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An explicit integral representation of Whittaker functions for the representations of the discrete series —The case of $SU(2,2)$—

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§1. Introduction

This paper is a supplement to a paper [Y-II] of Yamashita. Also it is an analogue of a result in [O].

We consider a Lie group $G = SU(2,2)$ and Whittaker functions of the large discrete series which have Whittaker model with respect to non-degenerate characters of a maximal unipotent subgroup N of G. Using Schmid operator, Yamashita [Y-II] explicitly computed the differential equations satisfied by the minimal K-type vectors in the Whittaker model of the discrete series representations.

The purpose of this paper is to push this computation one step further to obtain an explicit integral representation of the Whittaker functions representing these vectors belonging to the minimal K-type. There is a general integral representation due to Jacquet for Whittaker functions. But this representation is sometimes intractable for higher rank groups. We hope our formula is useful for investigation of L-factors of automorphic representations of the discrete series at the real places.

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§2. The group $SU(2,2)$ and its discrete series

2.1 *Structure of Lie group and Lie algebra.*

Let G be the special unitary group $SU(2,2)$ realized as

$$G = \{g \in SL(4,\mathbf{C}) \mid g^* I_{2,2} g = I_{2,2}\}, \quad I_{2,2} = \mathrm{diag}(1,1,-1,-1),$$

where $g^* = {}^t\bar{g}$ denotes the adjoint of a matrix g. We fix some notation for this group and its discrete series representations, used throughout this paper.

Let $U(4)$ be the unitary group of degree 4 in $SL(4,\mathbf{C})$. Take a maximal compact subgroup $K = G \cap U(4) = S(U(2) \times U(2))$. We set

$$\mathfrak{a}_p = \mathbf{R}H_1 + \mathbf{R}H_2 \quad \textit{with} \quad H_1 = X_{23} + X_{32}, \quad H_2 = X_{14} + X_{41},$$

where X_{ij} are elementary matrices given by

$$X_{ij} = (\delta_p^i \delta_q^j)_{1 \le p,q \le 4} \quad \textit{with Kronecker's} \quad \delta_p^j.$$

Then $\mathfrak{a}_p$ is a maximally split abelian subalgebra of $\mathfrak{g}$. Let Ψ denote the (restricted) root system of $(\mathfrak{g}, \mathfrak{a}_p)$. Then Ψ is of type C_2, and is expressed as

$$\Psi = \{\pm(\psi_1 \pm \psi_2)/2, \pm\psi_1, \pm\psi_2\}, \psi_i(H_j) = 2\delta_j^i \quad (i,j = 1,2).$$

Choose a positive system $\Psi^+ = \{(\psi_2 \pm \psi_1)/2, \psi_1, \psi_2\}$ having ψ_1 and $(\psi_2 - \psi_1)/2$ as its simple roots, and let

$$\mathfrak{n}_m = \sum_{\psi \in \Psi^+} \mathfrak{g}(\psi)$$

be the corresponding maximal nilpotent Lie subalgebra of $\mathfrak{g}$. Here $\mathfrak{g}(\psi)$ is the root subspace of $\mathfrak{g}$ corresponding to $\psi \in \Psi$. Then one obtains an Iwasawa decomposition of $\mathfrak{g}$ and G:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_p \oplus \mathfrak{n}_m, \quad G = KA_pN_m \quad \textit{with} \quad A_p = \exp \mathfrak{a}_p, \quad N_m = \exp \mathfrak{n}_m.$$

Now let

$$E_1 = \sqrt{-1}(H'_{23} - X_{23} + X_{32})/2, \qquad E_2 = \sqrt{-1}(H'_{14} - X_{14} + X_{41})/2,$$
$$E_3^{\pm} = (X_{13} + X_{43} \mp X_{12} \mp X_{42})/2, \quad E_4^{\pm} = (X_{24} - X_{21} \pm X_{34} \mp X_{31})/2,$$

where $H'_{kl} = X_{kk} - X_{ll}$ *for* $1 \leq k,\ l \leq 4$. *Then it is easily seen that*

$$E_i \in \mathfrak{g}(\psi_i), \quad E_j^{\pm} \in \mathfrak{g}((\psi_2 \pm \psi_1)/2) \otimes_{\mathbf{R}} \mathbf{C} \subset \mathfrak{n}_{m,\mathbf{C}}$$

for $i = 1, 2,\ j = 3, 4$, *and these six elements form a basis of the complexification* $\mathfrak{n}_{m,\mathbf{C}}$ *of* $\mathfrak{n}_m$.

By a direct computation we obtain the following expression of non-compact root vectors of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$ *along the complexified Iwasawa decomposition.*

Lemma 2.1.

$$\begin{aligned}
X_{23} &= \frac{1}{2}H'_{23} + \sqrt{-1}E_1 + \frac{1}{2}H_1, & X_{32} &= -\frac{1}{2}H'_{23} - \sqrt{-1}E_1 + \frac{1}{2}H_1,\\
X_{14} &= \frac{1}{2}H'_{14} + \sqrt{-1}E_2 + \frac{1}{2}H_2, & X_{42} &= -\frac{1}{2}H'_{14} - \sqrt{-1}E_2 + \frac{1}{2}H_2,\\
X_{13} &= -X_{43} + (E_3^+ + E_3^-), & X_{31} &= X_{34} + (E_4^- - E_4^+),\\
X_{24} &= X_{21} + (E_4^+ + E_4^-), & X_{42} &= -X_{12} + (E_3^- - E_3^+).
\end{aligned}$$

The above decomposition is used to compute the radial A_p*-part of the differential operator* $\mathcal{D}_{\lambda,\eta}$.

2.2 *Parametrization of the discrete series.*

Let us now parametrize the discrete series of $SU(2,2)$. *Take a compact Cartan subalgebra* $\mathfrak{t}$ *of* $\mathfrak{g}$ *consisting of all diagonal matrices in* $\mathfrak{k}$. *Then the root system* Δ *of* $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$, *of type* A_3, *is expressed as*

$$\Delta = \{\beta_{ij} \mid 1 \leq i, j \leq 4,\ i \neq j\},$$

where

$$\beta_{ij}(\mathrm{diag}(h_1, h_2, h_3, h_4)) = h_i - h_j$$

for $\mathrm{diag}(h_1, h_2, h_3, h_4) \in \mathfrak{t}_{\mathbf{C}}$. *Further the set of compact roots is given by* $\Delta_c = \{\pm\beta_{12}, \pm\beta_{34}\}$. *We identify the Weyl group* W *of* Δ *with the symmetric group* $\mathfrak{S}_4$ *of degree 4 acting on* $\mathfrak{t}_{\mathbf{C}}$ *by permutation of the diagonal entries. The compact Weyl group* W_c *is identified canonically with the subgroup* $\mathfrak{S}_2 \times \mathfrak{S}_2$.

