数理解析研究所講究録
第 836 巻 1993 年 88-97

Discretized Modeling for Shape Finding of Soft Bodies

Institute of Industrial Science,
Uinversity of Tokyo

中桐 滋 Shigeru Nakagiri
Yasuhiko Nakanishi
Nobuhiro Yoshikawa

1. Introduction

The minimal surface equation of z=f(x,y) due to the Cartesian coordinates is found by Lagrange in the following form of partial differential equation (Morgan).

$$(1+f_y^{\,2})f_{xx}-2f_xf_yf_{xy}+(1+f_x^{\,2})f_{yy}=0 \qquad (1)$$

The finding of the minimal surface determined by the boundary condition of a given closed curve had attracted interest of researchers after a Belgian mathematician J.A.F.Plateau (1801-1883). Helicoid and catenoid are famous example of the minimal surface thus found.

Modern engineering tends to demand the knowledge about shape of minimal surface as meta-knowledge for membrane structures with minimum weight. Attention has been paid to shape of droplet from the viewpoint of condensation technique for heavy metal separation. In such cases, the magnitude of contained volume inside a surface is more important than the boundary shape of closed curve, and the effect of surface tension, in other words, surface free energy and gravitation energy must be taken into account. This paper deals with discretized models and variational calculus proposed to find shape of soft bodies like foam and droplet. Stationary condition of the functionals in terms of surface tension and gravitation energy and derived from the discretized models is employed for the shape finding, instead of solving the aforementioned partial differential equation. Common assumptions are made for foam and droplet as follows.

1. Their surface tension coefficient and mass density are homogeneous, isotropic and constant.
2. No mass transportation takes place.
3. Foam and droplet are in equilibrium at a standstill.

The numerical examples are concerned with the shapes of solitary foams and droplets.

2. Surface Discretization into Segments

The surface under interest is discretized by segments. The shaded segments illustrated in Figs. 1 and 2 cover all the surface without overlapping and gap. The segments of truncated conical shell type are employed in case of axisymmetric surface, and those of planar triangle type in case of tree-dimensional surface. The connection between the segments are called node. The radius from the origin to the n-th node r_n is taken unknown,

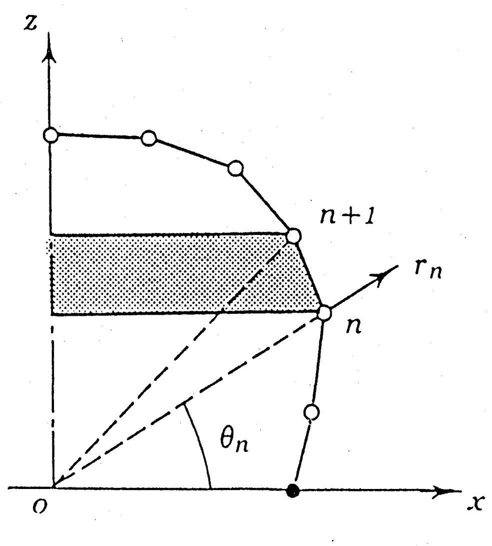

Fig.1 Axisymmetric case

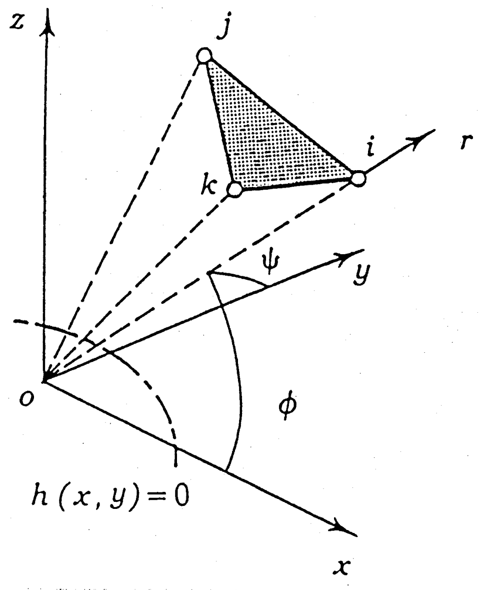

Fig.2 Three-dimensional case

while the direction angle of the nodal coordinates is taken known, because they can be determined at the time of segment discretization. Shape finding is considered completed when the nodal coordinates are made known. The surface area S_n of a segment can be epressed as a nonlinear function of the nodal coordinates as well as the volume V_n of the region corresponding to the segment which is indicated by broken lines in the figures.

