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GL equation and solutions with magnetic effect

神保 秀一 (Shuichi JIMBO, Okayama Univ.)

森田 善久 (Yoshihisa MORITA, Ryukoku Univ.)

We deal with the GL equation and an elliptic equation which are models in the super-conductivity phenomena. In particular, we consider their stable solutions. The energy defined for a physical state (Φ, A) is the following
(1)

$$\mathcal{H}(\Phi, A) = \int_\Omega \left\{ \frac{1}{2}|(h\nabla - iA)\Phi|^2 + \frac{\lambda}{4}(1 - |\Phi|^2)^2 \right\} dx + \int_{\mathbf{R}^n} \frac{1}{2}|\mathrm{rot}A|^2 dx$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with C^3 boundary and $\Phi : \Omega \longrightarrow \mathbf{C}$ $A : \Omega \longrightarrow \mathbf{R}^n$. $n = 2$ or $n = 3$ and $\lambda > 0$ and $h > 0$ are parameters. Φ and A are the (macro-)wave function of electrons and the vector potential of magnetic field, respectively. A stable state correspponds to a local minimizer of the above functional $\mathcal{H}$. In some situation of the superconductivity, the magnetic field is small and the model without A is used. We also deal with the model by removing A from $\mathcal{H}$ which is given by the the functional

$$\mathcal{H}_0(\Phi) = \int_\Omega \left\{ \frac{1}{2}|\nabla\Phi|^2 + \frac{\lambda}{4}(1 - |\Phi|^2)^2 \right\} dx. \tag{2}$$

Remark. It is easy to see

$$\mathcal{H}(\Phi, A) = \mathcal{H}(e^{i\rho}\Phi, A - h\nabla\rho) \tag{P}$$

for any smooth function $\rho : \mathbf{R}^n \longrightarrow \mathbf{R}$. In other word, the functional is invariant under the above translation (P). Hence the GL equation describing critical points is also translation invariant under (P). We use this property essentially later in §3.

Now we show the GL equation. By a simple calculation, we get the variational equations corresponding to (1) and (2) (respectively). Both of them are called GL equations.

$$
(3) \qquad \begin{cases} (h\nabla - iA)^2\Phi + \lambda(1-|\Phi|^2)\Phi = 0 \quad \text{in} \quad \Omega, \\ h\partial\Phi/\partial\nu - i(A\cdot\nu)\Phi = 0 \quad \text{on} \quad \partial\Omega, \\ \operatorname{rot}\operatorname{rot} A + ih(\overline{\Phi}\nabla\Phi - \Phi\nabla\overline{\Phi})/2 + |\Phi|^2 A = 0 \quad \text{in} \quad \Omega, \\ \operatorname{rot}\operatorname{rot} A = 0 \quad \text{in} \quad \Omega^c. \end{cases}
$$

$$
(4) \qquad \begin{cases} \Delta\Phi + \lambda(1-|\Phi|^2)\Phi = 0 \quad \text{in} \quad \Omega, \\ \partial\Phi/\partial\nu = 0 \quad \text{on} \quad \partial\Omega. \end{cases}
$$

(We deal with (4) for general n.)

Our basic problems are the followings,

"In what kind of domain Ω, is there a **non-trivial stable solution** in (3) and (4) ?"

"Is there any relation between the **structure of stable solutions** of (3) and that of (4) ?"

In this note, we answer very partially to these questions. For the detals of the results in §1 and §2, see Jimbo and Morita [10].

§1. Solutions in convex domain

It is natural to guess that any complicated stable steady state does not exist in a simple domain. The following result is affirmative to this concept.

Theorem 1. If Ω is convex, any non-constant solution of (4) is unstable.

Remark. It is known from Casten and Holland [5] and Matano [12] earlier works that any non-constant real valued solution of (4) is unstable.

However the situation in Thm.1 will be much different from the case of real valued problem when we deal with a more complicated domain. See also [11] for similar topics on the competition system.

(Sketch of the proof of Theorem 1) For simplicity of notation we put $h = 1$ and $\lambda = 1$ (This is not essential in the proof). Let $\Phi = u + iv$ be nonconstant solution of (4) and we consider the linearized eigenvalue problem at Φ. We deal with the problem in the real vector form.

$$\Delta \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \begin{pmatrix} 1 - 3u^2 - v^2 & -2uv \\ -2uv & 1 - u^2 - 3v^2 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} + \mu \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with Neumann boundary condition on $\partial\Omega$.
This is a self-adjoint problem with discrete spectrum bounded from below. The lowest eigenvalue μ_1 is characterized as follows,

$$\mu_1 = \inf\{J(\phi, \psi) \mid \int_\Omega (\phi^2 + \psi^2) dx = 1\}$$

and the corresponding minimizers are eigenfunctions, where

$$J(\phi, \psi) = \int_\Omega \left(|\nabla\phi|^2 + |\nabla\psi|^2 + K(\phi, \psi)\right) dx$$

and

$$K(\phi, \psi) = -(1 - 3u^2 - v^2)\phi^2 + 4uv\phi\psi - (1 - u^2 - 3v^2)\psi^2.$$

We will prove $\mu_1 < 0$ which will yield the conclusion of Th. 1. By a direct calculation, we have

$$I = \sum_{k=1}^{N} J(\frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_k}) = \frac{1}{2} \int_{\partial\Omega} \frac{\partial}{\partial \nu} (|\nabla u|^2 + |\nabla v|^2)\, dS.$$

From the convexity and the Neumann boundary condition of u, v on $\partial\Omega$, we have $I \le 0$. Hence $\mu_1 \le 0$ from $(*)$. We assume $\mu_1 = 0$. From $I \le 0$ and $(*)$, we have

$$J(\frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_k}) = 0$$

for any $k = 1, 2, \ldots, n$. Hence each ${}^t(\partial u/\partial x_k, \partial v/\partial x_k)$ is an eigenfunction corresponding to $\mu_1 = 0$. Therefore it satisfies Neumann B.C.

