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ROBUST ANALYSIS OF BOUNDARY SHAPE IDENTIFICATION ARISING IN THERMAL TOMOGRAPHY

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Abstract: This paper is concerned with a choice of regularization parameter for output least square identifications of distributed parameter systems. A method for adjusting the "optimal" regularization parameter is proposed based on the graphical approach. The idea is applied to a geometrical heat inverse problem arising in thermal testing of materials. The effectiveness and validity of the method proposed are shown in both computational experiments and experiments with laboratory data.

1. INTRODUCTION

The output least square identification (OLSI) is a fundamental technique for solving inverse problems. The method has been applied to various kinds of engineering problems including oil exploration surveys, oil recoveries, computed impedance tomographies, etc. The essential difficulties in engineering applications of OLSI come from the fact that real data are heavily corrupted by observation noise and it often occurs that the corresponding model output data are far from those practical data. This means most practical inverse problems exhibit a lack of continuous dependence of the estimated parameters on the data. Tikhonov regularization is one possible technique for avoiding these serious difficulties in computational efforts. Let E be a parameter space and C be the compact admissible subset of E. The data space is denoted by F. Let ϕ be a continuous mapping from C into F. Then the OLSI problem is to seek the minimum solution $\hat{q} \in C$ of the fit to data function

$$J(q) = \frac{1}{2} \|\phi(q) - y_d\|_F^2 . \tag{1}$$

Tikhonov's method is to replace the cost (1) by one well-behaved regularized function

$$J_\eta(q) = J(q) + \frac{\eta}{2} \|q - q_o\|_E^2 \tag{2}$$

where $\|\cdot\|_E$ is the norm (or semi-norm) defined in E and where q_o denotes *a priori* estimate of q. It is well-known that the above method ensures the robustness with respect to perturbation of the data y_d. This paper is concerned with the choice of regularization parameter η. In Section 2, the graphical method is introduced for the choice of regularization parameter. In Section 3, this scheme is applied to boundary estimation problems arising in thermal tomography. In Section 4, the validity of the proposed method is demonstrated through computational experiments. Finally, the method is applied to laboratory data for the corrosion profiles in thermal testing of materials.

2. TIKHONOV REGULARIZATION

To solve the minimization problem of OLSI by regularization is now a fundamental technique for parameter estimation problems. Tikhonov regularization ensures interesting properties of the OLSI problem, such as existence of a minimum, stability with respect to perturbation of the data y_d. Let $\hat{q}(\eta) \in C$ be a solution of

$$J_\eta(\hat{q}(\eta)) = \min_{q \in C} J_\eta(q) \tag{3}$$

such that

$$\lim_{\eta \to 0} \hat{q}(\eta) = \hat{q}(0). \tag{4}$$

Tikhonov showed that there exists a constant $\gamma_o \in [0, \bar{\gamma}]$ such that, for all $\delta > 0$,

$$\|\phi(q_o) - y_d\|_F \leq M\sqrt{\eta} \leq \gamma_o \quad \Longrightarrow \quad \|\hat{q}(\eta) - q_o\|_E \leq \delta. \tag{5}$$

This result gives us a useful insight on the choice of regularization parameter η. Miller [2] suggested that the constant M is set as an *a priori* upper bound on the parameter error δ. This means the regularization parameter can be chosen as

$$\eta = \left(\frac{\gamma_o}{M}\right)^2 \left(\text{or } = \left(\frac{\gamma_o}{\sigma}\right)^2\right). \tag{6}$$

The crucial point of this method is to require an *a priori* upper bound on the measurement error. Tikhonov and Arsenine [1] proposed another useful choice that was often called "Morozov principle":

> Given the upper bound of measurement error
>
> $$\|\phi(q_o) - y_d\|_F \leq \gamma, \tag{7}$$
>
> choose the regularization parameter η in such a way that the residual norm is equal to
>
> $$\|\phi(\hat{q}(\eta)) - y_d\|_F = \gamma. \tag{8}$$

