutions to (E).

Theorem 1. Suppose that $G(r) \not\equiv 0$ on $(0, \infty)$. Then the structure of solutions to (E) is as follows.

(a) If $r_G = \infty$, then the structure is of Type Z : $u(r;\alpha)$ is a zero-hit solution for every $\alpha > 0$.

(b) If $r_G < \infty$ and $r_H = 0$, then the structure is of Type S : $u(r;\alpha)$ is a slow-decay solution for every $\alpha > 0$.

(c) If $0 < r_H \leqq r_G < \infty$, then the structure is of Type M : There exists a unique positive number α_f such that $u(r;\alpha)$ is a zero-hit solution for every $\alpha \in (\alpha_f, \infty)$, $u(r;\alpha)$ is a fast-decay solution if and only if $\alpha = \alpha_f$, and $u(r;\alpha)$ is a slow-decay solution for every $\alpha \in (0, \alpha_f)$.

We should note that, if $G(r) \equiv 0$ on $(0, \infty)$, $u(r;\alpha)$ is a fast-decay solution for every $\alpha > 0$.

The next theorem implies that the condition $r_H \leqq r_G$ is sharp.

Theorem 2. Let a and b be any given numbers with $0 \leqq a < b \leqq \infty$. Then there exists $K(r)$ with $r_G = a$ and $r_H = b$ such that the structure of solutions to (E) is neither of Type Z, Type S, Type M and Type F.

The above Theorem 1 is so powerful that it covers almost all known results as colloralies and can be applied to the prescribing scalar curvature equation

$$\Delta u + Ku^{(n+2)/(n-2)} = 0 \quad \text{in } \mathbf{R}^n.$$

Theorem 3. Let $p = \dfrac{n+2}{n-2}$ and suppose that $K(r) \not\equiv$ Constant.

(a) If $K(r)$ is non-decreasing on $(0, \infty)$, then the structure is of Type Z.

(b) If $K(r)$ is non-increasing on $(0, \infty)$, then the structure is of Type S.

(c) If there exists $R \in (0, \infty)$ such that $K(r)$ is non-decreasing on $(0, R)$ and non-increasing on (R, ∞), and if $\lim_{r \to 0} K(r) = \lim_{r \to \infty} K(r)$, then the structure is of Type M. Moreover, if u is a slow-decay solution, then there exist positive constants c_1 and c_2 such that $c_1 r^{-(n-2)/2} \leqq u \leqq c_2 r^{-(n-2)/2}$ for every sufficiently large r.

Our proofs of Theorems 1 and 2 are based on the shooting method. A main difficulty in the shooting method lies in the fact that the asymptotic behavior of $u(r;\alpha)$ as $r \to \infty$ must be studied carefully. In order to overcome this difficulty, we employ useful characterizations of zero-hit, slow-decay and fast-decay solutions, by using the variants of well-known Pohozaev identity

$$P(r;u) \equiv G(r) u^+(r;\alpha)^{p+1} - (p+1) \int_0^r G(s) u^+(s;\alpha)^p u_r(s;\alpha) ds,$$

and

$$P(r;u) \equiv H(r) \{r^{n-2} u^+(r;\alpha)\}^{p+1} - (p+1) \int_0^r H(s) \{s^{n-2} u^+(s;\alpha)\}^p \{s^{n-2} u(s;\alpha)\}_s ds,$$

where

$$P(r;u) := r^{n-1} u_r \{r u_r + (n-2) u\} + \frac{2}{p+1} r^n K(r) (u^+)^{p+1}.$$

The following lemma is fundamental characterizations of solutions.

Lemma.

(a) *If* $u = u(r;\alpha)$ *is a zero-hit solution, then* $P(r;u) > 0$ *for* $r \in [z(\alpha), \infty)$, *where* $z(\alpha)$ *is a zero of* $u(r;\alpha)$.

(b) *If* $u = u(r;\alpha)$ *is a slow-decay solution, then there exists a sequence* $\{\hat{r}_i\}$ *such that* $\hat{r}_i \to \infty$ *as* $i \to \infty$ *and* $P(\hat{r}_i;u) < 0$ *for every* i.

(c) *If* $u = u(r;\alpha)$ *is a fast-decay solution, then there exists a sequence* $\{\bar{r}_i\}$ *such that* $\bar{r}_i \to \infty$ *and* $P(\bar{r}_i;u) \to 0$ *as* $i \to \infty$.

For the proofs of Theorems 1 and 2, we also use recent results assuring the existence of zero-hit, slow-decay and fast-decay solutions obtained by [YY1].

