

Asymptotic expansions of the mean values of Dirichlet L -functions IV

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1 Introduction

Let χ be a Dirichlet character mod q ($q \geq 2$) and let $L(s, \chi)$ with a complex variable $s = \sigma + it$ denotes the Dirichlet L -function attached to χ . Let $\varphi(n)$ denote Euler's function. The aim of this article is to consider the asymptotical property of the mean square

$$(1.1) \quad \sum_{\chi(\bmod q)} |L(s, \chi)|^2,$$

where the summation is taken over all the characters mod q .

Let $Q \geq 2$ be a real number. In 1971, P. D. T. A. Elliott [3] proved the asymptotic formula

$$\sum_{p \leq Q} \sum_{\substack{\chi(\bmod p) \\ \chi \neq \chi_0}} |L(s, \chi)|^2 = \frac{Q^2}{2 \log Q} \zeta(2\sigma) + O \left\{ \frac{Q^2}{(\log Q)^2} \right\} \quad (Q \rightarrow +\infty)$$

for $\Re s = \sigma > \frac{1}{2}$, where p runs through all prime numbers not exceeding Q , and χ_0 denotes the principal character with its respective modulus. Let $\zeta(s, \alpha)$ denote the Hurwitz zeta-function defined by

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n + \alpha)^s}, \quad \alpha > 0, \sigma > 1.$$

Using the relation

$$(1.2) \quad \sum_{\chi(\bmod q)} |L(s, \chi)|^2 = \frac{\varphi(q)}{q^{2\sigma}} \sum_{\substack{a=1 \\ (a, q)=1}}^q \left| \zeta \left(s, \frac{a}{q} \right) \right|^2,$$

P. X. Gallagher [4], in 1975, proved the asymptotic bound

$$\sum_{\chi(\bmod q)} |L(\tfrac{1}{2} + it, \chi)|^2 = O \{ (q + |t|) \log q (|t| + 2) \}$$

for arbitrary $q \geq 2$ and real t . This was improved by R. Balasubramanian [2], in 1980, to

$$\sum_{\chi(\bmod q)} |L(\tfrac{1}{2} + it, \chi)|^2 = \frac{\varphi^2(q)}{q} \log qt + O\{q(\log \log q)^2\} + O\left(te^{10\sqrt{\log q}}\right) \\ + O\left(q^{\frac{1}{2}}t^{\frac{2}{3}}e^{10\sqrt{\log q}}\right)$$

uniformly for all $q \geq 2$ and $t \geq 3$. In the special case $s = \frac{1}{2}$, the existence of a more explicit asymptotic formula was shown by D. R. Heath-Brown [5]. In 1981, he proved

$$\sum_{\chi(\bmod q)} |L(\tfrac{1}{2}, \chi)|^2 = \frac{\varphi(q)}{q} \sum_{k|q} \mu\left(\frac{q}{k}\right) T(k),$$

where k runs over all positive divisors of q and $T(k)$ can be expressed by the asymptotic form

$$T(k) = k\left(\log \frac{k}{8\pi} + \gamma\right) + 2\zeta^2\left(\frac{1}{2}\right)k^{\frac{1}{2}} + \sum_{n=0}^{2N-1} c_n k^{-\frac{n}{2}} + O(k^{-N})$$

for any integer $N \geq 1$, with some numerical constants c_n and Euler's constant γ . In particular, when $q = p$ is a prime, his formula yields an asymptotic series with respect to $p^{-\frac{1}{2}}$, because the term corresponding to $k = 1$ can be calculated explicitly. For the proof, he investigated the function $\sum_{\chi(\bmod q)} L(s, \chi)L(1-s, \bar{\chi})$, instead of using (1.2).

During 1989–1991, on the same lines as Gallagher and Balasubramanian, Zhang Wen-peng [13]–[17] obtained more precise asymptotic results for the following various mean values:

$$\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi(\bmod q)} |L(\tfrac{1}{2} + it, \chi)|^2, \quad \sum_{\chi(\bmod q)} |L^{(h)}(\tfrac{1}{2} + it, \chi)|^2 \quad (h = 0, 1), \\ \sum_{\substack{\chi(\bmod q) \\ \chi \neq \chi_0}} |L(1, \chi)|^2, \quad \sum_{\chi(\bmod q)}^* L'(\sigma + it, \chi)L'(1 - \sigma - it, \bar{\chi}) \quad (0 < \sigma < 1),$$

where $*$ means that the summation is restricted to the primitive characters $(\bmod q)$. For example, he proved