Although this method requires more computational efforts rather than Miller's method, no *a priori* upper bound M is necessary. Adjustment can be done only by trials and errors. Moreover, Kravaris and Seinfeld [3] showed the following asymptotic behavior:

$$\|\hat{q}(\eta(\gamma)) - q_o\|_E \quad \to 0 \qquad \text{as} \quad \|\phi(q_o) - y_d\|_F \leq \gamma \to 0. \tag{9}$$

This approach has been applied to many practical optimization algorithms. However, the range of regularization parameter η^o satisfying the above statement sometimes becomes too large, so that, in practical applications, there are many cases where the feasible regularization parameter $\eta(\gamma)^o$ could not determined by computational efforts. In other words, the "optimal" parameter η^o must be chosen over $\eta(\gamma) \in \Theta$ where Θ denotes the set satisfying the above statement. One possible method for finding the optimal parameter η^o is a graphical approach by means of L-curve. The outline of this approach is summarized as follows:

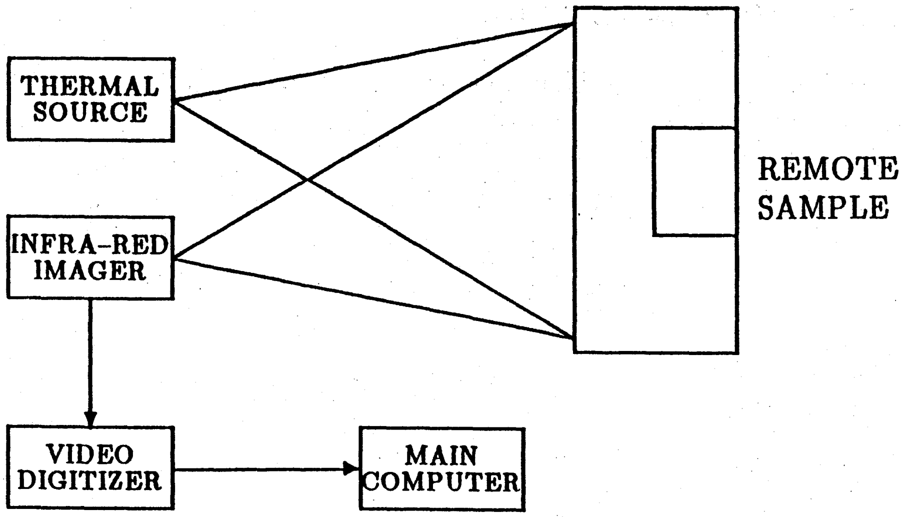


Fig. 1. Experimental set-up of thermal testing of materials

Given η, solve

$$J_\eta(\hat{q}(\eta)) = \min_q J_\eta(q), \tag{10}$$

for $\eta \in [0, \infty)$, collect the points

$$(\|\phi(\hat{q}(\eta)) - y_d\|_F, \|\hat{q}(\eta)\|_E) \tag{11}$$

and plot the above points. Then the "optimal" parameter η^o can be chosen as a point that lies in the "corner" of the resulting curve.

Existence of such "L"-shaped curve was well-known for the ill-posed algebraic linear problems, such as the discrete solver of integral equation of the first kind. Recently, Hansen [4] overviewed this graphical approach for the discrete ill-posed linear problems. More careful discussions have been done by Kitagawa [5]. The plotting curve was first named "L"-curve by Hansen in his paper[1]. Our attempt is to use this method for the nonlinear least square problems.

3. THERMAL TESTING OF MATERIALS

In this section, the method is applied to a geometrical heat inverse problem arising in thermal testing of materials. The problem is motivated by the non-destructive evaluation in aerospace structures. The overall configulation of the evaluation system is illustrated in Fig. 1. As shown in Fig. 1, the problem is to estimate the backsurface corrosion shape using the thermographical data from the front surface. The corresponding inverse problem is formulated as a parameter estimation problem. We restrict our attention in estimating the shape of backsurface in Fig. 2. The corrosion shape is approximated by

[1]The author thanks Professor Kitagawa for my learning this fact from him.