Fix a positive system $\Delta_c^+ = \{\beta_{12}, \beta_{34}\}$ *of* Δ_c. *Then* Δ *admits precisely six positive systems* $\Delta_I^+, \Delta_{II}^+, \ldots, \Delta_{VI}^+$, *containing* Δ_c^+:

$$\Delta_J^+ = w_J \Delta_I^+ \quad \textit{with} \quad \Delta_I^+ = \{\beta_{ij} \mid i < j\},$$

where the elements $w_J \in W$ *are given as*

$$w_I = 1, \quad w_{II} = s_2, \quad w_{III} = s_2 s_3,$$

$$w_{IV} = s_2 s_1, \quad w_V = s_2 s_3 s_1 = s_2 s_1 s_3, \quad w_{IV} = s_2 s_1 s_3 s_2$$

in terms of the transpositions $s_i = (i, i+1)$ $(i = 1, 2, 3)$.

Let Ξ_c^+ *be the set of linear forms* Λ *on* $\mathfrak{t}_{\mathbb{C}}$ *satisfying the following three conditions:*

(1) $(\Lambda, \alpha) \neq 0$ *for any* $\alpha \in \Delta$, *i.e.* Λ *is* Δ*-regular,*

(2) $(\Lambda, \beta) \geq 0$ *for any* $\beta \in \Delta_c^+$, *i.e.* Λ *is* Δ_c^+*-dominant,*

(3) the map $\exp H \mapsto \exp\langle \Lambda + \rho, H\rangle$ $(H \in \mathfrak{t})$ *gives a unitary character of* $T = \exp \mathfrak{t} \subset K$, *i.e.* $\Lambda + \rho$ *is* K*-integral.*

Then this space $\Xi_c^+ \subset \mathfrak{t}_{\mathbb{C}}^*$ *of Harish-Chandra parameters are divided into six parts:*

$$\Xi_c^+ = \coprod_{I \leq J \leq VI} \Xi_J^+, \quad \Xi_J^+ = \{\Lambda \in \Xi_c^+ \mid \Lambda \textit{ is } \Delta_J^+\textit{-dominant}\}.$$

We note that Ξ_I^+ *(resp.* Ξ_{VI}^+*) corresponds to the holomorphic (resp. anti-holomorphic) discrete series.*

As determined in [Y-I], the Gelfand-Kirillov dimensions of the discrete series representations π *are given as follows,*

$$GK\text{-}\dim(\pi) = 4, \qquad \textit{if} \quad [\pi] \in \Xi_I^+ \cup \Xi_{VI}^+;$$

$$GK\text{-}\dim(\pi) = 6 = \dim \mathfrak{n}_m, \qquad \textit{if} \quad [\pi] \in \Xi_{II}^+ \cup \Xi_V^+;$$

$$GK\text{-}\dim(\pi) = 5, \qquad \textit{if} \quad [\pi] \in \Xi_{III}^+ \cup \Xi_{IV}^+.$$

(Recently a more general result is obtained by [Y-III]).

Therefore the representations π *belonging to* $\Xi_{II}^+ \cup \Xi_V^+$ *is a large representation in the sense of Vogan [V], hence has a Whittaker model (cf. Kostant [K] for quasi-split groups).*

For our later use, we employ a coordinates expression of elements $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$:

$$\alpha = (\alpha_1, \alpha_2, \alpha_3) \quad \textit{with} \quad \alpha_j = \alpha(H'_{j,j+1}).$$

Here $H'_{ij} = X_{ii} - X_{jj}$.

2.3 *Representation of the maximal compact subgroup.*

We want to give explicit realization of irreducible finite dimensional representation of the maximal compact group $K = S(U(2) \times U(2))$. *Since* $\mathfrak{k} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{sl}(2,\mathbf{C}) \oplus \mathfrak{sl}(2,\mathbf{C}) \oplus \mathbf{C}$, *we first fix the realization of representation of* $\mathfrak{sl}(2,\mathbf{C})$.

Choose a Cartan basis

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \bar{X} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

in $\mathfrak{sl}(2,\mathbf{C})$. *The set of integral dominant weight is identified with the set of non-negative integers* $\mathbf{N}$ *via correspondence*

$$d \in \mathbf{N} \longmapsto \{H' \mapsto d \in \mathrm{Hom}(\mathbf{Z}H', \mathbf{Z})\}.$$

Let (τ_d, V_d) *be the unique irreducible representation with highest weight* d. *Then* V_d *has a basis* $f_n = f_n^{(d)}$ $(0 \leq n \leq d)$ *consisting of weight vectors satisfying*

$$\begin{cases} \tau_d(X) f_n = f_{n+1}; \\ \tau_d(H') f_n = (2n-d) f_n; \\ \tau_d(\bar{X}) f_n = n(d-n+1) f_{n-1}. \end{cases}$$

For convenience, we use the convention that $f_{d+1} = f_{-1} = 0$. *Now the parametrization of the representation of* K *is given as follows. Let* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *be a* Δ_c^+*-dominant integral linear form on* $\mathfrak{t}$, *i.e.* $\lambda_i \in \mathbf{Z}$ $(i = 1,2,3)$ *and* $\lambda_1, \lambda_3 \geq 0$. *Let* $\mathfrak{z}$ *be the center of* $\mathfrak{t}_{\mathbf{C}}$ *and* $\mathfrak{t}'_{\mathbf{C}} = [\mathfrak{t}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}}] \simeq \mathfrak{sl}(2,\mathbf{C}) \oplus \mathfrak{sl}(2,\mathbf{C})$ *the derived algebra. Then* $\mathfrak{t}_{\mathbf{C}} = \mathfrak{z} \oplus \mathfrak{t}'_{\mathbf{C}}$.

Then the irreducible $\mathfrak{t}_{\mathbf{C}}$*-module* $(\tau_\lambda, V_\lambda)$ *with highest weight* λ *is realized on* $V_\lambda = V_{\lambda_1} \otimes V_{\lambda_3}$ *with the action*

$$\tau_\lambda(Y) = \tau_{\lambda_1}(Y_1) \otimes id_{V_{\lambda_3}} + id_{V_{\lambda_1}} \otimes \tau_{\lambda_3}(Y_2)$$

for $Y = \mathrm{diag}(Y_1, Y_2) \in \mathfrak{t}'_{\mathbf{C}}$ *with* $Y_i \in \mathfrak{sl}(2,\mathbf{C})$ $(i = 1,2)$. *Moreover the action of the center* $\mathfrak{z}$ *is determined by the action of the generator* $I_{2,2}$:

$$\tau_\lambda(I_{2,2}) = (\lambda_1 + 2\lambda_2 + \lambda_3) \cdot id_{V_\lambda}.$$

For our later computation, another coordinates expression for $\lambda = [r, s; u]$ *with*

$$r \equiv \lambda_1 = \lambda(H'_{12}), \quad s \equiv \lambda_3 = \lambda(H'_{34}),$$

$$u \equiv \lambda_1 + 2\lambda_2 + \lambda_3 = \lambda(I_{2,2})$$

is useful.

