数理解析研究所講究録
第 830 巻 1993 年 8-11

On Type II and Type III Principal Graphs

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§0. Introduction

Let us consider a pair of type III_1 factors with finite index. Then it is known that there exists the simultaneous continuous crossed product decomposition of the pair. Therefore we have two principal graphs, so-called the type II principal graph and the type III principal graph [KL]. In this note we shall characterize the condition that above two graphs are different, in terms of Longo's sectors.

In the case of type $\mathrm{III}_\lambda (0 < \lambda < 1)$ factors, we can show the same type of characterization when the pair has simultaneous discrete crossed product decomposition.

§1. The type III_1 case

Let $M \supset N$ be a pair of type III_1 factors with finite index. Then we can construct the simultaneous continuous crossed product decomposition as follows. Let ω be a faithful normal state on N and $E : M \longrightarrow N$ the minimal conditonal expectation of Hiai [H]. We define a pair of type II_∞ factors $\widetilde{M} \supset \widetilde{N}$ as follows.

$$\widetilde{M} = M \underset{\sigma^{\omega\cdot E}}{\rtimes} \mathbf{R} \supset \widetilde{N} = N \underset{\sigma^{\omega}}{\rtimes} \mathbf{R}.$$

Let τ be the trace on $\widetilde{M}$ and θ the dual action of $\sigma^{\omega\cdot E}$, which coincides with the dual action of σ^{ω} restricted to $\widetilde{N}$. Thanks to Takesaki duality we can identify $M \supset N$ with

$\widetilde{M} \rtimes_\theta \mathbf{R} \supset \widetilde{N} \rtimes_\theta \mathbf{R}$. Let

$$M_n \supset M_{n-1} \supset \cdots \supset M \supset N$$

$$\widetilde{M}_n \supset \widetilde{M}_{n-1} \supset \cdots \supset \widetilde{M} \supset \widetilde{N}$$

be the towers associated with $M \supset N$ and $\widetilde{M} \supset \widetilde{N}$. In the same way as above M_n can be identified with $\widetilde{M}_n \rtimes_\theta \mathbf{R}$. (Note that θ is canonically extended to $\widetilde{M}_n$.)

Defintion 1.1. *We call the principal graph of $\widetilde{M} \supset \widetilde{N}$ the type II principal graph and that of $M \supset N$ the type III principal graph.*

It is known that the following relation holds [KL].

$$(\widetilde{M}_n \cap \widetilde{N}^{'})_\theta = M_n \cap N^{'}.$$

So in general two principal graphs do not coincide. Actually there exist examples having different principal graphs [S]. We show the necessary and sufficient condition that this phenomenon happens in terms of Longo's sectors. (For the definition of the sectors, see [L2,I].)

Theorem 1.2. *Let $M \supset N$ be a pair of type III_1 factors with finite index and $\gamma : M \longrightarrow N$ the canonical endomorphism [L3]. The type II and the type III principal graphs of $M_1 \supset M$ do not coincide if and only if there appears the modular automorphism ${}_M[\sigma_t^{\hat{\tau}}]_M$ $(t \neq 0)$ in ${}_M[\gamma^n]_M$ for some $n \in \mathbf{N}$.*

As a corollary we have the following.

Corollary 1.3. *Moreover if the depth of $M \supset N$ is finite the type II and type III principal graphs coincide.*

It is easy to show the sufficiency. So we sketch the proof of the necessity. Assume that the two graphs are different. Since a one-parameter action can not move any central projections in the higher relative commutant algebras, we can see that there exist two different ($M-M$ or $M-N$) sectors $[\rho_1], [\rho_2]$ which coincide restricted to $\widetilde{M}$. This means that there exists $t \neq 0$ and $[\rho_2] = [\sigma_t^{\hat{\tau}} \cdot \rho_1]$. Therefore

$$[\rho_2]\overline{[\rho_1]} = [\sigma_t^{\hat{\tau}}][\rho_1\overline{\rho_1}]$$

contains $[\sigma_t^{\hat{\tau}}]$. Corollary 2.3 follows from the fact that no aperiodic automorphism can appear in the decsendent $M-M$ sectors when the depth is finite.

§2. The type III$_\lambda$ $(0<\lambda<1)$ case

In the case of type III$_\lambda$ $(0<\lambda<1)$ factors, we have to assume that there sxist a pair of type II$_\infty$ factors $\widetilde{M} \supset \widetilde{N}$ and $\theta \in Aut(M)$ scaling the trace and globally preserving N such that

$$M = \widetilde{M} \underset{\theta}{\rtimes} \mathbf{Z} \supset N = \widetilde{N} \underset{\theta}{\rtimes} \mathbf{Z}.$$

Note that the assumption is satified if M is isomorphic to N and the center of $M \cap N'$ is trivial [Li].

We define the two principal graphs in the same way as before using the discrete crossed product decompositon. Then the same type of theorem holds.

Theorem 1.2. *Let $M \supset N$ be a pair of type III$_\lambda$ $(0<\lambda<1)$ factors with finite index and $\gamma : M \longrightarrow N$ the canonical endomorphism. We assume that the pair has the simultaneous discrete crossed product decomposition. Then The type II and the type III principal graphs of $M_1 \supset M$ do not coincide if and only if there appears the modular automorphism ${}_M[\sigma_t^{\hat{\tau}}]_M$ $(t \notin T(M))$ in ${}_M[\gamma^n]_M$ for some $n \in \mathbf{N}$.*

In contrast to the type III$_1$ case the discrete action can move central projections in the higher relative commutant algebras. So we have to treat an additional case. Assume that θ moves some central projections. Then we can show that there exists a decsendant ($M - M$ or $M - N$) sector $[\rho]$ satisfying $[\sigma_t^{\hat{\tau}} \rho] = [\rho]$ $(t \notin T(M))$. Hence,

$$[\rho\bar{\rho}] = [\sigma_t^{\hat{\tau}} \rho\bar{\rho}]$$

contains $[\sigma_t^{\hat{\tau}}]$.

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