3. Formulation Based on Minimal Surface Tension

It is well known that sphere has minimal surface for a certain volume. Surface tension acts in the tangential direction of surface to diminish the area. A spherical foam floating in the air may be degraded into a point, unless the other natural element is present. The inner pressure of a foam keeps its shape, corresponding to the volume of inside gas. Laplace's formula of Eq. (2) gives the relationship between the pressure difference p_1-p_2 between the inside and outside and the principal radii R_1 and R_2 of the surface for the surface tension coefficient of the foam a_L (Landau and Lifshitz).

$$p_1-p_2=a_L(1/R_1+1/R_2) \qquad (2)$$

This equation implies that the surface tension makes the surface of foam as small as possible for a certain volume inside the surface, and the surface shape is governed by the inner pressure and surface tension in equilibrium. The smallest surface gives rise to the smallest surface tension under the assumption of constant surface tension coefficient. The effect of gravity can be assumed to be small negligibly for thin foam. It turns out that the shape of foam can be found by the premise that the shape is determined so that the surface tension is minimal for a certain volume inside the surface. The following functional can be constituted on the basis of the said premise by means of introducing the equality constraint condition standing for the said volume C incorporated by the Lagrange multiplier method,

$$\Pi=a_L\sum_{n=1}^{N} S_n+\mu\left(\sum_{n=1}^{N} V_n-C\right) \tag{3}$$

where N denotes the number of segments, and μ the Lagrange multiplier. In the case of axisymmetric surface, the unknown radii should be remain positive. Such a transformation as $r_n=u_n^2$ is employed to ensure the positiveness. The following formulae are used when the stationary condition of the functional is derived with respect to the unknown UN and MU in the case of axisymmetric shape.

$$S_n=\pi(r_{n+1}\cos\theta_{n+1}+r_n\cos\theta n)l_n \tag{4}$$

$$l_n=\{r_{n+1}^2+r_n^2-2r_{n+1}r_n\cos(\theta_{n+1}-\theta_n)\}^{1/2} \tag{5}$$

$$V_n=\pi r_{n+1}r_n(r_{n+1}\cos\theta_{n+1}+r_n\cos\theta_n)\sin(\theta_{n+1}-\theta_n)/3 \tag{6}$$

$$\partial S_n/\partial u_n=2\pi u_n\{l_n\cos\theta_n+(u_{n+1}^2\cos\theta_{n+1}+u_n^2\cos\theta_n) \times(u_n^2-u_{n+1}^2\cos(\theta_{n+1}-\theta_n))/l_n\} \tag{7}$$

$$\partial V_n/\partial u_n=2\pi u_n u_{n+1}^2(u_{n+1}^2\cos\theta_{n+1}+2u_n^2\cos\theta_n) \times\sin(\theta_{n+1}-\theta_n)/3 \tag{8}$$

The governing equations of u_n and μ thus obtained are nonlinear so that an iterative solution is devised by means of introducing the initial guess indicated by the upper bar and small unknowns indicated by the triangular mark as follows.

$$u_n=\bar{u}_n+\Delta u_n \tag{9}$$

$$\mu=\bar{\mu}+\Delta\mu \tag{10}$$

Then linear simultaneous equations are obtained for the Δu_n and $\Delta\mu$. Iteration is to be repeated by solving the linear equations and modifying the unknowns until they converge.

In case of three-dimensional surface, the derivation of the said linear equations is so cumbersome that symbolic manipulation by REDUCE is employed.