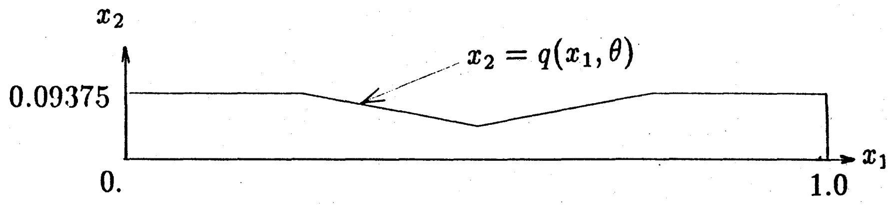


Fig. 2. Sample material with corrosion in computational experiments

linear B-spline

$$x_2 = q(x_1, \theta^M) = \sum_M \theta_i^M B_i^M(x_1). \tag{12}$$

Thus a sample material is parametrized by the finite dimensional vector $\theta^M = \{\theta_i^M\}$. The model output of the thermographical data is obtained by

$$\phi(q) = u(q(\theta^M))\Big|_S \tag{13}$$

that is the trace value of the solution of the following two-dimensional heat diffusion equation:

$$\frac{\partial u}{\partial t} - c \Delta u = 0 \qquad \text{in} \quad \mathcal{T} \times G(q(\theta^M)) \tag{14}$$

with the initial and boundary conditions,

$$\begin{aligned} u(0) &= u_0 \qquad \text{on} \quad \mathcal{T} \times G(q(\theta^M)) & (15) \\ \frac{\partial u}{\partial n} &= g \qquad \text{on} \quad \mathcal{T} \times S & (16) \\ \frac{\partial u}{\partial n} &= 0 \qquad \text{on} \quad \mathcal{T} \times \partial G(q(\theta^M))/S. & (17) \end{aligned}$$

In the above equations, $G(q(\theta^M))$ and $\partial G(q(\theta^M))$ denote a material shape with corrosion and its boundary, respectively. S denotes the front surface that contains $\partial G(q(\theta^M))$. The identification problem is stated as follows:

> Given $\{u_0, g, y_d\}$, determine geometrical structure of the sample material, in other words, estimate θ^M.

The computational techniques we have employed is based on the use of a finite element Galerkin approach to construct a sequence of finite dimensional approximating identification problems. Let us choose $\cup_N\{B_i^N\}$ as a set of basis functions in $H^1(C)$ and approximate the finite Galerkin solution of (14) by

$$u^N(t,q) = \sum_N w_i^N(t,q) B_i^N. \tag{18}$$

Then the coefficient vector $w^N(t,q) = \{w_i^N\}$ yields linear systems of equations:

$$C^N \frac{dw^N(t)}{dt} + A^N(q) w^N(t) = F^N(t,q) \tag{19}$$

with

$$w^N(0,q) = \bar{w}_0^N(q). \tag{20}$$

The computational method for the identification problem is given by

(IDP) Find $\hat{\theta}^{M,N}$ which minimizes

$$J_\eta^{M,N}(\theta^M) = \frac{1}{2}\left\|\phi^{M,N}(q) - y_d\right\|_F^2 + \frac{\eta}{2}\left\|q(\theta^M)\right\|_E^2 \tag{21}$$

subject to

$$\phi^{M,N}(q) = \sum_N w_i^N(t, q(\theta^M))\, B_i^N\Big|_S . \tag{22}$$

and the linear ordinary differential equation (19) with (20).

The purpose of this regularization is to smooth the estimated curve related to corrosion shape. The stabilization of differential components of the estimated curve play an essential role in this smoothing. Taking into account this, the semi-norm of $H^1(0,1)$ is taken as the regularization term:

$$\left\|q(\theta^M)\right\|_E^2 = \left\|\frac{dq}{dx_1}(\theta^M)\right\|_{L^2(0,1)}^2 = \sum_i \left|\theta_i^M - \theta_{i-1}^M\right|^2 . \tag{23}$$

4. COMPUTATIONAL EXPERIMENTS

In our computational experiments, a corrosion shape is assumed to be a steel sample with damage as depicted in Fig. 2. A sample was assumed with dimensions 0.09375 × 1. inch. A test example is constructed as follows. A true parameter vector θ_o^M in (12) is given. The corresponding solution is computed by solving (19). Simulated data are generated by adding random noise to the numerical solution of (19):

$$y_d = \phi^{M,N}(q(\theta_o^M)) + \epsilon \tag{24}$$

where ϵ implies additive disturbance given by Gaussian random sequence with $N(0.\,,\sigma^2)$. In this procedure, the dimensions of the unknown parameter vectors were taken as $dim(\theta^M) = 7$. The value of the true parameters were chosen as

$$\begin{aligned}\theta_o^M \;=\; & [0.08203125, 0.0703125, 0.05859375, 0.46875,\\ & 0.05859375, 0.0703125, 0.08203125].\end{aligned}$$