$$\sum_{\chi(\bmod q)} |L(\tfrac{1}{2} + it, \chi)|^2 = \frac{\varphi^2(q)}{q} \left\{ \log \left(\frac{qt}{2\pi} \right) + 2\gamma + \sum_{p|q} \frac{\log p}{p-1} \right\} \\ + O(qt^{-\frac{1}{2}}) + O\left[\left(t^{\frac{5}{6}} + q^{\frac{1}{2}}t^{\frac{5}{12}} \right) \exp \left\{ \frac{2 \log(qt)}{\log \log(qt)} \right\} \right]$$

and

$$\sum_{\substack{\chi(\bmod q) \\ \chi \neq \chi_0}} |L(1, \chi)|^2 = \frac{\pi^2}{6} \varphi(q) \prod_{p|q} \left(1 - \frac{1}{p^2} \right) \\ - \frac{\varphi^2(q)}{q^2} \left(\log q + \sum_{p|q} \frac{\log p}{p-1} \right)^2 + O(\log \log q)$$

for all $q \geq 3$ and $t \geq 3$, where p runs through all prime divisors of q .

On the other hand, F. V. Atkinson [1] developed a new method which enabled him to treat $\zeta(u)\zeta(v)$ as a function of two independent variables to deduce the explicit formula for the error term

$$E(T) = \int_0^T |\zeta(\frac{1}{2} + it)|^2 dt - T \log(T/2\pi) - (2\gamma - 1)T.$$

In spite of its importance and applicability, Atkinson's formula had long been neglected. In 1985, Y. Motohashi [11], inspired by this Atkinson's work and investigated the function

$$Q(u, v; q) = \varphi(q)^{-1} \sum_{\chi(\bmod q)} L(u, \chi) L(v, \bar{\chi}).$$

In his article, Atkinson's method was enlightened from a viewpoint of the theory of complex functions and the following "decomposition" of $Q(u, v; q)$ was proved:

$$Q(u, v; q) = L(u + v, \chi_0) + \varphi(q) q^{-u-v} \Gamma(u + v - 1) \zeta(u + v - 1) \cdot \left\{ \frac{\Gamma(1-u)}{\Gamma(v)} + \frac{\Gamma(1-v)}{\Gamma(u)} \right\} + g(u, v; q) + g(v, u; q),$$

where $g(u, v; p)$ can be expressed by certain infinite series which involves the confluent hypergeometric functions. From this formula in case $q = p$ is a prime, he obtained the asymptotic expansion

$$\begin{aligned} & (p-1)^{-1} \sum_{\chi(\bmod p)} |L(\frac{1}{2} + it, \chi)|^2 \\ &= \log \frac{p}{2\pi} + 2\gamma + \Re \frac{\Gamma'}{\Gamma}(\frac{1}{2} + it) + 2p^{-\frac{1}{2}} |\zeta(\frac{1}{2} + it)|^2 \cos(t \log p) \\ & \quad - p^{-1} |\zeta(\frac{1}{2} + it)|^2 + O(p^{-\frac{3}{2}}) \end{aligned}$$

for arbitrary fixed $t \in \mathbf{R}$.

More detailed utilization of the method of Atkinson and Motohashi improve the above asymptotic formula. In what follows we state this improvement in a more general form. Let

$$\binom{s}{n} = \frac{s(s-1)\cdots(s-n+1)}{n!} \quad (n = 0, 1, 2, \dots)$$

as usual and set

$$F(w; q) = q^{1-w} \Gamma(w-1) \zeta(w-1), \quad G(u, v) = \frac{\Gamma(1-u)}{\Gamma(v)},$$

$$S_N(u, v; k) = \sum_{n=0}^{N-1} \binom{-v}{n} \zeta(u-n) \zeta(v+n) k^{u-n} \quad (N \geq 1), \quad P(w; q) = \prod_{p|q} (1 - p^{-w}).$$

Our main theorem is stated as follows:

Theorem ([8, Theorem] and [9, Theorem 1 and 2]) Let $\mathbf{Z}_{\leq 1}$ denote the set of all integers not greater than 1 and define

$$E = \{\sigma + it; 2\sigma - 1 \in \mathbf{Z}_{\leq 1} \text{ or } \sigma + it \in \mathbf{Z}\},$$