§ 2. Results

The aim of this report is to generalize the above Theorem 1 to more general equation

$$(E) \qquad \begin{cases} (g(r)u_r)_r + g(r)K(r)(u^+)^p = 0, & r > 0, \\ u(0) = \alpha > 0, \end{cases}$$

where $g(r)$ is a given function satisfying the following conditions:

$$(g) \qquad \begin{cases} g(r) \in C^1([0, \infty)); \\ g(r) > 0 \quad \text{on} \quad (0, \infty); \\ 1/g(r) \notin L^1(0, 1); \\ 1/g(r) \in L^1(1, \infty). \end{cases}$$

We impose the following conditions on $K(r)$:

$$(K) \qquad \begin{cases} K(r) \quad \text{is continuous on} \quad (0, \infty); \\ K(r) \geqq 0 \quad \text{and} \quad K(r) \not\equiv 0 \quad \text{on} \quad (0, \infty); \\ h(r)K(r) \in L^1(0, 1); \\ g(r)(h(r)/g(r))^p K(r) \in L^1(1, \infty), \end{cases}$$

where

$$h(r) := g(r) \int_r^{\infty} \frac{1}{g(s)}\, ds.$$

It is shown that, under the first, second and third conditions in (K), the initial value problem (E) has a unique solution $u(r) \in C([0, \infty)) \cap C^2((0, \infty))$. We will denote the unique solution by $u(r;\alpha)$. We note that, if the last condition in (K) is not satisfied, then $u(r;\alpha)$ has a zero in $(0, \infty)$ for every $\alpha > 0$.

We classify each solution of (E) according to its behavior as $r \to \infty$. We say that

(i) $u(r;\alpha)$ is a zero-hit solution if $u(r;\alpha)$ has a zero in $(0, \infty)$,

(ii) $u(r;\alpha)$ is a slow-decay solution if $u(r;\alpha) > 0$ on $[0, \infty)$ and $\lim_{r \to \infty} (g(r)/h(r))u(r;\alpha) = \infty$,

(iii) $u(r;\alpha)$ is a fast-decay solution if $u(r;\alpha) > 0$ on $[0, \infty)$, and $\lim_{r \to \infty} (g(r)/h(r))u(r;\alpha)$ exists and is finite and positive.

It can be shown that, if $u(r;\alpha) > 0$ on $[0, \infty)$, then $(g(r)/h(r))u(r;\alpha)$ is non-decreasing in r. This implies that any solution of (E) is classified into one of the above three types.

We otain the following generalized Pohozaev identity

$$P(r;u) \equiv G(r)u^+(r;\alpha)^{p+1} - (p + 1)\int_0^r G(s)u^+(s;\alpha)^p u_r(s;\alpha)ds$$

and its variant

$$P(r;u) \equiv H(r)\{(g/h)u^+(r;\alpha)\}^{p+1}$$
$$- (p + 1)\int_0^r H(s)\{(g/h)u^+(s;\alpha)\}^p\{(g/h)u(s;\alpha)\}_s ds,$$

where

$$P(r;u) := g(r)u_r\{h(r)u_r + u\} + \frac{2}{p+1} g(r)h(r)K(r)(u^+)^{p+1}.$$

$$G(r) := \frac{2}{p+1} g(r)h(r)K(r) - \int_0^r g(s)K(s)ds,$$

$$H(r) := \frac{2}{p+1} h(r)^2 \left(\frac{h(r)}{g(r)}\right)^p K(r) - \int_r^\infty h(s)\left(\frac{h(s)}{g(s)}\right)^p K(s)ds.$$

By (g) and (K), the integrals in the definitions of $G(r)$ and $H(r)$ are well-defined.

We define r_G and r_H as in the case $g(r) = r^{n-1}$ in the previous section. Under the condition $r_H \leqq r_G$, we can completely classify the structure of solutions as Theorem 1.

§ 3. Applications.

Let us consider scalar field equations

$$\text{(S)} \qquad \Delta u - u + Q(|x|)u^p = 0, \qquad x \in \mathbf{R}^n, \qquad (n \geqq 3).$$

The existence of solutions have been studied in many papers. Ding-Ni [DN2] showed that (S) has at least one positive radial solutions if $Q(r) > 0$ and bounded by r^ℓ with $0 < \ell < (n-1)(p-1)/2$. On the other hand Li [L] has proved that (S) has no positive solution if $Q(r) \geqq 0$ and $Q(r)r^{-(n-1)(p-1)/2}$ is nondecreasing. However the structure of positive radial solutions is not known.