$\langle$*The adjoint representation*$\rangle$

The adjoint representation $\mathrm{Ad}\,\mathfrak{p}_{\mathbf{C}}$ *of* K *on* $\mathfrak{p}_{\mathbf{C}}$ *is decomposed into a direct sum of two irreducible subrepresentations:* $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$, *where*

$$\mathfrak{p}_{\pm} = \sum_{\beta \in \Delta^+_{I,n}} (\mathfrak{g}_{\mathbf{C}})_{\pm\beta} \quad \textit{and} \quad \Delta^+_{I,n} = \{\beta_{13}, \beta_{14}, \beta_{23}, \beta_{24}\}$$

is the set of non-compact roots in Δ^+_I. *The highest weights of* $\mathfrak{p}_+$, *and* $\mathfrak{p}_-$ *are* $\beta_{14} = [1,1;2]$ *and* $\beta_{32} = [1,1;-2]$, *respectively. For later use, we describe the* K*-isomorphisms* $\iota_{\pm} : \mathfrak{p}_{\pm} \simeq V_{[1,1;\pm 2]}$ *explicitly. The 4 elements* $f_{kl} = f^{(1)}_k \otimes f^{(1)}_l$ *with* $k, l \in \{0,1\}$ *form a basis of* $V_1 \otimes V_1 = V_{[1,1;\pm 2]}$. *Note that* $X_{23} \in \mathfrak{p}_+$, *and* $X_{41} \in \mathfrak{p}_-$ *are the lowest weight vectors. Then*

$$(X_{23}, X_{13}, X_{24}, X_{14}) \xmapsto{2+} (f_{00}, f_{10}, -f_{01}, -f_{11}),$$
$$(X_{41}, X_{31}, X_{42}, X_{32}) \xmapsto{2-} (f_{00}, f_{01}, -f_{10}, -f_{11}).$$

$\langle$*Decomposition of the tensor product* $\tau_\lambda \otimes \mathrm{Ad}_{\mathfrak{p}_{\mathbf{C}}}$$\rangle$

We decompose the $\mathfrak{t}_{\mathbf{C}}$*-module* $V_\lambda \otimes \mathfrak{p}_{\mathbf{C}}$ *into irreducible components, giving the projectors explicitly.*

Lemma 2.2.

(i) *The tensor product* $(\tau_d \otimes \tau_1, V_d \otimes V_1)$ of $\mathfrak{sl}(2,\mathbf{C})$*-modules decomposes as*

$$V_d \otimes V_1 \simeq V_{d+1} \otimes V_{d-1}.$$

(ii) *The projectors* $P^{\pm}_d : V_d \otimes V_1 \to V_{d\pm 1}$ are *up to scalar multiples, given respectively by* the *formulae:*

$$P^+_d(f^{(d)}_n \otimes f^{(1)}_0) = (d+1-n) f^{(d+1)}_n, \quad P^+_d(f^{(d)}_n \otimes f^{(1)}_1) = f^{(d+1)}_{n+1},$$
$$P^-_d(f^{(d)}_n \otimes f^{(1)}_0) = -n f^{(d-1)}_{n-1}, \quad P^-_d(f^{(d)}_n \otimes f^{(1)}_1) = f^{(d-1)}_n$$

for $0 \le n \le d$. Here $\{f^{(k)}_n\}_n$ *is the basis of* V_k *given above.*

This is easy to check (cf. Lemma 4.1 of [Y-I]).

The irreducible decomposition of $\mathfrak{t}_{\mathbf{C}}$*-module* $V_\lambda \otimes \mathfrak{p}_{\mathbf{C}}$ *is given as follows:*

$$V_\lambda \otimes \mathfrak{p}_{\mathbf{C}} = (V_\lambda \otimes \mathfrak{p}_+) \oplus (V_\lambda \otimes \mathfrak{p}_-), \quad V_\lambda \otimes \mathfrak{p}_\pm \simeq \bigoplus_{\varepsilon_1, \varepsilon_2 \in \{+1,-1\}} V_{[r+\varepsilon_1, s+\varepsilon_2; u\pm 2]}.$$

Furthermore the operator $P^{\varepsilon}_{rs} = P^{\varepsilon_1}_r \otimes P^{\varepsilon_2}_s$ *with* $\varepsilon = (\varepsilon_1, \varepsilon_2)$ *and* $P^{\pm 1}_d = P^{\pm}_d$ $(d \geq 0)$*, give a* $\mathfrak{t}_{\mathbf{C}}$*-module homomorphism from* $V_\lambda \otimes \mathfrak{p}_\pm$ *to the irreducible constituent* $V_{[r+\varepsilon_1, s+\varepsilon_2; u\pm 2]}$ *under the identification*

$$V_\lambda \otimes \mathfrak{p}_\pm = (V_r \otimes V_1) \otimes (V_s \otimes V_1), \quad V_{[r+\varepsilon_1, s+\varepsilon_2; u\pm 2]} = V_{r+\varepsilon_1} \otimes V_{s+\varepsilon_2}$$

as $\mathfrak{t}'_{\mathbf{C}}$*-modules. Noting the coordinates expressions of elements in* $\Delta^+_{I,n}$*:*

$$\beta_{ij} = [(-1)^{i+1}, (-1)^j; 2] \quad (i = 1, 2;\ j = 3, 4)$$

we can confirm the following.

Lemma 2.3.

(i) *Let* V_λ *be the irreducible* K*-module with highest weight* $\lambda = [r, s; u]$*. Then the tensor products* $V_\lambda \otimes \mathfrak{p}_+$ *and* $V_\lambda \otimes \mathfrak{p}_-$ *decompose into irreducibles as*

$$(\#) \qquad V_\lambda \otimes \mathfrak{p}_\pm \simeq \bigoplus_{\beta \in \Delta^+_{I,n}} V_{\lambda \pm \beta}.$$

(ii) *For each* $\beta \in \Delta^+_{I,n}$*, denote by* $\varepsilon(\beta) = (\varepsilon_1, \varepsilon_2)$ *the element of* $\{\pm 1\} \times \{\pm 1\}$ *corresponding to* β *through* $\beta = [\varepsilon_1, \varepsilon_2; 2]$*. Then the operators* $P^{\pm\varepsilon}_{rs}(\beta)$ *give projections from* $V_\lambda \otimes \mathfrak{p}_\pm$ *onto* $V_{\lambda \pm \beta}$ *along* $(\#)$*.*

§3. Radial A_p-part of the differential operator $\mathcal{D}_{\lambda,n}$

Let (τ, V) *be any finite dimensional representation of* K*, and* $(\eta, \mathcal{F})$ *a continuous Fréchet space representation of* N_m*. Then the space* $\mathcal{F}^\infty \subset \mathcal{F}$ *of* C^∞*-vectors for* η *is stable under the* N_m*-action, and the representation* η *on* $\mathcal{F}^\infty$ *with the usual Fréchet topology is smooth. The induced action of* $(\mathfrak{n}_m)_{\mathbf{C}}$ *on* $\mathcal{F}^\infty$ *is denoted by the same symbol* η.