then for any integer $N \geq 1$, in the region

$$(1.3) \quad \{\sigma + it; -N + 1 < \sigma < N, t \in \mathbf{R}\}$$

with the exception of the points of E , we have

$$\begin{aligned} & \varphi(q)^{-1} \sum_{\chi(\bmod q)} |L^{(h)}(\sigma + it, \chi)|^2 \\ &= \frac{d^{2h}}{dw^{2h}} \zeta(w) P(w; q) \Big|_{w=2\sigma} \\ &+ 2P(1; q) \sum_{\mu, \nu=0}^h \binom{h}{\mu} \binom{h}{\nu} F^{(2h-\mu-\nu)}(2\sigma; q) \Re \left\{ \frac{\partial^{\mu+\nu} G}{\partial u^\mu \partial v^\nu}(\sigma + it, \sigma - it) \right\} \\ &+ 2q^{-2\sigma} \sum_{\mu, \nu=0}^h \binom{h}{\mu} \binom{h}{\nu} (-\log q)^{2h-\mu-\nu} \sum_{k|q} \mu \left(\frac{q}{k}\right) T^{(\mu, \nu)}(\sigma + it; k), \end{aligned}$$

where $T^{(\mu, \nu)}(\sigma + it; k)$ has the asymptotic expression

$$T^{(\mu, \nu)}(\sigma + it; k) = \Re \left\{ \frac{\partial^{\mu+\nu} S_N}{\partial u^\mu \partial v^\nu}(\sigma + it, \sigma - it; k) + E_N^{(\mu, \nu)}(\sigma + it; k) \right\}.$$

Here $E_N^{(\mu, \nu)}(\sigma + it; k)$ is the error term satisfying the estimate

$$(1.4) \quad E_N^{(\mu, \nu)}(\sigma + it; k) = O \left[k^{\sigma-N} (|t| + 1)^{2N+\frac{1}{2}-\sigma} \log^{\mu+\nu} \{2k(|t| + 1)\} \right]$$

in the region (1.3), with the O -constant depending only on σ , N and h . In particular, when $q = p$ is a prime, we have the asymptotic expansion

$$\begin{aligned} & (p-1)^{-1} \sum_{\chi(\bmod p)} |L^{(h)}(\sigma + it, \chi)|^2 \\ &= \zeta^{(2h)}(2\sigma) - \frac{\partial^{2h}}{\partial u^h \partial v^h} \{p^{-u-v} \zeta(u) \zeta(v)\} \Big|_{(u, v) = (\sigma + it, \sigma - it)} \\ &+ 2 \sum_{\mu, \nu=0}^h \binom{h}{\mu} \binom{h}{\nu} F^{(2h-\mu-\nu)}(2\sigma; p) \Re \left\{ \frac{\partial^{\mu+\nu} G}{\partial u^\mu \partial v^\nu}(\sigma + it, \sigma - it) \right\} \\ &+ 2p^{-2\sigma} \sum_{\mu, \nu=0}^h \binom{h}{\mu} \binom{h}{\nu} (-\log p)^{2h-\mu-\nu} T^{(\mu, \nu)}(\sigma + it; p). \end{aligned}$$

From Stirling's formula and the functional equation of $\zeta(s)$, we have

$$\binom{-\sigma + it}{n} \zeta(\sigma + it - n) \zeta(\sigma - it + n) k^{\sigma+it-n} = O \{k^{\sigma-n} (|t| + 1)^{2n+\frac{1}{2}-\sigma}\},$$

for $-n + 1 < \sigma < n$ ($n \geq 1$), and this estimate is best-possible because

$$\zeta(\sigma + it) = \Omega(1)$$

for $\sigma > 1$ as $|t| \rightarrow +\infty$. Hence, when $h = 0$, the upper bound in (1.4) cannot be replaced by a smaller one.

Moreover, the asymptotic expressions for (1.1), where $\sigma + it$ lies in the exceptional set E , can be deduced as the limiting cases of our Theorem. For example, we have the following corollaries:

Corollary 1 ([9, Theorem 1 and 2])

$$\begin{aligned} & \varphi(q)^{-1} \sum_{\chi(\bmod q)} |L(\tfrac{1}{2} + it, \chi)|^2 \\ &= \frac{\varphi(q)}{q} \left\{ \log \frac{q}{2\pi} + 2\gamma + \sum_{p|q} \frac{\log p}{p-1} + \Re \frac{\Gamma'}{\Gamma}(\tfrac{1}{2} + it) \right\} \\ & \quad + 2q^{-1} \sum_{k|q} \mu\left(\frac{q}{k}\right) T^{(0,0)}(\tfrac{1}{2} + it; k). \end{aligned}$$