4. Formulation Based on Minimal Sum of Surace Tension and Gravitation Energy

The shape of catenary is so determined that the center of gravity of the catenary takes the lowest position for a certain tension. The shape of droplet is affected by the surface tension, surface condition of the wall, to which the droplet adheres, and gravity. The wall is supposed horizontal in the following case studies. The surface tension makes the shape of droplet spherical, and the gravitation energy acts to lower its center of gravity for a certain mass. Then, a functional can be devised as follows under the premise that the shape of droplet is governed by the two actions for smallest surface tension and lowest center of gravity,

$$\Pi=a_L\sum_{n=1}^{N}S_n+(a_{SL}-a_S)S_0+\rho g\sum_{n=1}^{N}V_n z_n+\mu\Big(\sum_{n=1}^{N}V_n-C\Big) \tag{11}$$

where a_{SL} denotes the surface tension coefficient between the droplet material and wall, a_S the one between the wall and environment, S_0 the surface area of the wall, to which the droplet adheres. The third term of the right hand side of Eq.(11) represents the gravitation energy expressed by the mass density ρ and z_n, which is the height of the center of gravity of the said region corresponding to a segment, and g the gravitational acceleration. The parameters appearing in the second term of the right hand side of Eq.(11) is replaced by the following Young's equation,

$$a_{SL}-a_S+a_L\cos\alpha=0 \qquad (12)$$

where α denotes contact angle between the wall and droplet surface at the boundary, which is employed as an index of wettability in engineering. The formulae and numerical technique described in Section 3 are applicable generally also to the shape finding of droplet, inspite of the term added to take the gravitational effect into account (Nakagiri et al.).

5. Numerical Examples

5.1 Shape of foams

Figure 3 illustrates the cross-scetion of water foams with a variety of inside volume. The surface tension coefficient is taken equal to 0.0728 N/m. The foams are supposed to inflate from a circular boundary of 6 mm in radius so that the shape is axisymmetric. The initial guess of the shape is given as a cone with the same volume. The cross-sections do not indicate the enlargement of a foam. Figure 4 depicts contour lines of one eighth of a foam surface analysed by use of 420 triangular segments. The foam is supposed to inflate from a square boundary of 12 mm in edge length. Figure 5 proves that the proposed method results in a spherical foam when the inside volume is so vast that the effect of boundary shape can be neglected. The radius