Let $C^\infty_{\eta,\tau}(G)$ *be the space of* $(\mathcal{F} \otimes V)$*-valued* C^∞*-functions* F *on* G *satisfying*

$$F(kgn) = \tilde{\eta}(n) \otimes \tau(k^{-1})F(g), \quad (k, g, n) \in K \times G \times N_m.$$

Since F *is smooth, the value* $F(g)$ *lies in* $\mathcal{F}^\infty \otimes V$ *for every* $g \in G$*. In view of Iwasawa decomposition* $G = KAN_m \simeq K \times A_p \times N_m$ *(as* C^∞*-manifold), one finds that the restriction map* $r_{\eta,\tau} : F \mapsto F|A_p$ *sets up a linear isomorphism:*

$$C^\infty_{\eta,\tau}(G) \simeq C^\infty(A_p, \mathcal{F}^\infty \otimes V).$$

Here $C^\infty(A_p, E)$ *denotes the space of* C^∞*-functions on a* C^∞*-manifold* A_p *with values in a Fréchet space* E.

Let (τ_i, V_i) $(i = 1, 2)$ *be* K*-modules and* $D : C^\infty_{\eta,\tau_1}(G) \to C^\infty_{\eta,\tau_2}(G)$*, a linear mapping. Set*

$$R(D) = r_{\eta,\tau_2} \circ D \circ r^{-1}_{\eta,\tau_1} : C^\infty(A_p, \mathcal{F}^\infty \otimes V_1) \longrightarrow C^\infty(A_p, \mathcal{F}^\infty \otimes V_2).$$

We call this linear map $R(D)$ *the radial* A_p*-part of* D.

We want to write down explicitly the radial A_p*-part* $R(\mathcal{D}_{\eta,\lambda})$ *of the differential operator* $\mathcal{D}_{\eta,\lambda} : C^\infty_{\eta,\tau_\lambda}(G) \to C^\infty_{\eta,\tau_\lambda^-}(G)$ *for each* λ*. By definition,* $\mathcal{D}_{\eta,\lambda}$ *is expressed as*

$$\mathcal{D}_{\eta,\lambda} F = (id_{\mathcal{F}^\infty} \otimes P_\lambda)((\nabla_{\eta,\lambda} F)(\cdot)), \quad F \in C^\infty_{\eta,\tau_\lambda}(G),$$

where $\nabla_{\eta,\lambda} : C^\infty_{\eta,\tau_\lambda}(G) \to C^\infty_{\eta,\tau_\lambda \otimes \mathrm{Ad}}(G)$ *with* $\mathrm{Ad} = \mathrm{Ad}_{\mathfrak{p}_{\mathbf{C}}}$*, is defined as*

$$\nabla_{\eta,\lambda} F = \sum_k L_{X_k} F(\cdot) \otimes X_k,$$

by means of an orthonormal basis (X_k) *of* $\mathfrak{p}$.

Moreover P_λ *is a projector from* $V_\lambda \otimes \mathfrak{p}_{\mathbf{C}}$ *to* $V_\lambda^- \simeq \bigoplus_{\beta\in\Delta_n^+} V_{\lambda-\beta}$. *Then*

$$R(\mathcal{D}_{\eta,\lambda})\phi = (id_{\mathcal{F}^\infty} \otimes P_\lambda)(R(\nabla_{\eta,\lambda})\phi(\cdot)) \quad \textit{for} \quad \phi \in C^\infty(A_p, \mathcal{F}^\infty \otimes V_\lambda).$$

Choose as an orthonormal basis, the elements

$$(X_{ij} + X_{ji})/2\sqrt{2}, \quad \sqrt{-1}(X_{ij} - X_{ji})/2\sqrt{2} \quad (i = 1, 2;\ j = 3, 4).$$

Then

$$4\nabla_{\eta,\lambda}F = \nabla^+_{\eta,\lambda}F + \nabla^-_{\eta,\lambda}F$$

with

$$\begin{cases} \nabla^+_{\eta,\lambda}F = \sum_{i,j} L_{X_{ji}}F(\cdot) \otimes X_{ij}, \\ \nabla^-_{\eta,\lambda}F = \sum_{i,j} L_{X_{ij}}F(\cdot) \otimes X_{ji}. \end{cases} \quad (i = 1, 2,\ j = 3, 4)$$

The operator $\nabla^\pm_{\lambda,\eta}$ *are from* $C^\infty_{\tau_\lambda,\eta}(G)$ *to* $C^\infty_{\tau_\lambda\otimes\mathrm{Ad}_\pm,\eta}(G)$, *respectively. Here* $\mathrm{Ad}_\pm$ *is the adjoint representation of* K *on* $\mathfrak{p}_\pm$. *Thus we have* $4R(\nabla_{\lambda,\eta}) = R(\nabla^+_{\lambda,\eta}) + R(\nabla^-_{\lambda,\eta})$.

To express $R(\nabla^\pm_{\lambda,\eta})$ *concisely, we introduce some notations.*

Notation 3.1.

(i) *For the basis* $\{E_i, E_j^\pm;\ i = 1, 2,\ j = 3, 4\}$ *of* $\mathfrak{n}_{m,\mathbf{C}}$, *denote the operators* $\eta(E_i)$ and $\eta(E_j^\pm)$ on $\mathcal{F}^\infty$ by η_i and $\eta_j^\pm$, respectively.

(ii) We set $\partial_i = (L_{H_i}$ restricted to $A_p)$.

(iii) The function $a \in A_p \mapsto a^\psi \equiv e^{\psi(\log a)}$ will be written as e^ψ for each $\psi \in (\mathfrak{a}_p)^*_{\mathbf{C}}$.

(iv) Furthermore, it is convenient to employ the convention: for a $\mathfrak{t}_{\mathbf{C}}$-module V, $X \in \mathfrak{t}_{\mathbf{C}}$ and $E \in \mathfrak{n}_{m,\mathbf{C}}$, express the operators $X \otimes id_{\mathcal{F}^\infty}$ and $id_V \otimes \eta(E)$ on $V \otimes \mathcal{F}^\infty$ simply by X and $\eta(E)$, respectively.