In particular, when $q = p$ is a prime, we have the asymptotic expansion

$$\begin{aligned} & (p-1)^{-1} \sum_{\chi(\bmod p)} |L(\tfrac{1}{2} + it, \chi)|^2 \\ &= \log \frac{p}{2\pi} + 2\gamma + \Re \frac{\Gamma'}{\Gamma}(\tfrac{1}{2} + it) - p^{-1} |\zeta(\tfrac{1}{2} + it)|^2 + 2p^{-1} T^{(0,0)}(\tfrac{1}{2} + it; p). \end{aligned}$$

Corollary 2 ([8, Corollary 1]) Let $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$ be the digamma-function and put

$$\begin{aligned} A_0(q) &= \log \frac{q}{2\pi} + \gamma_0, & A_1(q) &= \frac{1}{2} \log^2 \frac{q}{2\pi} + \gamma_0 \log \frac{q}{2\pi} + \gamma_1 + \frac{\pi^2}{8}, \\ A_2(q) &= \frac{1}{6} \log^3 \frac{q}{2\pi} + \frac{\gamma_0}{2} \log^2 \frac{q}{2\pi} + \left(\gamma_1 + \frac{\pi^2}{8}\right) \log \frac{q}{2\pi} + \frac{\pi^2}{8} \gamma_0 + \gamma_2, \end{aligned}$$

where $\gamma_0 (= \gamma)$, γ_1 and γ_2 are the coefficients of the Laurent expansion of $\zeta(s)$ at $s = 1$ defined by

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 + \gamma_1(s-1) + \gamma_2(s-1)^2 + \gamma_3(s-1)^3 + \dots$$

Then we have

$$\begin{aligned} & \varphi(q)^{-1} \sum_{\chi(\bmod q)} |L'(\tfrac{1}{2} + it, \chi)|^2 \\ &= P(1; q) \left\{ 2\gamma_2 + 2\gamma_1 \frac{P'}{P}(1; q) + \gamma_0 \frac{P''}{P}(1; q) + \frac{1}{3} \frac{P'''}{P}(1; q) \right. \\ & \quad - \frac{1}{6} \Re \psi''(\tfrac{1}{2} + it) + \frac{1}{3} \Re \psi^3(\tfrac{1}{2} + it) + A_0(q) \Re \psi^2(\tfrac{1}{2} + it) \\ & \quad \left. + 2A_1(q) \Re \psi(\tfrac{1}{2} + it) + 2A_2(q) \right\} \\ & \quad + 2q^{-1} \sum_{\mu, \nu=0}^1 (-\log q)^{2-\mu-\nu} \sum_{k|q} \mu\left(\frac{q}{k}\right) T^{(\mu, \nu)}(\tfrac{1}{2} + it; k). \end{aligned}$$

If $q = p$ is a prime, then

$$\begin{aligned}
& (p-1)^{-1} \sum_{\chi(\bmod p)} |L'(\tfrac{1}{2} + it, \chi)|^2 \\
&= 2\gamma_2 - \frac{1}{6} \Re \psi''(\tfrac{1}{2} + it) + \frac{1}{3} \Re \psi^3(\tfrac{1}{2} + it) + A_0(p) \Re \psi^2(\tfrac{1}{2} + it) \\
&\quad + 2A_1(p) \Re \psi(\tfrac{1}{2} + it) + 2A_2(p) - \frac{\partial^2}{\partial u \partial v} \{p^{-u-v} \zeta(u) \zeta(v)\} \Big|_{(u,v)=(\frac{1}{2}+it, \frac{1}{2}-it)} \\
&\quad + 2p^{-1} \sum_{\mu, \nu=0}^1 (-\log p)^{2-\mu-\nu} T^{(\mu, \nu)}(\tfrac{1}{2} + it; p).
\end{aligned}$$

We note here

$$\begin{aligned}
\frac{P'}{P}(1; q) &= \sum_{p|q} \frac{\log p}{p-1}, \quad \frac{P''}{P}(1; q) = \left(\sum_{p|q} \frac{\log p}{p-1} \right)^2 - \sum_{p|q} \frac{p \log^2 p}{(p-1)^2}, \\
\frac{P'''}{P}(1; q) &= \left(\sum_{p|q} \frac{\log p}{p-1} \right)^3 + \sum_{p|q} \frac{p(p+1) \log^3 p}{(p-1)^3} - 3 \left(\sum_{p|q} \frac{\log p}{p-1} \right) \left(\sum_{p|q} \frac{p \log^2 p}{(p-1)^2} \right).
\end{aligned}$$