$$\frac{\partial}{\partial \nu}\left(\frac{\partial u}{\partial x_k}, \frac{\partial v}{\partial x_k}\right) = (0,0) \quad \text{on} \quad \partial\Omega.$$

By using the Neumann B.C. of u and v, we have $\nabla u = 0$ and $\nabla v = 0$ at any point x_0 such that the Gaussian curvature of $\partial\Omega$ at $x = x_0$ is not 0. The set of all such points form a nonempty open subset Γ of the boundary $\partial\Omega$. Therefore for each $k = 1, 2, \ldots, n$, $\Psi = {}^t(\partial u/\partial x_k, \partial v/\partial x_k)$ satisfies

$$\begin{cases} \Delta\Psi + \begin{pmatrix} 1 - 3u^2 - v^2 & -2uv \\ -2uv & 1 - u^2 - 3v^2 \end{pmatrix} \Psi = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in} \quad \Omega \\ \Psi = \partial\Psi/\partial\nu = {}^t(0,0) \quad \text{on} \quad \Gamma \end{cases}$$

By applying a kind of the Holmgren's uniqueness theorem, $\Psi \equiv {}^t(0,0)$ in Ω. This is equivalent to say that u and v are constant functions. This is a contradiction to the assumption we have. ∎

§2. Stable solutions in a rotational domain

We consider what kind and how complicated the domain should be so that we have a nontrivial stable solutions for (3) and (4). In this section we deal with (4).

$$\text{(4)} \qquad \begin{cases} \Delta\Phi + \lambda(1 - |\Phi|^2)\Phi = 0 \quad \text{in} \quad \Omega, \\ \partial\Phi/\partial\nu = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Let D a bounded domain in $\mathbf{R}^{n-1}$ with a smooth boundary and $R > 0$ a constant. We set a domain $\Sigma(\epsilon) \subset \mathbf{R}^{n-1}$, $\Omega(\epsilon) \subset \mathbf{R}^n$ $(\epsilon > 0)$ as follows,

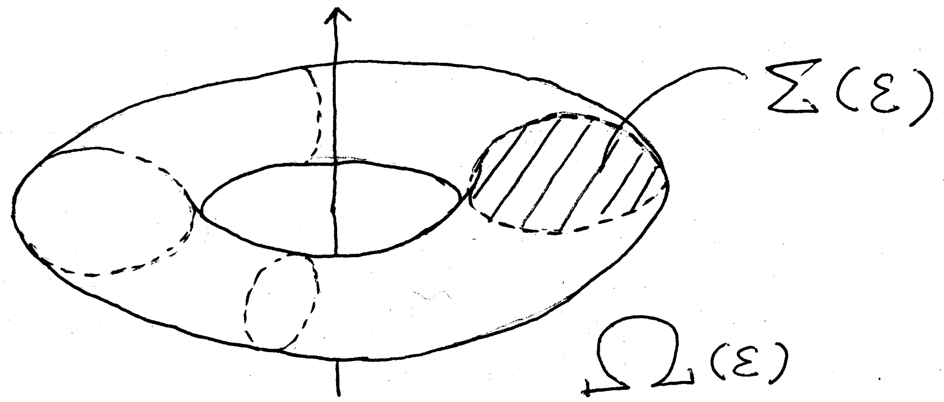
$$\Sigma(\epsilon) = \{(R + \epsilon y_1, \epsilon y_2, \epsilon y_3, \ldots, \epsilon y_{n-1}) \in \mathbf{R}^{n-1} \mid (y_1, \ldots, y_{n-1}) \in D\},$$

$$\Omega(\epsilon) = \{(y_1 \cos\theta, y_1 \sin\theta, y') \in \mathbf{R}^n \mid y = (y_1, y') \in \Sigma(\epsilon),\ \theta \in S^1\},$$

where $y' = (y_2, \ldots, y_{n-1})$ and $S^1 = [0, 2\pi)/\sim$ is a circle.

We introduce the cylindrical coordinate system, $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_k = z_{k-2}$ $(3 \le k \le n)$ and the notation $\widehat{\Delta} = \sum_{k=1}^{n-2} \partial^2/\partial z_k^2$. The equation (4) is rewritten as follows,

$$(5)\qquad \begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial\Phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2\Phi}{\partial\theta^2} + \widehat{\Delta}\Phi + \lambda(1-|\Phi|^2)\Phi = 0 \quad \text{in} \quad \Omega(\epsilon), \\ \dfrac{\partial\Phi}{\partial\nu} = 0 \quad \text{on} \quad \partial\Omega(\epsilon). \end{cases}$$



We seek for a solution of (4) for $\Omega = \Omega(\epsilon)$ in a particular form.

$$(6)\qquad \Phi(x) = W(r,z)e^{im\theta}$$

where $z = (z_1, ..., z_{n-2})$ and m is a non-zero integer. Substituting (6) into (5) yields the equation of $W = W(r,z)$:

$$(7)\qquad \begin{cases} LW + \lambda(1-W^2)W = 0 \quad \text{in} \quad \Sigma(\epsilon), \\ \partial W/\partial n = 0 \quad \text{on} \quad \partial\Sigma(\epsilon), \end{cases}$$

where

$$LW = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial W}{\partial r}\right) - \frac{m^2}{r^2}W + \widehat{\Delta}W$$

and n is the unit outward vector on $\partial\Sigma(\epsilon)$.

We will see the unique existence of positive solution to (7).