(v) We define linear differential operators $\mathcal{L}_i^\pm$ and $\mathcal{S}_j^\pm$ on $C^\infty(A_p, V_\lambda \otimes \mathcal{F}^\infty)$ by

$$\mathcal{L}_i^\pm\phi = (\partial_i \pm 2\sqrt{-1}\, e^{-\psi_i}\eta_i)\phi \quad (i = 1, 2),$$
$$\mathcal{S}_j^\pm\phi = (e^{-(\psi_2+\psi_1)/2}\eta_j^+ \pm e^{-(\psi_2-\psi_1)/2}\eta_j^-)\phi \quad (j = 3, 4),$$

for $\phi \in C^\infty(A_p, V_\lambda \otimes \mathcal{F}^\infty)$.

Under the above notation we have the following

Proposition 3.1. *The operators* $R(\nabla^{\pm}_{\lambda,\eta}) : C^{\infty}(A_p, V_\lambda \otimes \mathcal{F}^{\infty}) \to C^{\infty}(A_p, V_\lambda \otimes \mathcal{F}^{\infty} \otimes \mathfrak{p}_{\pm})$, *are expressed as*

$$\text{(i)} \qquad R(\nabla^{+}_{\lambda,\eta}) = \frac{1}{2}(\mathcal{L}_1^- + H'_{23} - 2)(\phi \otimes X_{23}) + (X_{12} - \mathcal{S}_3^-)(\phi \otimes X_{24}) - (X_{34} + \mathcal{S}_4^-)(\phi \otimes X_{13}) + \frac{1}{2}(\mathcal{L}_2^- + H'_{14} - 6)(\phi \otimes X_{14}).$$

$$\text{(ii)} \qquad R(\nabla^{-}_{\lambda,\eta}) = \frac{1}{2}(\mathcal{L}_2^+ - H'_{14} - 6)(\phi \otimes X_{41}) + (X_{43} + \mathcal{S}_3^+)(\phi \otimes X_{31}) + (-X_{21} + \mathcal{S}_4^+)(\phi \otimes X_{42}) + \frac{1}{2}(\mathcal{L}_1^+ - H'_{23} - 2)(\phi \otimes X_{32}).$$

This is the proposition 5.1 of [Y-I].

§4. Differential difference equations for the minimal K-type

Retain the notation of §§2.3, and realize the representation $(\tau_\lambda, V_\lambda)$ $(\lambda = [r, s; u])$ *of* K *as in there* $V_\lambda = V_r \otimes V_s$ *with a basis*

$$f^{(rs)}_{kl} = f^{(r)}_k \otimes f^{(s)}_l \quad (0 \le k \le r,\ 0 \le l \le s)$$

consisting of weight vectors. Expand a function $\phi \in C^{\infty}(A_p, V_\lambda \otimes \mathcal{F}^{\infty})$ *as*

$$\phi(a) = \sum_{k,l} f^{(rs)}_{kl} \otimes c_{kl}(a) \quad (a \in A_p) \quad \textit{with} \quad c_{kl} \in C^{\infty}(A_p, \mathcal{F}^{\infty}).$$

We are going to write down the differential equation $R(\mathcal{D}_{\lambda,\eta})\phi = 0$ *by means of these coefficients* (c_{kl}). *Let* $\beta = [\varepsilon_1, \varepsilon_2; 2]$ *be a non-compact root in* $\Delta^{\pm}_I$ *with* $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$, *and recall the* K*-homomorphism* $P^{\pm(\varepsilon_1,\varepsilon_2)}_{rs}$ *from* $V_\lambda \otimes \mathfrak{p}_{\pm}$ *onto* $V_{\lambda \pm \beta}$ *given in Lemma (2.2). For simplicity, we denote the operators* $P^{\pm(\varepsilon_1,\varepsilon_2)}_{rs} \otimes id_{\mathcal{F}^{\infty}}$ *by* $P^{\pm(\varepsilon_1,\varepsilon_2)}_{rs}$.

Lemma 4.1. *Let* $(\tau_\lambda, V_\lambda)$ *be the minimal* K*-type of a discrete series, and* $\phi \in C^{\infty}(A_p, V_\lambda \otimes \mathcal{F}^{\infty})$. *Then* $R(\mathcal{D}_{\lambda,\eta})\phi = 0$ *if and only if*

$$P^{(\delta_1,\delta_2)}_{rs}(R(\nabla^{+}_{\lambda,\eta})\phi) = 0 \quad \text{and} \quad P^{(\varepsilon_1,\varepsilon_2)}_{rs}(R(\nabla^{-}_{\lambda,\eta})\phi) = 0.$$

Here (δ_1, δ_2) *and* $(\varepsilon_1, \varepsilon_2)$ *run over the elements of* $\{\pm 1\} \times \{\pm 1\}$ *in the following table.*

	(δ_1, δ_2)	$(\varepsilon_1, \varepsilon_2)$
$\langle I \rangle$	none	arbitrary, i.e. $(\pm 1, \pm 1)$
$\langle II \rangle$	$(-1, -1)$	$(-1, \pm 1)$, $(-1, -1)$
$\langle III \rangle$	$(-1, \pm 1)$	$(-1, \pm 1)$
$\langle IV \rangle$	$(\pm 1, -1)$	$(\pm 1, -1)$
$\langle V \rangle$	$(-1, \pm 1), (1, -1)$	$(-1, -1)$
$\langle VI \rangle$	arbitrary	none

where $\langle J \rangle$ means the case when $\Lambda = \lambda + \rho_c - \rho_n$ is Δ_J^+-dominant.

This is Lemma 5.2 of [Y1].

We modify some of the above differential equations in the following manner:

$$(C_1^\pm) \quad P_{rs}^{(-1,-1)}(R(\nabla_{\lambda,\eta}^\pm)\phi) = 0;$$

$$(C_2^\pm) \quad (P_r^- \otimes id_{V_s})(R(\nabla_{\lambda,\eta}^\pm)\phi) = 0 \quad \textit{with} \quad P_r^- \otimes id_{V_s} = P_{rs}^{(-1,-1)} \oplus P_{rs}^{(-1,1)};$$

$$(C_3^\pm) \quad (id_{V_r} \otimes P_s^-)(R(\nabla_{\lambda,\eta}^\pm)\phi) = 0 \quad \textit{with} \quad id_{V_r} \otimes P_s^- = P_{rs}^{(1,-1)} \oplus P_{rs}^{(-1,-1)};$$

$$(C_4^\pm) \quad R(\nabla_{\lambda,\eta}^\pm)\phi = 0.$$

Then $R(\mathcal{D}_{\lambda,\eta})\phi = 0$ *is equivalent to*

$$(C_1^+), (C_2^-), (C_3^-) \quad \textit{for the case } \langle II \rangle \textit{ and}$$

$$(C_1^-), (C_2^+), (C_3^+) \quad \textit{for the case } \langle V \rangle.$$

Now let us rewrite $(C_i^\pm)$ $(i = 1, 2, 3, 4)$ *more explicitly in terms of the component* c_{kl} *of* ϕ.