Corollary 3 ([10, Theorem 1]) Let χ_0 be the principal character mod q and define

$$\tilde{S}_N(u, v; k) = S_N(u, v; k) - \zeta(u) \zeta(v) k^u,$$

then we have

$$\begin{aligned}
& \varphi(q)^{-1} \sum_{\substack{\chi(\bmod q) \\ \chi \neq \chi_0}} |L(1, \chi)|^2 \\
&= \zeta(2) \prod_{p|q} (1 - p^{-2}) \\
&\quad + q^{-2} \varphi(q) \left\{ \gamma_0^2 - 2\gamma_1 - 2\zeta(2) - \left(\log q + \sum_{p|q} \frac{\log p}{p-1} \right)^2 \right\} \\
&\quad + 2q^{-2} \sum_{k|q} \mu\left(\frac{q}{k}\right) \Re \left\{ \tilde{S}_N(1, 1; k) + O(k^{1-N}) \right\}.
\end{aligned}$$

In particular, if $q = p$ is a prime, then we have the asymptotic expansion

$$\begin{aligned}
& (p-1)^{-1} \sum_{\substack{\chi(\bmod p) \\ \chi \neq \chi_0}} |L(1, \chi)|^2 \\
&= \zeta(2) - \frac{\log^2 p}{p-1} + p^{-1} \left\{ \gamma_0^2 - 2\gamma_1 - 2\zeta(2) \right\} \\
&\quad + 2p^{-2} \Re \left\{ \tilde{S}_N(1, 1; p) + O(p^{1-N}) \right\},
\end{aligned}$$

Corollary 3 can be applied to deduce upper estimates for class numbers of cyclotomic fields. Furthermore we have

Corollary 4 ([8, Corollary 2])

$$\begin{aligned}
& \varphi(q)^{-1} \sum_{\substack{\chi(\bmod q) \\ \chi \neq \chi_0}} |L'(1, \chi)|^2 \\
&= q^{-1} P(1; q) \left[-\frac{1}{4} \log^4 q + \gamma_0 \log^3 q \right. \\
&\quad \left. + \left\{ \gamma_0 \frac{P'}{P}(1; q) + \frac{1}{2} \frac{P''}{P}(1; q) - 2\gamma_1 - 2\zeta(2) \right\} \log^2 q \right. \\
&\quad \left. + \left\{ -\gamma_0 \frac{P''}{P}(1; q) - 2\gamma_0 \gamma_1 + 6\gamma_2 + 4\zeta(2)\gamma_0 - 2\zeta(3) \right\} \log q \right. \\
&\quad \left. - \gamma_0 \frac{P'}{P}(1; q) \frac{P''}{P}(1; q) - \frac{1}{4} \left(\frac{P''}{P}(1; q) \right)^2 - \zeta(4) - \zeta^2(2) + 2\gamma_0 \zeta(3) \right. \\
&\quad \left. + 2\zeta(2)(\gamma_0^2 - 2\gamma_1) + (\gamma_1^2 - 6\gamma_3) \right] + \frac{d^2}{dw^2} \zeta(w) P(w; q) \Big|_{w=2} \\
&\quad + 2q^{-2} \sum_{\mu, \nu=0}^1 (-\log q)^{2-\mu-\nu} \sum_{k|q} \mu\left(\frac{q}{k}\right) \tilde{T}^{(\mu, \nu)}(1; k).
\end{aligned}$$

In particular, if $q = p$ is a prime, then

$$\begin{aligned}
& (p-1)^{-1} \sum_{\substack{\chi(\bmod p) \\ \chi \neq \chi_0}} |L'(1, \chi)|^2 \\
&= \zeta''(2) - \frac{1}{4} \frac{\log^4 p}{p-1} + \gamma_0 \frac{\log^3 p}{p-1} - \gamma_0^2 \frac{\log^2 p}{p-1} \\
&\quad + p^{-1} \left\{ (\gamma_0^2 - 2\gamma_1 - 2\zeta(2)) \log^2 p + 2(2\zeta(2)\gamma_0 - \zeta(3) - \gamma_0 \gamma_1 + 3\gamma_2) \log p \right. \\
&\quad \left. - \zeta(4) - \zeta^2(2) + 2\gamma_0 \zeta(3) + 2\zeta(2)(\gamma_0^2 - 2\gamma_1) + (\gamma_1^2 - 6\gamma_3) \right\} \\
&\quad + 2p^{-2} \sum_{\mu, \nu=0}^1 (-\log p)^{2-\mu-\nu} \tilde{T}^{(\mu, \nu)}(1; p).
\end{aligned}$$

where $\tilde{T}^{(\mu, \nu)}(1; k)$ has the asymptotic expression

$$\tilde{T}^{(\mu, \nu)}(1; k) = \Re \left\{ \frac{\partial^{\mu+\nu} \tilde{S}_N}{\partial u^\mu \partial v^\nu}(1, 1; k) + E_N^{(\mu, \nu)}(1; k) \right\}$$

for any integer $N \geq 1$, with the error estimate for $E_N^{(\mu, \nu)}(1; k)$ in (1.4).