Proposition 2. Assume that $\lambda R^2 > m^2$. Then (7) has a unique positive solution $W = W_\epsilon(r,z)$ such that $0 < W_\epsilon(r,z) < 1$, $(0 < \epsilon < \epsilon_0)$. Moreover the solution $W_\epsilon(r,z)$ satisfies

$$\lim_{\epsilon \to 0} \sup_{(r,z)\in\Sigma(\epsilon)} |W_\epsilon(r,z) - (1 - \frac{m^2}{\lambda R^2})^{1/2}| = 0.$$

(Proof of Proposition 2) We construct a positive solution by the upper-lower solution method. Let

$$w_{1,\epsilon}(r,z) = \left(1 - \frac{m^2}{\lambda(R-\epsilon d)^2}\right)^{1/2}, \quad w_{2,\epsilon}(r,z) = \left(1 - \frac{m^2}{\lambda(R+\epsilon d)^2}\right)^{1/2}$$

where $d > 0$ is a constant such that $B_d(\mathbf{o}) \supset D$ where $B_d(\mathbf{o}) \subset \mathbf{R}^{n-1}$ is the ball of radius d centered at the origin $\mathbf{o}$. It is easy to see by the aid of a little calculation that $w_{1,\epsilon}$ and $w_{2,\epsilon}$ are a upper solution and a lower solution of (7), respectively for small $\epsilon > 0$. Hence we have a solution W_ϵ of (7) satisfying,

$$w_{1,\epsilon}(r,z) \le W_\epsilon(r,z) \le w_{2,\epsilon}(r,z).$$

We prove the uniqueness of positive solution. Let us assume that there are two positive solutions W, Y. Taking $\eta > 0$ very small so that

$$\eta \le W(r,z) \le 1-\eta, \ \eta \le Y(r,z) \le 1-\eta \quad \text{in} \quad \Sigma(\epsilon),$$

(We know from the maximum principle that the maximum of any positive solution is less than 1). We can take $w_1(r,z) = \eta$, $w_2(r,z) = 1-\eta$ as lower solution and upper solution by taking η smaller if necessary. Applying again the same argument as above, we get maximum solution $\widehat{w}_2$ and minimum solution $\widehat{w}_1$ ($\widehat{w}_1(r,z) \le \widehat{w}_2(r,z)$ in $\Sigma(\epsilon)$).

$$0 = \int_{\Sigma(\epsilon)} \left(\widehat{w}_2(L\widehat{w}_1 + \lambda\widehat{w}_1(1-\widehat{w}_1^2)) - \widehat{w}_1(L\widehat{w}_2 + \lambda\widehat{w}_2(1-\widehat{w}_2^2))\right) r\,dr\,dz$$

$$= \lambda \int_{\Sigma(\epsilon)} \widehat{w}_2\widehat{w}_1 \left(\widehat{w}_2^2 - \widehat{w}_1^2\right) r\,dr\,dz \ge 0.$$ This implies uniqueness. ∎

Analysis of the equation on S^1

We consider

$$S(\theta) = \overline{w}e^{im\theta} = p(\theta) + iq(\theta),$$

where we put

$$\overline{w} = \left(1 - \frac{m^2}{\lambda R^2}\right)^{1/2}.$$

This is a solution of the following ODE on S^1,

$$\frac{1}{R^2}\frac{d^2S}{d\theta^2} + \lambda(1-|S|^2)S = 0 \quad \text{in} \quad S^1, \tag{8}$$

which we expect to be the limit equation of (4) ($\Omega = \Omega(\epsilon)$) as $\epsilon \to 0$. The linearized eigenvalue problem of (8) at $S = S(\theta)$ is

$$L_0\Psi + \mu\Psi = 0 \quad \text{in} \quad S^1, \tag{9}$$

where $\Psi(\theta) = {}^t(\psi_1(\theta), \psi_2(\theta))$ and the operator

$$L_0\Psi \equiv \frac{1}{R^2}\frac{d^2}{d\theta^2}\Psi + \lambda(1-|S(\theta)|^2)\Phi - 2\lambda \begin{pmatrix} p(\theta)^2 & p(\theta)q(\theta) \\ p(\theta)q(\theta) & q(\theta)^2 \end{pmatrix} \Psi.$$

Proposition 3 ([11]). We denote the set of eigenvalues of (9) by $\{\mu_k\}_{k=1}^{\infty}$ arranged in increasing order with counting multiplicity. Let us suppose that $2\lambda R^2 > 6m^2 - 1$. Then we have $\mu_1 = 0$, $\mu_2 > 0$. Equivalently, there exists a constant $\delta_0 > 0$ such that the following coercive inequality holds,

$$\begin{aligned} &\int_{S^1} \left[\frac{1}{R^2}\left\{(d\psi_1/d\theta)^2 + (d\psi_2/d\theta)^2\right\} - \frac{m^2}{R^2}(\psi_1^2+\psi_2^2) \right. \\ &\left. + 2\overline{w}^2(\psi_1\cos m\theta + \psi_2 \sin m\theta)^2\right] d\theta \geq \delta_0 \int_{S^1}(\psi_1^2+\psi_2^2)d\theta, \end{aligned} \tag{10}$$

for any $\psi_1, \psi_2 \in H^1(S^1)$ satisfying

$$\int_{S^1}(\psi_1 \sin m\theta - \psi_2 \cos m\theta)d\theta = 0. \tag{11}$$

From this information on solution on the circle S^1, we can deduce the stability of (nearby) solution on the thin rotational domain $\Omega(\epsilon)$.

Theorem 4. Suppose that $2\lambda R^2 > 6m^2 - 1$. Then there exists a $\epsilon_0 > 0$ such that (4) ($\Omega = \Omega(\epsilon)$) has a stable solution $\Phi_\epsilon(x) = W_\epsilon(r,z)e^{im\theta}$ for $0 < \epsilon < \epsilon_0$.