We put

$$b_0 = (r + s + u)/2, \qquad b_1 = (-r + s + u)/2 = b_0 - r,$$

$$b_2 = (r - s + u)/2 = b_0 - s, \qquad b_3 = (r + s - u)/2 = -b_0 + r + s,$$

which are integers by the integrability of $\lambda = [r, s; u]$.

In the following definition, we understand the undefined coefficients, say $c_{k,-1}$ *and* $c_{k,s+1}$ *are zero.*

Definition 4.1.

(i) *First, we define the equation* $(C_1^+) = (C_1^+ : 1)$ *on the coefficients* (c_{kl}) *by*

$$(C_1^+) \qquad (k+1)(l+1)(\mathcal{L}_1^- - k - l + b_0 - 2)\, c_{k+1,l+1} - 2(k+1)\mathcal{S}_3^- c_{k+1,l} + 2(l+1)\mathcal{S}_4^- c_{k,l+1} - (\mathcal{L}_2^- - k - l - b_3 - 4)\, c_{k,l} = 0,$$

where $0 \le k \le r-1$ *and* $0 \le l \le s-1$.

(ii) *Second we set*

$$(C_2^+ : 1) \qquad (k+1)(\mathcal{L}_1^- - k - l + b_0 - 1)\, c_{k+1,l} + 2c_{k,l-1} + 2\mathcal{S}_4^- c_{k,l} = 0;$$

$$(C_2^+ : 2) \qquad (\mathcal{L}_2^- - k + l - b_3 - 2)\, c_{k,l} + 2(k+1)\mathcal{S}_3^- c_{k+1,l} = 0,$$

for $0 \le k \le r-1$ *and* $0 \le l \le s$.

(iii) *Moreover we put*

$$(C_3^+ : 1) \qquad (l+1)(\mathcal{L}_1^- - k - l + b_0 - 1)\, c_{k,l+1} + 2c_{k-1,l} - 2\mathcal{S}_3^- c_{k,l} = 0;$$

$$(C_3^+ : 2) \qquad (\mathcal{L}_2^- + k - l - b_3 - 2)\, c_{k,l} - 2(l+1)\mathcal{S}_4^- c_{k,l+1} = 0,$$

for $0 \le k \le r$ *and* $0 \le l \le s-1$.

(iv) *Finally we set*

$$(C_4^+ : 1) \qquad (\mathcal{L}_1^- - k - l + b_0)\, c_{k,l} = 0;$$

$$(C_4^+ : 2) \qquad c_{k-1,l} - \mathcal{S}_3^- c_{k,l} = 0;$$

$$(C_4^+ : 3) \qquad c_{k,l-1} + \mathcal{S}_4^- c_{k,l} = 0;$$

$$(C_4^+ : 4) \qquad (\mathcal{L}_2^- + k + l - b_3)\, c_{k,l} = 0,$$

where $0 \le k \le r$ *and* $0 \le l \le s$.

Remark. *We note that* $(C_3^+ : i)$ *is obtained from* $(C_2^+ : i)$ *through the replacements:* $(k, r; l, s) \mapsto (l, s; k, r)$ *and* $(\mathcal{S}_3^-, \mathcal{S}_4^-) \mapsto (-\mathcal{S}_4^-, -\mathcal{S}_3^-)$.

Definition 4.2. The equation $(C_m^- : q)$ is given as follows in relation to $(C_m^+ : q)$. We put

$$d_{k,l} = \left(\prod_{n=1}^{k} n(r-n+1) \cdot \prod_{h=1}^{l} h(s-h+1) \right)^{-1} \cdot c_{r-k,s-l}.$$

Rewrite $(C_m^+ : q)$ to a system of differential equations for $(d_{k,l})$, and then replace the operators $\mathcal{S}_3^\pm$, $\mathcal{S}_4^\pm$, $\mathcal{L}_i^\pm$ $(i = 1, 2)$ and the constant u, respectively by $\mathcal{S}_4^\mp$, $\mathcal{S}_3^\mp$, $\mathcal{L}_i^\mp$ and $-u$. We name the resulting system of equations $(C_m^- : q)$.

Remark. For instance, $(C_2^- : 2)$ *is given as*

$$(C_2^- : 2) \qquad (k+1)(\mathcal{L}_2^+ + k - l - r - b_2 - 1)\, c_{k+1,l} + 2\mathcal{S}_4^+ c_{k,l} = 0 \quad (0 \le k \le r-1,\ 0 \le l \le s).$$

It should be noticed that $(C_m^+ : q)$ *is regained from* $(C_m^- : q)$ *by the same procedure as in the above definition.*

Proposition 4.2. *Let* m $(1 \le m \le 4)$ *be an integer and* $\varepsilon' \in \{+, -\}$. *A function* $\phi = \sum_{k,l} c_{kl} f_{kl}^{(rs)} \in C^\infty(A_p, V_\lambda \otimes \mathcal{F}^\infty)$ *fulfills* $(C_m^{\varepsilon'})$ *if and only if its coefficients* (c_{kl}) *satisfy the system of differential difference equations on* A_p : $(C_m^{\varepsilon'} : q)$ *with* $1 \le q \le \kappa_m$, *defined in Definition 4.1 and 4.2. Here* $\kappa_m = 1$ $(m = 1)$; $\kappa_m = 2$ $(m = 2, 3)$; $\kappa_m = 4$ $(m = 4)$.

§5. Solution of differential equation for a character η in the case II

Let η *be a one-dimensional representation of* N_m. *Then we solve explicitly the system of differential equations* C_1^+, C_2^-, C_3^- *for the minimal* K*-type* τ_λ *of a discrete series representation* π_Λ *with* $\Lambda \in \Xi_{II}$. *In particular we have an integral formula for the highest weight vector in the minimal* K*-type of the Whittaker realization of* π_Λ.

In what follows, we identify the vector group A_p *with* $\mathbf{R}^2$ *via*

$$(t_1, t_2) \in \mathbf{R}^2 \longmapsto \exp(-t_1 H_1 - t_2 H_2) \in A_p,$$

using the basis $\{H_i\}_{i=1,2}$ *of* $\mathfrak{a}_p$ *in (2.1). Then the differential operator* ∂_i *and the function* $e^{-\psi_i}$ *in (3.1) turn out to be* $\partial/\partial t_i$ *and* e^{2t_i} *respectively.*

Note that

$$\eta_2 = \eta(E_2) = 0, \quad \eta_j^+ = \eta(E_j^+) = 0 \quad (j = 3, 4)$$

because $E_2, E_j^+ \in [\mathfrak{n}_{m,\mathbf{C}}, \mathfrak{n}_{m,\mathbf{C}}]$. *This in turn implies that*

$$\mathcal{L}_2^+ = \mathcal{L}_2^- = \partial/\partial t_2, \quad \mathcal{S}_j^+ = -\mathcal{S}_j^- = e^{t_2 - t_1}\eta_j^-,$$

which we denote respectively by L_2 and S_j from now on.