The number of nodes related to finite elements in the computational experiments were set as $N = 165$. The estimated sequence $\hat{\theta}^M$ was computed by using the optimization routine for nonlinear least square problems. The trust region method is adopted to the problem **(IDP)**. For the implementation of the trust region algorithm, a Fortran software package "OPT2" created by Dr. R.G. Carter was used (see [7] for details). Our aim of

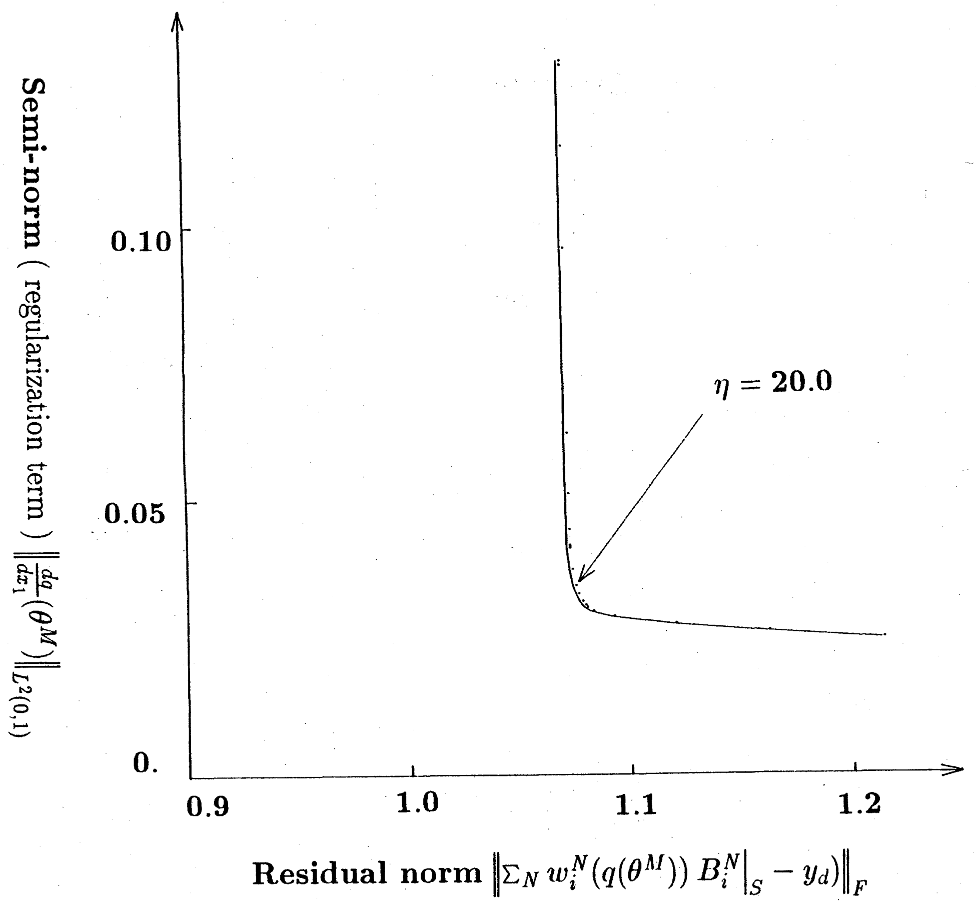


Figure 3. Tikhonov "L"-curve in computational experiments

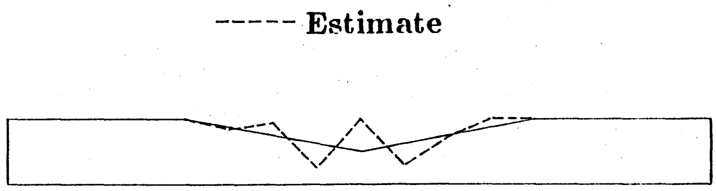
this computational experiments is to confirm the existence of distinct "corner" drawing the curve,

$$\left(\left\|\sum_N w_i^N(q(\theta^M))\; B_i^N\Big|_S - y_d\right\|_F,\; \left\|\frac{dq}{dx_1}(\theta^M)\right\|_{L^2(0,1)}\right).$$

