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A conjugacy class of regular operators

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1- Let $X = \mathbb{C}^{1+n}_{(t,x)}$ with coordinates $t \in \mathbb{C}$, $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, $(t, x; \tau, \xi)$ the symplectic coordinate of T^*X and $\mathcal{E}_X$ the sheaf of microdifferential operators on T^*X. In Kashiwara and Oshima's study of regular systems (cf. [K-O]), the following definition occurs (with a slightly different vocabulary): a matrix of microdifferential operators $A(x, D_x)$ is *essentially of nonpositive order* if there exists $\nu > 0$ such that the coefficients of *any* power of A are microdifferential operators of order at most ν. It is shown in [K-O] that any regular system of microdifferential equations $\mathcal{M}$ with regular singularities along $V = \{t = \xi_1 = \cdots = \xi_r = 0\}$, is a quotient of a system of the form $(tD_t - A(x, D_x))u = D_{x_1} = \cdots = D_{x_r} = 0$, with A essentially of nonpositive order, which enables one to define the monodromy of $\mathcal{M}$. Actually, it is shown that A may be chosen such that it has the following slightly more precise form:

$$(1) \qquad \begin{cases} [A, D_t] = [A, t] = 0, \quad \text{and} \quad A \text{ consists of blocks } \; A_{ij} \text{ of } \; N_i \times N_j \\ \text{matrices of differential operators such that } A_{ij} = 0 \text{ for } \; i > j \\ \text{and } A_{ii} = A_{ii}(x) \text{ is a matrix of holomorphic functions.} \end{cases}$$

When investigating e.g. distribution solutions of regular systems from a microlocal point of view, that is, solutions with values in the sheaf $\mathcal{C}^f_{\mathbb{R}^n}$ of tempered microfunctions on $\mathbb{R}^n$, one is led to look for the simplest normal form over some extension of $\mathcal{E}_X$ to a ring of operators acting on $\mathcal{C}^f_{\mathbb{R}^n}$ (see remark 2) below).

In particular one has on $\mathcal{C}^f_{\mathbb{R}^n}$ an action of $\mathcal{E}^{\mathbb{R},f}_X$, the ring of tempered microlocal operators (cf. [An]) and the purpose of this note is to prove the

Theorem 1. *Let $A = A(x, D_x)$ be a matrix of differential operators such that* (1) *holds. Then*

(i) *$(D_t I_N)^{A(x,D_x)}$ is a well defined invertible matrix operator over $\mathcal{E}^{\mathbb{R},f}_{X,(0;dt)}$, with inverse $(D_t I_N)^{-A(x,D_x)}$, and*

(ii) *one has $(D_t I_N)^{A(x,D_x)}(tD_t I_N - A(x, D_x))\,(D_t I_N)^{-A(x,D_x)} = tD_t I_N$.*

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2- Before going into the proof let us recall the following facts about $\mathcal{E}_X^{\mathbb{R},f}$. Let X be a n-dimensional complex manifold. The sheaf of rings $\mathcal{E}_X^{\mathbb{R},f}$ is the tempered version defined in [An] of the sheaf of rings $\mathcal{E}_X^{\mathbb{R}}$ on T^*X of holomorphic microlocal operators of [S-K-K], of which it is a subsheaf.

One has $\gamma^{-1}\gamma_*\mathcal{E}_X^{\mathbb{R},f} = \mathcal{E}_X$. Also $\mathcal{E}_X^{\mathbb{R},f}$ is faithfully flat over $\mathcal{E}_X$.

We are going to make use of a few topics of the theory of symbols of holomorphic microlocal operators as developped by Kataoka and Aoki (see [Ao] and the literature quoted there), which we adapt to the framework of $\mathcal{E}_X^{\mathbb{R},f}$.

Let $x_0^* = (x_0;\xi_0) \in \overset{\circ}{T}{}^*X$ and U a $\mathbb{R}_{>0}$-conical open neighborhood of x_0^*. Denote by

$$S(U) \quad (\text{resp. } S^f(U), \text{resp. } R(U)),$$

the space of holomorphic functions $p(x,\xi)$ on U such that for any compactly generated cone $U' \subset U$ one has :

$$\begin{cases} \text{for any } \ \varepsilon > 0 \quad p(x,\xi) = O(e^{\varepsilon|\xi|}) \text{ on } U' \\ (\text{resp. there exists } m > 0 \text{ such that } p(x,\xi) = O(|\xi|^m) \text{ on } U'), \\ (\text{resp. there exists } \delta > 0 \text{ such that } p(x,\xi) = O(e^{-\delta|\xi|}) \text{ on } U'). \end{cases}$$

The notations $S(U)$, $R(U)$ are borrowed from [Ao].