We transfer the system (C_1^+), $(C_2^- : i)$, $(C_3^- : i)$, $(i = 1, 2)$ for (c_{kl}), $c_{kl} \in C^\infty(\mathbf{R}^2)$ $(0 \leq k \leq r,\ 0 \leq l \leq s)$, into a more convenient form to handle.

Definition 5.1. Set for each c_{kl},

$$h_{kl} = k!\, l!\, \exp\{\sqrt{-1}\, e^{2t_1}\eta_1 + (k + l - b_0)t_1 + (b_3 - k - l - 2)t_2\} \cdot c_{kl}$$

where $\eta_1 = \eta(E_1)$ and r, s, b_j $(0 \leq j \leq 3)$ are integers before Definition (4.1)

Proposition 5.1. *The system of functions (c_{kl}) is a solution of (C_1^+), (C_2^-) (C_3^-), if and only if (h_{kl}) satisfy the following differential equations:*

(i) $$e^{2(t_2-t_1)}(L_1 + 2L_2 - 4\sqrt{-1}\, e^{2t_1}\eta_1 - 2b_3)h_{k+1,l+1} - (L_2 - 2b_3 - 2)h_{kl} = 0$$
$$(0 \leq k \leq r - 1,\ 0 \leq l \leq s - 1),$$

(ii) $$e^{2(t_2-t_1)}(L_2 + 2)h_{k+1,l+1} + L_1 h_{kl} = 0 \qquad (0 \leq k \leq r - 1,\ 0 \leq l \leq s - 1),$$

(iii) $$(L_2 + 2(k + 1 - r))h_{k+1,l} + 2\eta_4^- h_{kl} = 0 \qquad (0 \leq k \leq r - 1,\ 0 \leq l \leq s)$$

(iv) $$(L_2 + 2(l + 1 - r))h_{k,l+1} - 2\eta_3^- h_{kl} = 0 \qquad (0 \leq k \leq r,\ 0 \leq l \leq s - 1),$$

(v) $$2e^{2(t_2-t_1)}\eta_3^- h_{k+1,s} + L_1 h_{ks} = 0 \qquad (0 \leq k \leq r - 1),$$

(vi) $$-2e^{2(t_2-t_1)}\eta_4^- h_{r,l+1} + L_1 h_{rl} = 0 \qquad (0 \leq l \leq s - 1),$$

where $L_i = \partial/\partial t_i$ *for* $i = 1, 2$.

This is Proposition 4.1 of [Y-II].

Now we assume that η is generic.

Assumption 5.1. $\eta_3^- \cdot \eta_4^- \neq 0$ and $\eta_1 \neq 0$.

In this case, any solution (h_{kl}) of (i)-(vi) in Proposition 5.1 is uniquely determined by $h = h_{rs}$ (i.e. the highest weight vector) through the relation (iii) and (iv). By (iv) and (vi), h should fulfil the equation

$$\text{(H-1)} \qquad (L_1 L_2 - 4S_3S_4)h = 0.$$

Further one get from (i), (iii) and (vi),

$$\text{(H-2)}' \qquad \{(L_2 - 2b_3 - 2)L_2^2 + 4S_3S_4(L_1 + 2L_2 - 4\sqrt{-1}\,e^{2t_1}\eta_1 - 2b_3)\}\, h = 0.$$

Conversely, it is easily checked that any $h \in C^\infty(\mathbf{R})$ *satisfying (H-1) and (H-2)′ can be extended uniquely to a solution* (h_{kl}) *of (i)–(vi) of Proposition (5.1) through (iii), (iv).*

Apply the operator L_1 *to (H-2)′ and use (H-1) to replace* L_2L_1h *by* $4S_3S_4h$*. Then we have*

$$\text{(H-2)} \qquad \{(L_1 + L_2)^2 + (-2b_3 - 2)(L_1 + L_2) + (-4\sqrt{-1}\,e^{2t_1}\eta_1)L_1\}\, h = 0.$$

Conversely, apply L_2 *to (H-2) and use (H-1), then we recover (H-2)′.*

Thus we get the following lemma.

Lemma 5.2. *The solutions* (h_{kl}) *of (i)–(vi) in Proposition (5.1) correspond bijectively to* $h \in C^\infty(\mathbf{R}^2)$ *satisfying (H-1) and (H-2) through* $h = h_{rs}$.

§6. Explicit integral formula for Whittaker functions

Now we want to solve the equations (H-1), (H-2). Changing the variables from t_i $(i = 1, 2)$ *to* $a_i = e^{t_i}$ $(i = 1, 2)$*, we put*

$$W(a_1, a_2) = h(\log a_1, \log a_2) \in C^\infty(\mathbf{R}^2_{\geq 0}).$$

Then (H-1) and (H-2) are replaced by

$$\text{(W-1)} \qquad \left\{ L_1L_2 - 4\eta_3^-\eta_4^- \left(\frac{a_2}{a_1}\right)^2 \right\} W = 0,$$

and

$$\text{(W-2)} \qquad \{(L_1 + L_2)^2 + (-2b_3 - 2)(L_1 + L_2) + (-4\sqrt{-1}\,\eta_1 a_1^2 L_1)\}\, W = 0,$$

where $L_i = a_i \frac{\partial}{\partial a_i}$ $(i=1,2)$*. Note that the system of equations (W-1), (W-2) is very similar to the system of partial differential equations (H-1), (H-2) of Lemma (8.1) of [O].*

Now we assume that the character $\eta : N_m \to \mathbf{C}$ *is unitary.*

Lemma 6.1. *When η is unitary, η_1 is a purely imaginary number, and $\sqrt{-1}\,\eta_3^-$ and $\sqrt{-1}\,\eta_4^-$ are mutually conjugate complex numbers. In particular $\eta_3^-\eta_4^- \leq 0$.*

Proof. Since $E_1 \in \mathfrak{n}_{m,\mathbf{R}}$, η_1 is purely imaginary. Because $E_3^- + E_4^- \in \mathfrak{n}_{m,\mathbf{R}}$ and $\sqrt{-1}E_3^- - \sqrt{-1}E_4^- \in \mathfrak{n}_{m,\mathbf{R}}$, $\eta_3^- + \eta_4^-$ and $\sqrt{-1}(\eta_3^- - \eta_4^-)$ are purely imaginary numbers. This settles the proof.