(Proof of Theorem 4) The linearized eigenvalue problem around the solution $\Phi_\epsilon = W_\epsilon e^{im\theta}$ is given by

$$\begin{cases} \mathcal{L}\mathbf{p} + \mu\mathbf{p} = 0 \quad \text{in} \quad \Omega(\epsilon), \\ \text{Neumann B.C.} \end{cases} \tag{12}$$

where

$$\mathcal{L}\mathbf{p} \equiv \Delta\mathbf{p} + \begin{pmatrix} 1 - 3\widetilde{u}^2 - \widetilde{v}^2 & -2\widetilde{u}\widetilde{v} \\ -2\widetilde{u}\widetilde{v} & 1 - \widetilde{u}^2 - 3\widetilde{v}^2 \end{pmatrix}\mathbf{p}$$

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \widehat{\Delta} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

This is written real vector form.
Denote the solution $\Phi_\epsilon(x) = \widetilde{u} + i\widetilde{v}$ ($\widetilde{u} = W_\epsilon \cos m\theta$, $\widetilde{v} = W_\epsilon \sin m\theta$). We have that the first eigenvalue $\mu_1(\epsilon)$ of (12) is 0 from the symmetry of equation and corresponding eigenfunction can be taken by ${}^t(-\widetilde{v}, \widetilde{u})$. We will prove the simplicity of $\mu_1(\epsilon) = 0$, that is, a coerciveness of the quadratic form given by the differential operator $\mathcal{L}$. We define the following

$$\mathcal{E}(\phi,\psi) = \int_{\Omega(\epsilon)} (|\nabla\phi|^2 + |\nabla\psi|^2 + H(\phi,\psi))dx \tag{13}$$

where

$$H(\phi,\psi) = (-1 + \widetilde{u}^2 + \widetilde{v}^2)(\phi^2 + \psi^2) + 2(\widetilde{u}\phi + \widetilde{v}\psi)^2.$$

What we have to prove is the coerciveness of $\mathcal{E}(\phi,\psi)$ for $(\phi,\psi) \in H^1(\Omega(\epsilon))$ such that

$$\int_{\Omega(\epsilon)} (-\widetilde{v}\phi + \widetilde{u}\psi)dx = 0, \tag{14}$$

equivalently

$$\int_{S^1\times\Sigma(\epsilon)} W_\epsilon(r,z)(-\sin m\theta\phi + \cos m\theta\psi) r dr dz d\theta = 0. \tag{15}$$

Let $(\phi,\psi)\in H^1(\Sigma(\epsilon)\times S^1)^2$ and define

$$\begin{pmatrix}\phi^\dagger\\ \psi^\dagger\end{pmatrix} \equiv \begin{pmatrix}\phi\\ \psi\end{pmatrix} - \alpha(r,z)\begin{pmatrix}-\sin m\theta\\ \cos m\theta\end{pmatrix}, \tag{16}$$

$$\alpha(r,z) \equiv \frac{1}{2\pi}\int_{S^1}(-\phi(r,z,\theta)\sin m\theta + \psi(r,z,\theta)\cos m\theta)d\theta. \tag{17}$$

Then $(\phi^\dagger,\psi^\dagger)$ satisfies (11) for any $(r,z)\in\Sigma(\epsilon)$ and so does it (10). After a little computation we obtain

$$\begin{aligned}&\int_{S^1}\Big(\frac{1}{R^2}(\phi_\theta^2+\psi_\theta^2) - \frac{m^2}{R^2}(\phi^2+\psi^2) + 2\overline{w}^2(\phi\cos m\theta + \psi\sin m\theta)^2\Big)d\theta\\ &\qquad \geq \delta_0\left\{\int_{S^1}(\phi^2+\psi^2)d\theta - 2\pi\alpha(r,z)^2\right\}.\end{aligned} \tag{18}$$

To bound the second eigenvalue $\mu_2(\epsilon)$ from below, we estimate the following quadratic form by the aid of (18)

$$\begin{aligned}\mathcal{E}(\phi,\psi) = &\int_{S^1\times\Sigma(\epsilon)}(\phi_r^2+\psi_r^2+|\nabla_z\phi|^2+|\nabla_z\psi|^2) r dr dx d\theta\\ &+\int_{S^1\times\Sigma(\epsilon)}\frac{1}{r^2}(\phi_\theta^2+\psi_\theta^2) r dr dz d\theta\\ &-\int_{S^1\times\Sigma(\epsilon)}(1-W_\epsilon^2)(\phi^2+\psi^2) r dr dz d\theta\end{aligned}$$

$$+2\int_{S^1\times\Sigma(\epsilon)} W_\epsilon^2(\phi\cos m\theta+\psi\sin m\theta)^2 rdrdzd\theta$$

$$\geq \int_{S^1\times\Sigma(\epsilon)}(\phi_r^2+\psi_r^2+|\nabla_z\phi|^2+|\nabla_z\psi|^2)rdrdzd\theta$$

$$+\int_{S^1\times\Sigma(\epsilon)}(\frac{m^2}{r^2}-(1-W_\epsilon^2))(\phi^2+\psi^2)rdrdzd\theta$$

$$+\int_{S^1\times\Sigma(\epsilon)}2(W_\epsilon^2-\frac{R^2}{r^2}\overline{w}^2)(\phi\cos m\theta+\psi\sin m\theta)^2 rdrdzd\theta$$

$$+\delta_0\int_{S^1\times\Sigma(\epsilon)}\frac{R^2}{r^2}(\phi^2+\psi^2)rdrdzd\theta$$

$$-2\pi\int_{\Sigma(\epsilon)}\frac{R^2\alpha(r,z)^2}{r^2}rdrdz$$

$$\equiv I_1+I_2+I_3+I_4+I_5.$$

From Proposition 2, I_2 and I_3 can be absorbed in I_4 for small $\epsilon>0$. Hence only the negative term $-\int_{\Sigma(\epsilon)}\alpha(r,z)^2rdrdz = I_5$ is the problem. By the second Poincare inequality, we estimate as follows

$$\int_{\Sigma(\epsilon)}\alpha(r,z)^2rdrdz\leq\frac{1}{\sigma_2(\epsilon)}\int_{\Sigma(\epsilon)}(\alpha_r^2+|\nabla_z\alpha|^2)rdrdz+\frac{(\int_{\Sigma(\epsilon)}\alpha(r,z)rdrdz)^2}{\int_{\Sigma(\epsilon)}rdrdz}.$$