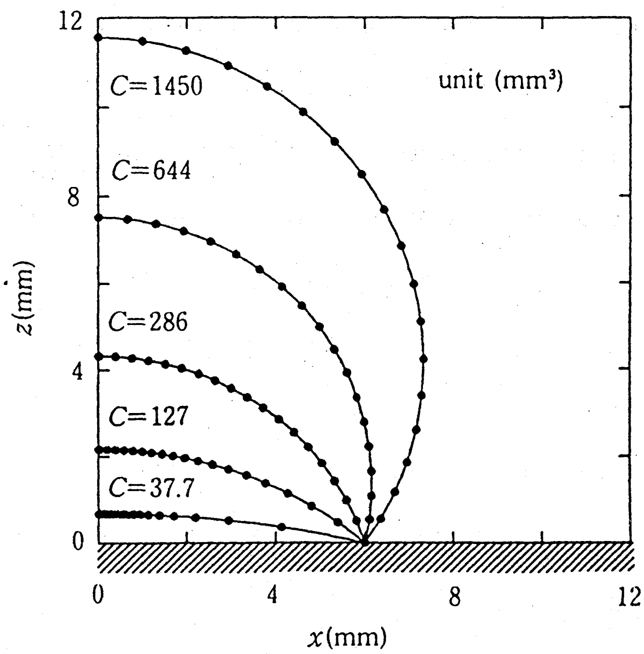

Figure 3 Cross-section of water foams

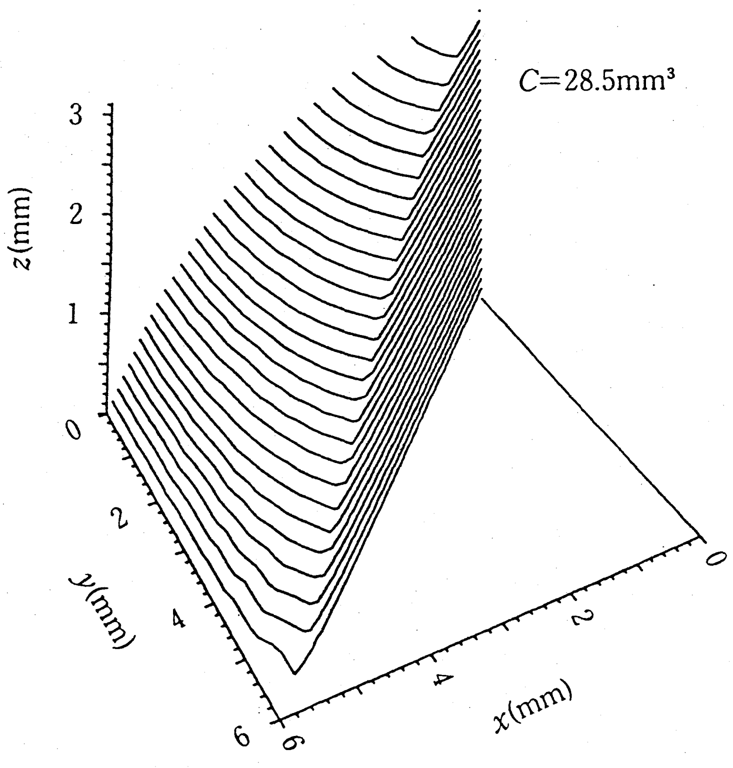

Figure 4 Contour lines a foam with square boundary

of a sphere and related Langrange multiplier are obtained numerically as 7.05 mm and 20.8 MPa for a vast volume of C=1450 mm3. The Laplace's formula results in 20.7 MPa for the pressure difference P1-P2 when the said radius and surface tension coeffi cient are input to the right hand side of Eq.(2). This means that the Lagrange multiplier used in the formulation gives the pressure difference. If the surface tension coefficient a_L is taken as nondimensional in Eq.(3), the Lagrange multiplier means curvature. The shape of foam remains the same irrespective to the value of AL, when the volume C is kept constant.

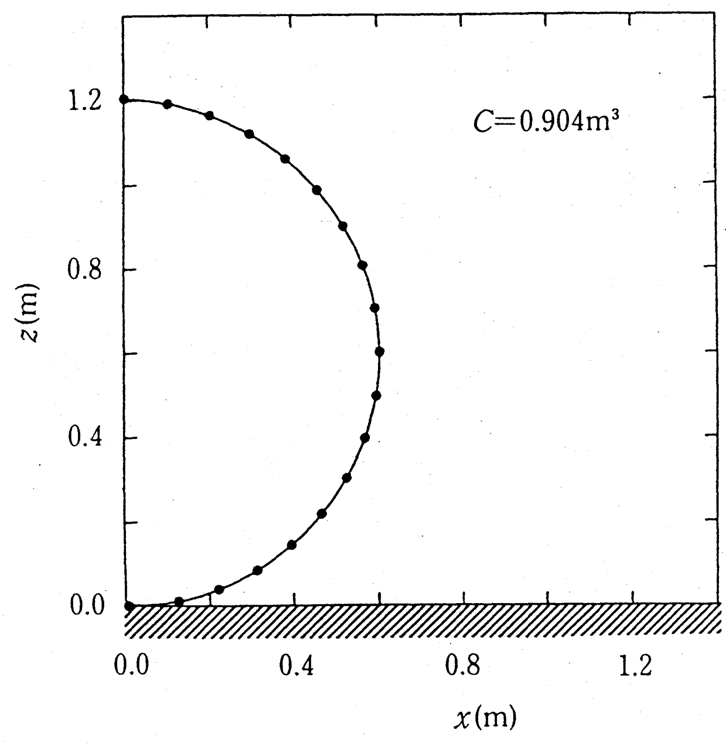

Figure 5 Spherical foam with vast volume

5.2 Shape of droplets

The material of the wall, to which droplet adheres, is supposed to be Teflon, and then the contact angle is given as 108 degrees in case of water droplet. When the contact angle is input, the contact area S0 is set free so that the radius of the wall boundary is taken as unknown. Figure 6 shows the cross-section of the droplets above Teflon wall, and Fig.7 the one of the droplets hanging down from the wall. When droplet is down-