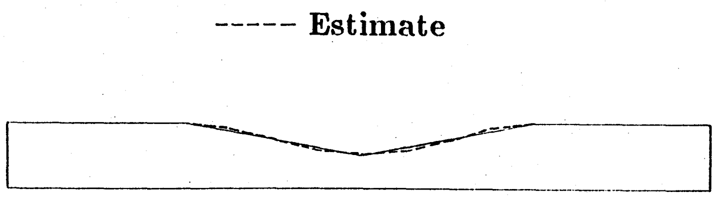
To this end, for each $\eta \in [0, \infty)$, the optimization problem **(IDP)** was solved numerically and the value of residual norm $\left\|\Sigma_N\, w_i^N(q(\theta^M))\; B_i^N\Big|_S - y_d)\right\|_F$ and the semi-norm (regularization term) $\left\|\frac{dq}{dx_1}(\theta^M)\right\|_{L^2(0,1)}$ were computed. Figure 3 depicts the corresponding "L"-curve for the data with the random noise $\sigma = 0.2$. The results illustrates that the curve has a distinct corner around a point corresponding to $\eta = 20$. Figure 4 shows the estimated shape without regularization and the regularized shape with $\eta = 20$.

5. EXPERIMENTS WITH LABORATORY DATA

In this section, the results of using our estimation procedures are reported with experimental data. All experiments were carried out in the Instrument Research Divisions,



(a) $\eta = 0.0$



(b) $\eta = 20.0$

Figure 4. Estimated shape in computational experiments

Nondestructive Measurement Science Branch, NASA Langley Research Center. The experimental data consisted of surface temperature distributions for a four second period after a thermal source was injected to a sample. For detailed discussions, we refer to [6]. A material sample was fabricated to simulate corrosion similar to that shown in Fig. 5 (in this case, a sample with actual dimensions $0.25 \times 1.$inch was used). Figure 6 depicts the Tikhonov "L"-curve for this problem. As shown in Fig. 6, there exists a corner, although the vertical part of the "L"-curve is not steep compared with the results in computational experiments. Nevertheless, the curve in Fig. 6 give us useful information for improving the estimated shape. The successful results is demonstrated in Fig. 7 that depicts the estimated shape without regularization and the regularized shape with $\eta = 10.0$.