2 Outline of the proof of Theorem

We define the contour \mathcal{C} which starts from infinity, proceeds along the real axis to δ ($0 < \delta < \pi$), rounds the origin counter-clockwise and returns to infinity. Let $h^{(N)}(z)$ denote the N -th derivative of the function

$$h(z) = \frac{e^z}{e^z - 1} - \frac{1}{z},$$

and define for $N \geq 1$

$$R_N(u, v; k) = \frac{1}{\Gamma(u)\Gamma(v)(e^{2\pi i u} - 1)(e^{2\pi i v} - 1)} \cdot \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} \int_{\mathcal{C}} \frac{y^{v+N-1}}{e^y - 1} \int_{\mathcal{C}} h^{(N)}\left(x + \frac{\tau y}{k}\right) x^{u-1} dx dy d\tau,$$

where $\Im x$ and $\Im y$ vary from 0 to 2π round \mathcal{C} . Here the contour integrals are absolutely convergent for $\Re u < N + 1$ and any $v \in \mathbf{C}$, since the inequality

$$h^{(N)}\left(x + \frac{\tau y}{k}\right) = O\left((1 + |x|)^{-N-1}\right)$$

holds uniformly for all $x, y \in \mathcal{C} \cup [0, +\infty[$ and $\tau \in [0, 1]$ (cf. [9, Lemma 1]). Then we have

Lemma 2.1 ([7, Lemma 1] or [8, Lemma 2.1])

$$Q(u, v; q) = \zeta(u+v)P(u+v; q) + P(1; q)F(u+v; q)\{G(u, v) + G(v, u)\} + q^{-u-v} \sum_{k|q} \mu\left(\frac{q}{k}\right)\{S(u, v; k) + S(v, u; k)\},$$

where

$$S(u, v; k) = S_N(u, v; k) + k^{u-N} R_N(u, v; k)$$

for any integer $N \geq 1$. In particular, if $q = p$ is a prime, then

$$Q(u, v; p) = \zeta(u+v) - p^{-u-v} \zeta(u)\zeta(v) + F(u+v; p)\{G(u, v) + G(v, u)\} + p^{-u-v} \{S(u, v; p) + S(v, u; p)\}.$$

The assertions of Lemma 2.1 are proved by the procedure of Motohashi [11] and by integrating by parts N -times of the contour integral expression of $g(u, v; q)$ in [7, (2.2)]. By applying certain residue calculus for $R_N(u, v; k)$, and then by using the transformation formula of the confluent hypergeometric functions, we can show the following alternative expressions which are useful for the deduction of the estimate (1.4):

Lemma 2.2 ([8, Lemma 2.2]) *Let $\sigma_a(n)$ denote the sum of the a -th powers of the positive divisors of n . Then $R_N(u, v; k)$ is expressed by the following absolutely convergent infinite series: For $\Re u < N$, $\Re v > -N + 1$ with $\Re(u+v) < 2$, we have*

$$R_N(u, v; k) = (-1)^N (2\pi)^{u+v-1} \frac{\Gamma(N+1-u)}{\Gamma(v)} \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} \sum_{l=1}^{\infty} \sigma_{u+v-1}(l) \cdot \left\{ e^{\frac{\pi i}{2}(u+v-1)} J_-(\tau, l; k) + e^{-\frac{\pi i}{2}(u+v-1)} J_+(\tau, l; k) \right\} d\tau,$$

where

$$J_{\pm}(\tau, l; k) = \int_0^{\infty} y^{v+N-1} \left(1 + \frac{\tau y}{k}\right)^{u-N-1} e^{\pm 2\pi i l y} dy.$$

On the other hand, if $\Re u < N$, $\Re v > -N + 1$ with $\Re(u + v) > 0$, then

$$R_N(u, v; k) = (-1)^N \frac{\Gamma(v + N)}{\Gamma(v)} \int_0^1 \frac{(1 - \tau)^{N-1}}{(N-1)!} \sum_{l=1}^{\infty} \sigma_{1-u-v}(l) \cdot \left\{ \tilde{J}_-(\tau, l; k) + \tilde{J}_+(\tau, l; k) \right\} d\tau,$$

where

$$\tilde{J}_{\pm}(\tau, l; k) = \int_0^{\infty} y^{-u+N} \left(1 + \frac{\tau y}{k}\right)^{-v-N} e^{\pm 2\pi i l y} dy.$$