Denote the right hand side by I_5'. $\sigma_2(\epsilon)$ is the second eigenvalue of the operator

$$Z\equiv\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\tilde{\Delta}\quad\text{in}\quad\Sigma(\epsilon)$$

with Neumann B.C. on $\partial\Sigma(\epsilon)$. We remark that $\lim_{\epsilon\to0}\sigma_2(\epsilon)\epsilon^2=\kappa_2$ where $\kappa_2>0$ is the second eigenvalue of $n-1$ dim Laplacian in D with Neumann B.C. and hence there exists a constant $c>0$ such that $\sigma_2(\epsilon)\geq c/\epsilon^2$ for small $\epsilon>0$. Using the definition of $\alpha(r,z)$ and the Schwarz's inequality,

$$\int_{\Sigma(\epsilon)}(\alpha_r^2+|\nabla_z\alpha|^2)rdrdz\leq\frac{1}{2\pi}\int_{S^1\times\Sigma(\epsilon)}(\phi_r^2+\psi_r^2+|\nabla_z\phi|^2+|\nabla_z\psi|^2)rdrdz.$$

Threrefore the first term in I_5' can be absorbed in I_1. From the condition, we estimate the sencond term of I_5'.

$$\overline{w}\int_{\Sigma(\epsilon)}\alpha(r,z)rdrdz = \frac{1}{2\pi}\int_{S^1\times\Sigma(\epsilon)}(-\phi\sin m\theta+\psi\cos m\theta)rdrdzd\theta$$

$$=\frac{1}{2\pi}\int_{S^1\times\Sigma_2(\epsilon)}(\overline{w}-W_\epsilon)(-\phi\sin m\theta+\psi\cos m\theta)rdrdzd\theta.$$

Here we used (15). From Proposition 2, this can also be absorbed in I_5. We have that

$$\mathcal{E}(\phi,\psi)\geq\frac{\delta_0}{2}\int_{S^1\times\Sigma(\epsilon)}(\phi^2+\psi^2)rdrdzd\theta \quad (\text{small} \quad \epsilon>0),$$

which concludes Theorem 4. ∎

§3. Solutions with magnetic effect

In this section we show that the solution Φ_ϵ in Theorem 2 can be approximation of solution of the original equation (3) when $\epsilon >$ is small. We will deal with only the two dimensional case.

In this case the domain $\Omega(\epsilon)\subset\mathbf{R}^2$ is expressed as follows.

$$\Omega(\epsilon)=\{(x_1,x_2)\in\mathbf{R}^2 \mid R-\epsilon<\sqrt{x_1^2+x_2^2}<R+\epsilon\}.$$

We denote $\partial\Omega(\epsilon)=\Gamma_1(\epsilon)\cup\Gamma_2(\epsilon)$ where $\Gamma_1(\epsilon)$ is the inner circle and $\Gamma_2(\epsilon)$ is the outer one. We seek for a (rotational) solution (Φ, A) of (3) in the following form,

$$A(r,\theta)=Y(r)\left(\frac{-\sin\theta}{r},\frac{\cos\theta}{r}\right),\quad \Phi(r,\theta)=W(r)e^{im\theta}.$$

By a direct calculation, W and Y should satisfy the following equations,

$$(19)\qquad \begin{cases} h^2\Delta W+\lambda W(1-W^2)-\dfrac{1}{r^2}(mh-Y)^2W=0 \quad \text{in} \quad \Omega(\epsilon), \\ \Delta Y-\dfrac{2}{r}\dfrac{\partial Y}{\partial r}+(mh-Y)W^2=0 \quad \text{in} \quad \Omega(\epsilon), \end{cases}$$

$$(20) \qquad \begin{cases} \Delta Y - \dfrac{2}{r}\dfrac{\partial Y}{\partial r} = 0 \quad \text{in} \quad \Omega(\epsilon)^c, \\ Y(0) = 0. \end{cases}$$

(20) is solved as follows, $Y(r) = c_1 r^2$ in $0 < r < R - \epsilon$ and $Y(r) = c_2$ in $r > R + \epsilon$ where c_1 and c_2 are unknown constants. Hence the system of equations (19)-(20) can be rewritten as follows,

$$(21) \qquad \begin{cases} h^2 \Delta W + \lambda W(1 - W^2) - \dfrac{1}{r^2}(mh - Y)^2 W = 0 \quad \text{in} \quad \Omega(\epsilon), \\ \dfrac{\partial W}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega(\epsilon), \\ \Delta Y - \dfrac{2}{r}\dfrac{\partial Y}{\partial r} + (mh - Y)W^2 = 0 \quad \text{in} \quad \Omega(\epsilon), \\ \dfrac{\partial Y}{\partial \nu} + \dfrac{2}{R - \epsilon} Y = 0 \quad \text{on} \quad \Gamma_1(\epsilon), \quad \dfrac{\partial Y}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_2(\epsilon). \end{cases}$$

Proposition 5. There exists a $\epsilon_2 > 0$ such that (21) has a solution (W_ϵ, Y_ϵ) for $0 < \epsilon < \epsilon_2$ such that

$$\lim_{\epsilon \to 0} \sup_{|r - R| \le \epsilon} |W_\epsilon(r) - (1 - \frac{h^2 m^2}{\lambda R^2})^{1/2}| = 0,$$

$$\lim_{\epsilon \to 0} \sup_{|r - R| \le \epsilon} |\partial W_\epsilon(r)/\partial r| = 0, \ \lim_{\epsilon \to 0} \sup_{r \ge 0} |Y_\epsilon(r)| = 0,$$

where $\overline{w} = (1 - h^2 m^2/\lambda R^2)^{1/2}$.