We will investigate the structure. We can apply new Theorem 1 with $g(r)$ to investigate (S). In fact, by putting $v(r) = u(r)/\varphi(r)$, $v(r)$ satisfies the equation

$$(E_Q) \qquad \begin{cases} v_{rr} + \left(\dfrac{2\varphi_r}{\varphi} + \dfrac{n-1}{r}\right) v_r + Q(r)\varphi(r)^{p-1}v^p = 0, & r > 0, \\ v(0) = \alpha > 0 \end{cases}$$

where

$$\begin{cases} (r^{n-1}\varphi_r)_r - r^{n-1}\varphi = 0, & r > 0, \\ \varphi(0) = 1, \quad \varphi_r(0) = 0. \end{cases}$$

Thus we may put

$$\varphi(r) = c_n r^{(2-n)/2} I_{(n-2)/2}(r), \qquad c_n = 2^{(n-2)/2}\Gamma(n/2),$$

$$g(r) = r^{n-1}\varphi(r)^2 = c_n^2 r I_{(n-2)/2}(r)^2,$$

$$h(r) = g(r)\int_r^\infty \frac{1}{g(s)}\,ds = rI_{(n-2)/2}(r)K_{(n-2)/2}(r),$$

$$K(r) = \varphi(r)^{p-1}Q(r) = c_n^{p-1}r^{(2-n)(p-1)/2}I_{(n-2)/2}(r)^{p-1}Q(r),$$

$$(p+1)c_n^{-(p+1)}G_r(r) = c_n^{-(p+1)}\{\ 2\{g(r)h(r)K(r)\}_r - (p+1)g(r)K(r)\ \}$$

$$= 2\{r^{4-n-(n-2)p}(r^{(n-2)/2}I_{(n-2)/2}(r))^{p+2}(r^{(n-2)/2}K_{(n-2)/2})Q(r)\}_r$$

$$- (p+1)r^{1-(n-2)p}(r^{(n-2)/2}I_{(n-2)/2}(r))^{p+1}Q(r),$$

where $I_j(r)$ and $K_j(r)$ are modified Bessel function of the first kind and second kind of j order, respectively. In particular, for $n = 3$,

$$\varphi(r) = r^{-1}\sinh(r),$$

$$g(r) = \sinh^2(r),\quad h(r) = \sinh(r)e^{-r},\quad K(r) = r^{1-p}\sinh^{p-1}(r)Q(r),$$

$$G_r(r) = \frac{2}{p+1}(r^{1-p}\sinh^{p+2}(r)e^{-r}Q(r))_r - r^{1-p}\sinh^{p+1}(r)Q(r).$$

Example 1. If $n = 3$ and $Q(r) = 1$, then the structure of solutions is as follows.

(a) <u>If</u> $1 < p < 5$, then the structure of (E_Q) is of Type M.

((S) has a uniqe positive radial solution u with $u \sim e^{-r}/r$ at ∞. See, [K] and [KL].)

(b) <u>If</u> $p \geqq 5$, then the structure is of Type S.

(All solutions of (S) are positive and $u \to 1$ as $r \to \infty$.)

In fact, we may note that

$$G_r(r) = \frac{1}{p+1}r^{-p}\sinh^{p+1}(r)\{(p+3)re^{-2r} + (p-1)e^{-2r} - (p-1)\}$$

and

$$\{(p+3)re^{-2r} + (p-1)e^{-2r} - (p-1)\}_r = e^{-2r}\{(5-p) - 2(p+3)r\}.$$

Example 2. If $n = 3$ and $Q(r) = r^{p-2}$, then the structure of solutions of (E_Q) is of Type M. ((S) has a uniqe positive radial solution u with $u \sim e^{-r}/r$ at ∞.) In fact, we may note that

$$G_r(r) = \frac{1}{p+1} r^{-2}\sinh^{p+1}(r)\{(p+3)re^{-2r} + e^{-2r} - 1\}$$

and

$$\{(p+3)re^{-2r} + e^{-2r} - 1\}_r = e^{-2r}\{(p+1) - 2(p+3)r\}.$$

Example 3. If $n = 3$ and $Q(r) = r^{p-1}$, then the structure of solutions of (E_Q) is of Type Z. ((S) has no positive solutions. See, [L].) In fact,

$$G_r(r) = \frac{p+3}{p+1} e^{-2r}\sinh^{p+1}(r) > 0 \qquad \text{for} \quad r > 0.$$

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