Successively differentiating both sides of the formulas in Lemma 2.1, we get

$$\begin{aligned} & \frac{\partial^{2h} Q}{\partial u^h \partial v^h}(u, v; q) \\ &= \frac{d^{2h}}{dw^{2h}} \zeta(w) P(w; q) \Big|_{w=u+v} \\ &+ P(1; q) \sum_{\mu, \nu=0}^h \binom{h}{\mu} \binom{h}{\nu} F^{(2h-\mu-\nu)}(u+v; q) \frac{\partial^{\mu+\nu} G^*}{\partial u^{\mu} \partial v^{\nu}}(u, v) \\ &+ q^{-u-v} \sum_{\mu, \nu=0}^h \binom{h}{\mu} \binom{h}{\nu} (-\log q)^{2h-\mu-\nu} \sum_{k|q} \mu \left(\frac{q}{k}\right) \frac{\partial^{\mu+\nu} S^*}{\partial u^{\mu} \partial v^{\nu}}(u, v; k) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^{2h} Q}{\partial u^h \partial v^h}(u, v; p) &= \zeta^{(2h)}(u+v) - \frac{\partial^{2h}}{\partial u^h \partial v^h} p^{-u-v} \zeta(u) \zeta(v) \\ &+ \sum_{\mu, \nu=0}^h \binom{h}{\mu} \binom{h}{\nu} F^{(2h-\mu-\nu)}(u+v; p) \frac{\partial^{\mu+\nu} G^*}{\partial u^{\mu} \partial v^{\nu}}(u, v) \\ &+ p^{-u-v} \sum_{\mu, \nu=0}^h \binom{h}{\mu} \binom{h}{\nu} (-\log p)^{2h-\mu-\nu} \frac{\partial^{\mu+\nu} S^*}{\partial u^{\mu} \partial v^{\nu}}(u, v; p), \end{aligned}$$

where we write $G^*(u, v) = G(u, v) + G(v, u)$ and $S^*(u, v; k) = S(u, v; k) + S(v, u; k)$ for brevity. If we specialize $u = \sigma + it$ and $v = \sigma - it$ in these formulas and write

$$E_N^{(\mu, \nu)}(\sigma + it; k) = \frac{\partial^{\mu+\nu}}{\partial u^{\mu} \partial v^{\nu}} k^{u-N} R_N(u, v; k) \Big|_{(u, v) = (\sigma + it, \sigma - it)},$$

then we obtain the right-hand expressions for $\varphi(q)^{-1} \sum_{\chi \pmod{q}} |L^{(h)}(s, \chi)|^2$ in our Theorem by noting that

$$\overline{\frac{\partial^{\mu+\nu} S}{\partial u^{\mu} \partial v^{\nu}}(u, v; k)} \Big|_{(u, v) = (\sigma + it, \sigma - it)} = \frac{\partial^{\mu+\nu} S}{\partial u^{\nu} \partial v^{\mu}}(v, u; k) \Big|_{(u, v) = (\sigma + it, \sigma - it)},$$

which is a consequence of the reflection principle.

The following lemma is essential in proving the estimate for $E_N^{(\mu, \nu)}(\sigma + it; k)$:

Lemma 2.3 ([8, Lemma 2.3]) *Let γ be a non negative integer, and let $\alpha, \beta, \delta, \kappa, a, b, t$ be real numbers such that $\alpha > -1, \delta \geq 0, \kappa \geq 1, 0 < a < \min(\frac{1}{2}, \frac{t}{8\pi\kappa})$ and $1 \leq t \leq b$. Then*

$$\begin{aligned} & \int_a^b x^\alpha (1+x)^\beta \log^\gamma x \log^\delta (1+x) \exp i \left\{ t \log \left(\frac{1+x}{x} \right) + 2\pi\kappa x \right\} dx \\ &= \left(U - \frac{1}{2} \right)^\alpha \left(U + \frac{1}{2} \right)^\beta \log^\gamma \left(U - \frac{1}{2} \right) \log^\delta \left(U + \frac{1}{2} \right) \frac{1}{2\kappa} \sqrt{\frac{t}{\pi}} U^{-\frac{1}{2}} \\ & \quad \cdot \exp i \left\{ tV + 2\pi\kappa \left(U - \frac{1}{2} \right) + \frac{\pi}{4} \right\} \\ & \quad + O(t^{-1} a^{\alpha+\delta+1} |\log a|^\gamma) + O(\kappa^{-1} b^{\alpha+\beta} \log^{\gamma+\delta}(2b)) + R(t, \kappa), \end{aligned}$$