(Proof of Prop. 5) First we remark that the equation (21) is a cooperation system in the region $0 \le W \le 1, \ 0 \le Y \le mh$ and hence we can apply the upper-lower solution method. We construct a upper solution (W_2, Y_2) and a lower solution (W_1, Y_1) as follows,

$$Y_2 = mh, \quad W_2 = 1$$

$$Y_1 = \eta_2(1 + \frac{2\epsilon}{R-\epsilon}) - \frac{\eta_2}{2\epsilon(R-\epsilon)}(r - R - \epsilon)^2, \quad W_1 = \eta_1.$$

in the region $|r - R| \leq \epsilon$. If $\eta_1 > 0$ and $\eta_2 > 0$ are small we have

$$0 < Y_1(r) \leq Y_2(r) \leq mh, \quad 0 < W_1 \leq W_2 \leq 1.$$

It is easy to see that (W_2, Y_2) is an upper solution. If we take $\eta_1 = \epsilon$ and $\eta_2 = \epsilon^4$, (W_1, Y_1) is a lower solution for small $\epsilon > 0$. Hence an existence of solutio of (21) yields. The asymptotic behavior of this solution as $\epsilon \to 0$ can be seen by the stretching at $r = R$. ∎

Theorem 6. Under the same situation, there exists a $\epsilon_1 > 0$ such that for $0 < \epsilon < \epsilon_1$, (3) $(\Omega = \Omega(\epsilon))$ has a stable solution $(\widehat{\Phi}_\epsilon, A_\epsilon)$ with $\widehat{\Phi}_\epsilon \sim \Phi_\epsilon$ in $\Omega(\epsilon)$ and $A_\epsilon \sim 0$.

Hereafter in this section, we describe a sketch of the proof of Theorem 6. However we only give ideas and the line of the proof. For the study of the stability of the solution constructed in Prop. 5, we need the following second variational formula. By showing positivity of this quadratic form in some sense, we prove the stability.

Formula of second variation of $\mathcal{H}(\Phi, A)$

(22) $$\mathcal{E}(p, q, \Psi) = \frac{1}{2}\frac{d^2}{d\epsilon^2}_{|\epsilon=0}\mathcal{H}(u + \epsilon p, v + \epsilon q, A + \epsilon\Psi) =$$

$$\int_\Omega \left\{h^2(|\nabla p|^2 + |\nabla q|^2) - \lambda(1 - u^2 - v^2)(p^2 + q^2) + 2\lambda(up + vq)^2\right\} dx$$

$$+ \int_\Omega \left\{A^2(p^2 + q^2) + 4\langle A \cdot \Psi\rangle(up + vq) - 2h(p\langle\nabla q \cdot A\rangle - q\langle\nabla p \cdot A\rangle)\right\} dx$$

$$+ \int_{\mathbf{R}^n} (|\mathrm{rot}\Psi|^2 + (u^2 + v^2)\Psi^2) dx$$

(23)

$$I(\psi_1,\psi_2) = \int_{S^1} (\frac{h^2}{R^2}((d\psi_1/d\theta)^2 + (d\psi_2/d\theta)^2)$$
$$- \lambda(1-\overline{w}^2)(\psi_1^2+\psi_2^2) + 2\lambda\overline{w}^2(\psi_1 \cos m\theta + \psi_2 \sin m\theta)^2 d\theta$$
$$= \int_{S^1} (\frac{h^2}{R^2}((d\psi_1/d\theta)^2 + (d\psi_2/d\theta)^2)$$
$$- \frac{h^2m^2}{R^2}(\psi_1^2+\psi_2^2) + 2(\lambda - \frac{h^2m^2}{R^2})(\psi_1 \cos m\theta + \psi_2 \sin m\theta)^2) d\theta.$$

for $\psi_1, \psi_2 \in H^1(S^1)$. From (10), we know that this form is coercive, we need more detailed information about the coerciveness to apply to the equation with magnetic effect. By putting

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \cos m\theta & -\sin m\theta \\ \sin m\theta & \cos m\theta \end{pmatrix} \begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix}, \; \widehat{I}(\widehat{\psi}_1, \widehat{\psi}_2) = I(\psi_1, \psi_2),$$

we have,

$$\widehat{I}(\widehat{\psi}_1, \widehat{\psi}_2) = \int_{S^1} (\frac{h^2}{R^2}((\frac{d\widehat{\psi}_1}{d\theta})^2 + (\frac{d\widehat{\psi}_2}{d\theta})^2 \tag{23}$$
$$+ 2m(-\widehat{\psi}_2 \frac{d\widehat{\psi}_1}{d\theta} + \widehat{\psi}_1 \frac{d\widehat{\psi}_2}{d\theta})) + 2(\lambda - \frac{h^2m^2}{R^2})\widehat{\psi}_1^2) d\theta,$$

Proposition 7. For any constants $\eta > 0$ and $\tau > 0$, there exists a constant $\lambda_* > h^2m^2/R^2$ such that for $\lambda > \lambda_*$

$$\widehat{I}(\widehat{\psi}_1, \widehat{\psi}_2) \geq \tau \int_{S^1} \widehat{\psi}_1^2 d\theta + \pi(\frac{h^2}{2R^2} - \eta) \int_{S^1} \widehat{\psi}_2^2 d\theta \tag{24}$$

for any $\widehat{\psi}_1, \widehat{\psi}_2 \in H^1(S^1)$.

$$-2h\int_{\Omega}\{p\langle\nabla v\cdot\Psi\rangle - q\langle\nabla u\cdot\Psi\rangle + u\langle\nabla q\cdot\Psi\rangle - v\langle\nabla p\cdot\Psi\rangle\}\,dx$$

where $p, q : \Omega \longrightarrow \mathbf{R}$, $\Psi : \mathbf{R}^n \longrightarrow \mathbf{R}^n$. n is 2 or 3. This is the term of degree 2 of the Taylor expansion of $\mathcal{H}(u+p, v+q, A+\Psi)$ at $(p, q, \Psi) = (0, 0, 0)$. To see the stability of the solution (u, v, A) we have only to check the positivity of this quadratic form $\mathcal{E}(p, q, \Psi)$ toward the normal direction to the (P) invariant subset through the solution (u, v, A).