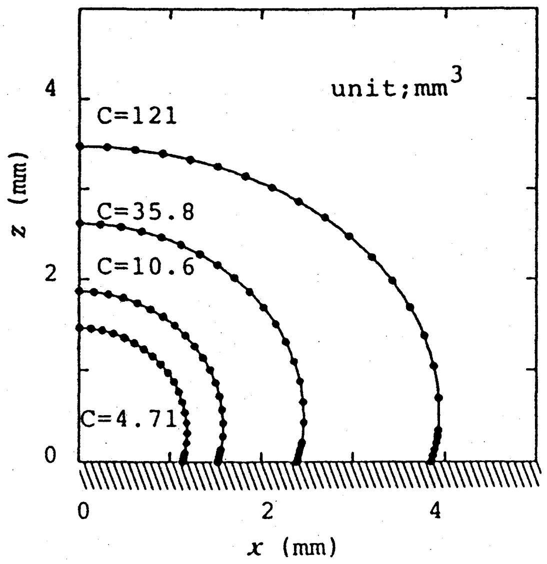

Fig.6 Upward droplets

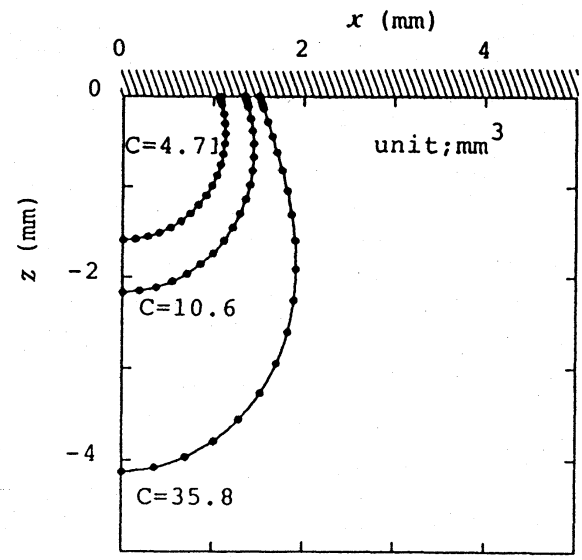

Fig.7 Downward droplets

ward, there is no limitation of the position of the center of gravity until it drops off, different from upward droplet. The slender cross-section of the downward droplets evidences the difference. It is found that the iterative solution did not converge, when vast mass is assigned to a droplet. Surface tension cannot sustain the weight of large downward droplet unlimitedly. This implies that the downward droplet is said to drop off because of short surface tension, when the iterative solution is not converged for the input mass.

6.Concluding Remarks

The formulation to find the shape of soft bodies is presented on the basis of variational calculus combined with discretized models of the minimal surface. The functionals are constituted for the minimal surface tension or the minimal sum of the surface tension and gravitation energy under the equality constraint condition of the constant volume of foam and the constant mass of droplet. The nodal coordinates to express the shape of foam and droplet are determined by the linearized stationary condition of the functional for the iterative solution. The numerical examples prove that the proposed method is effective for the shape finding of solitary foam and droplet.

REFERENCES

Morgan,F.; Geometric Measure Theory-A Beginner's Guide-, Academic Press, 1988.

Landau,L.D. and Lifshitz,E.M.; Fluid Mechanics, Course of Theoretical Physics, Vl.6, Pergamon Press, 1959.

Nakagiri,S., Nakanishi,Y. and Yoshikawa,N.; Numerical Shape-Finding of Soft Bodies Based on Discretized Modeling (in Japanese), Trans. JSME., Ser.A, Vol.57, No.541(1991), pp.2104-2210.