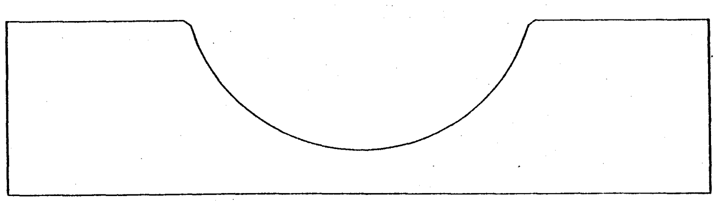


Figure 5. Sectional plan of steel sample used in the experiments

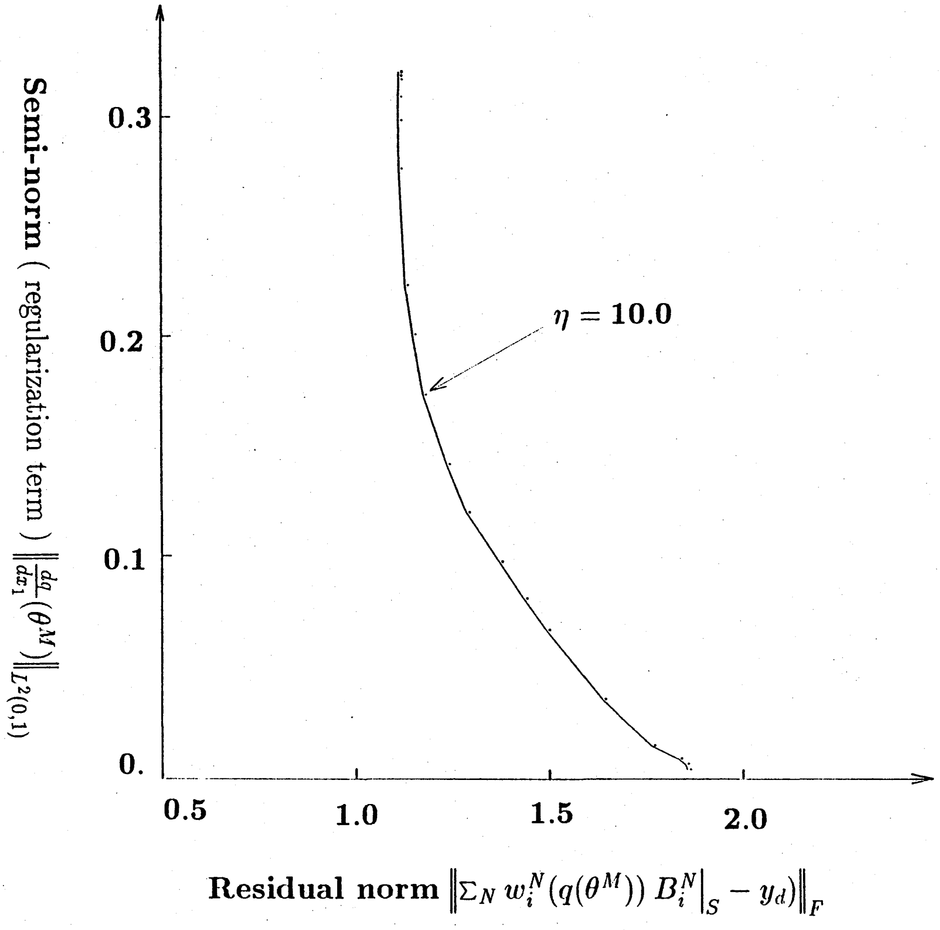
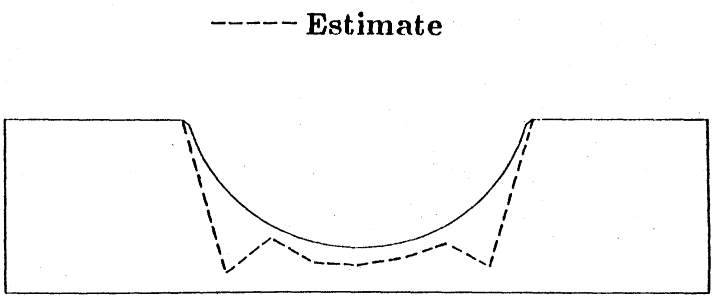
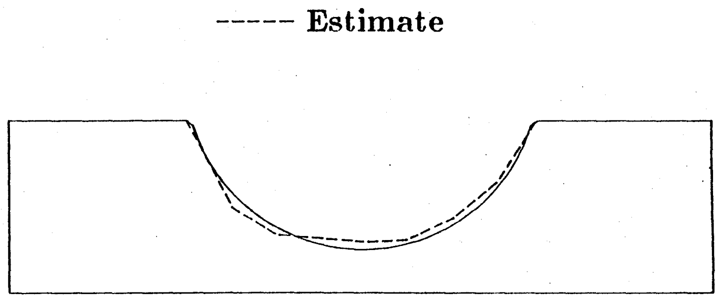


Figure 6. Tikhonov "L"-curve in experiments with laboratory data



(a) $\eta = \mathbf{0.0}$



(b) $\eta = \mathbf{10.0}$

Figure 7. Estimated shape in experiments with laboratory data

6. CONCLUDING REMARKS

A graphical method was proposed for determining the "optimal" Tikhonov parameter by using L-curve. The proposed scheme is quite simple, easy to apply and no *a priori* statistical information is necessary. For each computational experiment with a different noise level, it was shown that L-curve yields a "distinct" corner on the residual versus the semi-norm plane, so that the "optimal" regularization parameter could be easily chosen. Although there are many open problems on the theoretical part of this approach, the validity and applicability of the proposed method were demonstrated through the use of both the computational and laboratory data.

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