where

$$R(t, \kappa) \ll \begin{cases} \kappa^{-\frac{1}{2}(\alpha+\beta) - \frac{5}{4}} t^{\frac{1}{2}(\alpha+\beta) - \frac{1}{4}} \log^{\gamma+\delta} \left(\frac{2t}{\kappa} \right), & (1 \leq \kappa \leq t), \\ \kappa^{-\alpha-\delta-1} t^{\alpha+\delta-\frac{1}{2}} \log^\gamma \left(\frac{2\kappa}{t} \right), & (\kappa \geq t), \end{cases}$$

and

$$U = \sqrt{\frac{1}{4} + \frac{t}{2\pi\kappa}}, \quad V = 2 \operatorname{Arcsinh} \sqrt{\frac{\pi\kappa}{2t}}.$$

Here the constants implied in the O - and Vinogradov's \ll symbols depend at most on α, β, γ and δ . A similar result holds for the corresponding integral with $-\kappa$ in place of κ , except that in this case the explicit term on the right-hand side is to be omitted.

This lemma is proved by the saddle-point method.

Since the first infinite series for $R_N(u, v; k)$ in Lemma 2.2 is compact uniformly convergent in the region $\Re u < N, \Re v > -N + 1$ with $\Re(u + v) < 2$, the term-by-term differentiation is permissible, and this gives for $N + 1 < \sigma < 1$

$$\begin{aligned} (2.1) \quad & E_N^{(\mu, \nu)}(\sigma + it; k) \\ &= (-1)^N \int_0^1 \frac{(1-\tau)^{N-1}}{(N-1)!} \left[\sum_{\substack{\mu_0 + \dots + \mu_5 = \mu \\ \nu_1 + \dots + \nu_5 = \nu}} \frac{\mu!}{\mu_0! \dots \mu_5!} \frac{\nu!}{\nu_1! \dots \nu_5!} k^{\sigma+it-N} \log^{\mu_0} k \right. \\ & \quad \cdot (2\pi)^{2\sigma-1} \log^{\mu_1+\nu_1} (2\pi) \frac{d^{\mu_2}}{du^{\mu_2}} \Gamma(N+1-u) \Big|_{u=\sigma+it} \frac{d^{\nu_2}}{dv^{\nu_2}} \frac{1}{\Gamma(v)} \Big|_{v=\sigma-it} \\ & \quad \cdot \sum_{l=1}^{\infty} \sigma_{2\sigma-1}^{(\mu_3+\nu_3)}(l) \left\{ e^{\frac{\pi i}{2}(2\sigma-1)} \left(\frac{\pi i}{2} \right)^{\mu_4+\nu_4} J_-^{(\mu_5, \nu_5)}(\tau, l; k) \right. \\ & \quad \left. + e^{-\frac{\pi i}{2}(2\sigma-1)} \left(-\frac{\pi i}{2} \right)^{\mu_4+\nu_4} J_+^{(\mu_5, \nu_5)}(\tau, l; k) \right\} \Big] d\tau, \end{aligned}$$

where we write

$$\sigma_a^{(n)}(l) = \frac{d^n}{da^n} \sigma_a(l) = \sum_{d|l} d^a \log^n d$$

and

$$J_{\pm}^{(\mu_5, \nu_5)}(\tau, l; k) = \frac{\partial^{\mu_5+\nu_5} J_{\pm}(\tau, l; k)}{\partial u^{\mu_5} \partial v^{\nu_5}} \Big|_{(u, v) = (\sigma+it, \sigma-it)}$$

Making use of Lemma 2.3, we can show

$$J_{\pm}^{(\mu_5, \nu_5)}(\tau, l; k) \ll \begin{cases} \left(\frac{k}{\tau}\right)^{N+\frac{1}{4}} l^{-\sigma+\frac{1}{4}} t^{\sigma-\frac{3}{4}} \log^{\mu_5+\nu_5}(2t) & \text{for } l \leq k^{-1}t\tau, \\ l^{-\sigma-N} t^{\sigma+N-\frac{1}{2}} \left\{ \log^{\nu_5}\left(\frac{2k}{\tau}\right) + \log^{\nu_5} l \right\} & \text{for } l \geq k^{-1}t\tau. \end{cases}$$

If we substitute these bounds into (2.1) and estimate term-by-term, then we consequently obtain the estimate (1.4) in case $-N+1 < \sigma < 1$. The deduction of (1.4) in case $0 < \sigma < N$ is the same as above except for the use of the second infinite series of $R_N(u, v; k)$ in Lemma 2.2. \square

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