We will prove that this form is positive definite on the space E which is normal to tangent space of the (P) invariant subset through the solution.

$$E = \{(p, q, \Psi) \mid \int_{\Omega}(pv - qu)dx = 0,\ \mathrm{div}\Psi = 0,\ \mathrm{in}\,\Omega,\ \langle\Psi\cdot\nu\rangle = 0,\ \mathrm{on}\,\partial\Omega\}$$

As in the case of the equation (4), we reduce the problem in $\Omega(\epsilon)$ ($\epsilon > 0$: small) to that in S^1. From Prop. 5, we know that the asymptotic behavior of

$$\Phi_\epsilon(r,\theta) = W_\epsilon(r)e^{im\theta}, \quad A_\epsilon(r,\theta) = {}^t\left(-\frac{Y_\epsilon(r)\sin\theta}{r}, \frac{Y_\epsilon(r)\cos\theta}{r}\right)$$

with the aid of

$$\lim_{\epsilon\to 0}\sup_{[R-\epsilon,R+\epsilon]}|W_\epsilon(r) - \overline{w}| = 0, \quad \lim_{\epsilon\to 0}\sup_{r\geq 0}|Y_\epsilon(r)| = 0.$$

We denote those lines of the right hand side of (22) (Variational formula) by J_1, J_2, J_3, J_4 respectively from the top to bottom. J_1 is the most important term which we should rely on and its positivity comes from the stability of the solution of (8). J_3 is a good term because it is coercive in E. J_2 looks complicated but as A converges uniformly to 0, so it can be absorbed by other coercive terms. J_4 is a problem. We have to deal with this very carefully with the coercivity of J_1. To see J_1, we investigate the reduced quadratic form of J_1. We introduce the following form I which is almost left hand side of (10).

(Sketch of the proof of Prop. 7) We use the Fourier expansion of $\widehat{\psi}_1$ and $\widehat{\psi}_2$ as follows

$$\begin{pmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\xi_0 + \sum_{k=1}^{\infty}(\xi_k \cos k\theta + \zeta_k \sin k\theta)$$

where $\xi_k = {}^t(\xi_{k,1}, \xi_{k,2})$, $\zeta_k = {}^t(\zeta_{k,1}, \zeta_{k,2}) \in \mathbf{R}^2$. Here we see

$$\|\widehat{\psi}_1\|^2_{L^2(S^1)} + \|\widehat{\psi}_2\|^2_{L^2(S^1)} = \pi\xi_0^2 + \pi\sum_{k=1}^{\infty}(\xi_k^2 + \zeta_k^2). \tag{25}$$

Substitute this into $\widehat{I}$, we have,

$$\widehat{I}(\{\xi_k\}_{k=0}^{\infty}, \{\zeta_k\}_{k=1}^{\infty}) = 2\pi(\lambda - \frac{h^2m^2}{R^2})\xi_{0,1}^2 + \pi\sum_{k=1}^{\infty}({}^t\xi_k, {}^t\zeta_k)\, B_k \begin{pmatrix} \xi_k \\ \zeta_k \end{pmatrix}$$

where each $B_k(\lambda)$ $(k \geq 1)$ is a 4×4 real matrix given by,

$$B_k(\lambda) = \begin{pmatrix} \frac{h^2k^2}{R^2} + 2\lambda - \frac{2h^2m^2}{R^2} & 0 & 0 & \frac{2h^2mk}{R^2} \\ 0 & \frac{h^2k^2}{R^2} & -\frac{2h^2mk}{R^2} & 0 \\ 0 & -\frac{2h^2mk}{R^2} & \frac{h^2k^2}{R^2} + 2\lambda - \frac{2h^2m^2}{R^2} & 0 \\ \frac{2h^2mk}{R^2} & 0 & 0 & \frac{h^2k^2}{R^2} \end{pmatrix}.$$

By a simple calculation, we obtain 2 distinct eigenvalues $\mu_{k,-}(\lambda) < \mu_{k,+}(\lambda)$ as follows,

$$\mu_{k,\pm}(\lambda) = \frac{h^2k^2}{R^2} + \lambda - \frac{h^2m^2}{R^2} \pm \sqrt{(\lambda - \frac{h^2m^2}{R^2})^2 + \frac{6h^4k^2m^2}{R^4}}. \tag{26}$$

It is easy to see that

$$\mu_{k,-}(\lambda) \geq \frac{h^2}{R^2}k(k - \sqrt{6}|m|) \tag{27}$$

if $k \geq \sqrt{6}|m|$ and $\lambda \geq h^2m^2/R^2$.

Corresponding eigenvectors can be chosen as follows,

$$\mu_{k,-}(\lambda) \longrightarrow {}^t(\alpha_k(\lambda), 0, 0, \beta_k(\lambda)),\ {}^t(0, -\beta_k(\lambda), \alpha_k(\lambda), 0),$$

$$\mu_{k,+}(\lambda) \longrightarrow {}^t(-\beta_k(\lambda), 0, 0, \alpha_k(\lambda)),\ {}^t(0, \alpha_k(\lambda), \beta_k(\lambda), 0).$$

where

$$\alpha_k(\lambda) = \frac{-2h^2mk/R^2}{\left\{\left(\lambda - \frac{h^2m^2}{R^2} + \sqrt{(\lambda - \frac{h^2m^2}{R^2})^2 + \frac{6h^4m^2k^2}{R^4}}\right)^2 + (\frac{2h^2mk}{R^2})^2\right\}^{1/2}}$$

$$\beta_k(\lambda) = \frac{\lambda - \frac{h^2m^2}{R^2} + \sqrt{(\lambda - \frac{h^2m^2}{R^2})^2 + \frac{6h^4m^2k^2}{R^4}}}{\left\{\left(\lambda - \frac{h^2m^2}{R^2} + \sqrt{(\lambda - \frac{h^2m^2}{R^2})^2 + \frac{6h^4m^2k^2}{R^4}}\right)^2 + (\frac{2h^2mk}{R^2})^2\right\}^{1/2}}.$$