By assumption η is generic. Hence $\eta_3^-\eta_4^- < 0$.

We first find a formal solution of (W-1), (W-2). Write W as a Laplace transformation of Φ:

$$W(a_1, a_2) = \int_{\mathbf{R}^2} \Phi(u_1, u_2)\, e^{(u_1 a_1^{-2} + u_2 a_2^2)} du_1 du_2.$$

Then

$$L_1 L_2 W = \int (-4u_1 u_2)\Phi(u_1, u_2)\, e^{(u_1 a_1^{-2} + u_2 a_2^2)} du_1 du_2.$$

Therefore (W-1) implies an equation for a distribution Φ:

$$\left(\frac{a_2}{a_1}\right)^2 (u_1 u_2 + \eta_3^-\eta_4^-)\Phi = 0.$$

Hence Φ has support on the hyperbola $u_1 u_2 = -\eta_3^-\eta_4^- > 0$. Thus with a function φ on $\mathbf{R} - \{0\}$, we can write

$$W(a_1, a_2) = \int_{\mathbf{R}} \varphi(u) \exp\left\{ c\left(\frac{u}{a_1^2} - \frac{\eta_3^-\eta_4^-}{u} a_2^2 \right) \right\} \frac{du}{u},$$

where c is a constant ± 1.

Note that

$$a_1 \frac{\partial}{\partial a_1} W = \int_{\mathbf{R}} \left\{ \frac{-2cu}{a_1^2} \right\} \varphi(u) \exp\left\{ c\left(\frac{u}{a_1^2} - \frac{\eta_3^-\eta_4^-}{u} a_2^2 \right) \right\} \frac{du}{u},$$

and

$$a_2 \frac{\partial}{\partial a_2} W = \int_{\mathbf{R}} \left\{ \frac{-2c\eta_3^-\eta_4^- a_2^2}{u} \right\} \varphi(u) \exp\left\{ c\left(\frac{u}{a_1^2} - \frac{\eta_3^-\eta_4^-}{u} a_2^2 \right) \right\} \frac{du}{u}.$$

Assume that

$$\varphi(u) \exp\left\{ c\left(\frac{u}{a_1^2} - \frac{\eta_3^-\eta_4^-}{u} a_2^2 \right) \right\} \longrightarrow 0$$

when $u \to 0$ or $u \to \infty$, then integration by part implies that

$$(L_1 + L_2)W = \int_{\mathbf{R}} (-2)\varphi(u)\frac{\partial}{\partial u} \exp\left\{c\left(\frac{u}{a_1^2} - \frac{\eta_3^- \eta_4^-}{u} a_2^2\right)\right\} \cdot \frac{du}{u}$$

$$= \int_{\mathbf{R}} \left\{2u\frac{d}{du}\varphi(u)\right\} \cdot \exp\left\{c\left(\frac{u}{a_1^2} - \frac{\eta_3^- \eta_4^-}{u} a_2^2\right)\right\} \cdot \frac{du}{u}.$$

Hence (W-2) implies a differential equation for φ:

$$\left\{\left(2u\frac{d}{du}\right)^2 + (-2b_3 - 2)\left(2u\frac{d}{du}\right) + 8c\sqrt{-1}\eta_1\right\}\varphi = 0.$$

Assume that φ has support in $\{u \in \mathbf{R} \mid u > 0\}$. Then we should choose $c = -1$, in order to justify the integration by part.

Write

$$\varphi(u) = v^{\frac{1}{2}+(b_3+1)}\varphi_0(v)$$

with $v = \sqrt{u}$. Then $\varphi_0(v)$ satisfies the differential equation

$$(*): \qquad v^2\frac{d^2\varphi_0(v)}{dv^2} + \left\{\frac{1}{4} - (b_3+1)^2 + (-8\sqrt{-1}\,\eta_1)v^2\right\}\varphi_0 = 0.$$

Assume further that $-8\sqrt{-1}\,\eta_1$ is a negative real number, i.e. $\sqrt{-1}\,\eta_1$ is a positive real number. Recall that Δ_{II}^+-dominancy of the Harish-Chandra parameter Λ implies that

$$r + s + 2 > -u > |r - s| + 2.$$

Hence $b_3 = \frac{1}{2}(r + s - u)$ satisfies inequalities

$$r + s + 1 > b_3 > 1 + \max(r, s).$$

In particular b_3 is a positive integer.

When $\mathrm{Re}\,(k - \frac{1}{2} - m) \leq 0$, an integral representation

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z} \cdot z^k}{\Gamma(\frac{1}{2} - k + m)} \int_0^\infty t^{-k-\frac{1}{2}+m}\left(1 + \frac{t}{z}\right)^{k-\frac{1}{2}+m} e^{-t}dt$$

defined for $z \notin (-\infty, 0)$ satisfies the Whittaker differential equation

$$z^2\frac{d^2W}{dz^2} + \frac{1}{4}\left\{\frac{1}{4} - m^2 + kz + \left(-\frac{1}{4}\right)z^2\right\}W = 0.$$

Set $k = 0$ and $m = b_3 + 1$, and

$$\varphi_0(v) = W_{0,b_3+1}(\sqrt{32|\eta_1|} \cdot v).$$

Then φ_0 satisfies the differential equation $(*)$. This gives an integral representation of the function $W(a_1, a_2)$.

Theorem 6.2. *Let π_Λ be a discrete series representation of $SU(2,2)$ with a Δ_{II}^{+}-dominant Harish-Chandra parameter Λ. Assume that the character $\eta : N_m \to \mathbf{C}$ is unitary and generic. Then*

(i) *π_Λ has a Whittaker model for η if and only if $\mathrm{Im}(\eta_1) < 0$.*

(ii) *In this case, the function $h(\log a_1, \log a_2) = W(a_1, a_2)$ has an integral representation*

$$W(a_1,a_2) = \text{const.} \int_0^\infty v^{\frac{1}{2}+(b_3+1)} W_{0,b_3+1}(\sqrt{32|\eta_1|}\, v) \times \exp\left\{-\left(\frac{v^2}{a_1^2} + \frac{(-\eta_3^{-}\eta_4^{-})}{v^2} a_2^2\right)\right\} \frac{dv}{v}.$$

Here $b_3 = \frac{1}{2}(r+s-u) = -\lambda_2 = -\Lambda_2 - 1$ with Harish-Chandra parameter $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$.

Outline of the proof. *The argument of the proof is completely similar to the case of $Sp(2,\mathbf{R})$. We note here some key points. When $\mathrm{Im}(\eta_1) < 0$, i.e. $\sqrt{-1}\,\eta_1$ is positive, the function h satisfies the differential equations (H-1), (H-2).*

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