We remark that

(28) $$\alpha_k(\lambda) \sim O(1/\lambda),\ \lim_{\lambda\to\infty} \beta_k(\lambda) = 1,$$

(29) $$\lim_{\lambda\to\infty} \mu_{k,+}(\lambda)/\lambda = 2,\ \lim_{\lambda\to\infty} \mu_{k,-}(\lambda) = \frac{h^2k^2}{R^2},$$

for each $k \geq 1$. The following expression will be complicated and we will denote $B_k(\lambda) = B_k$, $\alpha_k(\lambda) = \alpha_k$, $\beta_k(\lambda) = \beta$ by dropping λ, $\mu_\pm(\lambda) = \mu_\pm$. We have

(30)

$$({}^t\xi_k, {}^t\zeta_k)B_k \begin{pmatrix} \xi_k \\ \zeta_k \end{pmatrix} = \mu_{k,-}\left\{(\alpha_k\xi_{k,1} + \beta_k\zeta_{k,2})^2 + (-\beta_k\xi_{k,2} + \alpha_k\zeta_{k,1})^2\right\}$$

$$+\mu_{k,+}\left\{(-\beta_k\xi_{k,1} + \alpha_k\zeta_{k,2})^2 + (\alpha_k\xi_{k,2} + \beta_k\zeta_{k,1})^2\right\}$$

$$\geq \mu_{k,-}\left\{(\beta_k\zeta_{k,2})^2/2-(\alpha_k\xi_{k,1})^2+(\beta_k\xi_{k,2})^2/2-(\alpha_k\zeta_{k,1})^2\right\}$$
$$+\mu_{k,+}\left\{(\beta_k\xi_{k,1})^2/2-(\alpha_k\zeta_{k,2})^2+(\beta_k\zeta_{k,1})^2/2-(\alpha_k\xi_{k,2})^2\right\}$$
$$=(\beta_k^2\mu_{k,-}/2-\alpha_k^2\mu_{k,+})(\zeta_{k,2}^2+\xi_{k,2}^2)$$
$$+(\beta_k^2\mu_{k,+}/2-\alpha_k^2\mu_{k,-})(\xi_{k,1}^2+\zeta_{k,1}^2).$$

Now we estimate $\widehat{I}$ from below. For arbitrary given $\tau > 0$, take k_* so large that $\pi h^2 k_*(k_* - \sqrt{6}|m|)/R^2 \geq \tau$. The important point is that k_* is independent of λ. Using (27) we can estimate the terms in $\widehat{I}$ corresponding to the parameter region $k \geq k_*$. To estimate the terms for $1 \leq k < k_*$, we apply (28) and (29) to (30). From (28) and (29), we have,

$$\lim_{\lambda\to\infty}\alpha_k(\lambda)^2\mu_{k,+}(\lambda)=0,\quad \lim_{\lambda\to\infty}\beta_k(\lambda)^2\mu_{k,-}(\lambda)=h^2k^2/R^2.$$

For any $\eta > 0$, if is λ is large,

$$({}^t\xi_k,{}^t\zeta_k)B_k(\lambda)\begin{pmatrix}\xi_k\\ \zeta_k\end{pmatrix}\geq\left(\frac{h^2}{2R^2}-\eta\right)(\zeta_{k,2}^2+\xi_{k,2}^2)+\tau/\pi(\xi_{k,1}^2+\zeta_{k,2}^2),$$

for $1 \leq k < k_*$. Using (25), we complete the proof of Prop. 7. ∎

Let us go to the estimate of J_4 for small epsilon. We remark that if $\mathrm{div}\Psi = 0$ in Ω and $\langle\Psi\cdot\nu\rangle = 0$ on $\partial\Omega$,

$$\int_\Omega\{p\langle\nabla v\cdot\Psi\rangle-q\langle\nabla u\cdot\Psi\rangle\rangle\}\,dx=\int_\Omega\{u\langle\nabla q\cdot\Psi\rangle-v\langle\nabla p\cdot\Psi\rangle\}\,dx.$$

Hence for any $(p,q,\Psi)\in E$,

$$J_4=-4h\int_\Omega\{p\langle\nabla v\cdot\Psi\rangle-q\langle\nabla u\cdot\Psi\rangle\}\,dx.$$

Now we express the vector field Ψ in the polar-like coordinate system as follows,

$$\Psi=\left(-\frac{\Psi_1\sin\theta}{r},\frac{\Psi_1\cos\theta}{r}\right)+(\Psi_2\cos\theta,\Psi_2\sin\theta).$$

Here Ψ_1 and Ψ_2 are scalar valued functions and moreover we have,

$$|\Psi|^2 = \Psi_1^2/r^2 + \Psi_2^2.$$

J_4 can be expressed

$$J_4 = -4h\left\{\frac{m\Psi_1 W_\epsilon}{r^2}(p\cos m\theta + q\sin m\theta) + \Psi_2\frac{\partial W_\epsilon}{\partial r}(p\sin m\theta - q\cos m\theta)\right\}.$$

From the property in Prop.5, the second term can be absorbed by other terms. The first term remains problem. For infinitesimal $\epsilon > 0$, the first term can be reduce to

$$\frac{m\Psi_1\overline{w}}{R^2}\widehat{\psi}_1$$

which can be dominated by a big term of $\widehat{\psi}_1$ controlled by λ. See Prop.7. Here it is a important point that the coefficient of the second term of (24) is bounded from 0 for infinitesimal $\epsilon > 0$.

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