Proposition 2.

(i) (cf. [Ao]) *There is an isomorphism of vector spaces*

$$\varinjlim_{U \ni x_0^*} S(U)/R(U) \xrightarrow{\sim} \mathcal{E}^{\mathbb{R}}_{X,x_0^*}.$$

(ii) *The isomorphism of* (i) *induces an isomorphism*

$$\varinjlim_{U \ni x_0^*} S^f(U)/R(U) \xrightarrow{\sim} \mathcal{E}^{\mathbb{R},f}_{X,x_0^*}.$$

In a local coordinate system $(x_1,\cdots,x_n)$, the above morphisms take x_i to x_i, ξ_i to D_{ξ_i} and $x_i\xi_i$ to $x_iD_{\xi_i}$. In fact (ii) is easily deduced from the calculation in [Ao].

A representative $p(x,\xi) \in S(U)$ of an operator $P \in \mathcal{E}^{\mathbb{R}}_{X,x_0^*}$ for a suitable neighborhood U of x_0^* is called a *symbol* of P, and the lower bound of the $m \in \mathbb{R}$ such that $p(x,\xi) = O(|\xi|^m)$ in a conical neighborhood of x_0^* is called the order of P at x_0^*, e.g. $m < \infty$ iff $P \in \mathcal{E}^{\mathbb{R},f}_{X,x_0^*}$. As noted by Aoki, (i) of Proposition 2 entails that if $p(x,\xi) \in S(U)$ satisfies $p(x,\xi) = o(|\xi|)$ in a conical neighborhood of x_0^* then $\exp(p(x,\xi))$ is a symbol of an operator of $\mathcal{E}^{\mathbb{R}}_{X,x_0^*}$. By (ii) we get also :

$$(2) \qquad \begin{cases} \text{if } p(x,\xi) \in S(U) \text{ satisfies } \ p(x,\xi) = O(\log|\xi|) \\ \text{in a conical neighborhood of } \ x_0^*, \\ \text{then } \ \exp(p(x,\xi)) \text{ is a symbol of an operator of } \mathcal{E}^{\mathbb{R},f}_{X,x_0^*} \end{cases}$$

For example, if $X = \mathbb{C}^n_x$ with coordinates $x = (x_1, \ldots, x_n)$, one has $\exp(x_1 \log D_{x_1}) \in \mathcal{E}^{\mathbb{R},f}_{X,dx_1}$ whereas $\exp(x_1\sqrt{D_{x_1}}) \in \mathcal{E}^{\mathbb{R}}_{X,dx_1} \setminus \mathcal{E}^{\mathbb{R},f}_{X,dx_1}$.

3- *Proof of theorem 1.* We have to estimate the growth order in τ of the symbol of $\exp(A(x;\xi)\log\tau)$. This is done by a straightforward use of norms of matrices of microdifferential operators. (Of course, if $N = 1$, the theorem is already implied by (2).)

For a differential operator $P(x, D_x)$ of order $\leq l$ we denote as usually by $N_l(P, s) = N_l(P(x;\xi), s)$ the Boutet de Monvel and Krée formal norm of P defined as the series

$$N_l(P,s) = \sum_{\alpha,\beta,k} \frac{2}{(2n)^k} \frac{k!}{(\mid \alpha \mid +k)!(\mid \beta \mid +k)!} |\partial_x^\alpha \partial_\xi^\beta P_{l-k}(x;\xi)| s^{2k+|\alpha+\beta|},$$

where $P_{l-k}(x;\xi)$ denotes the symbol of the homogenous part of order $l-k$ of P, and s is an independant variable.

If $P = (P_{ij})$ is $N \times N$ matrix whose entries are (micro-) differential operators of order at most l, we denote, as usually by $N_l(P, s)$ the matrix whose entries are the $N_l(P_{ij}, s)$. Recall then, that if Q is another $N \times N$ matrix of (micro-) differential operators of order at most l' one has

$$N_{l+l'}(PQ, s) << N_l(P, s) N_{l'}(Q, s),$$

where the symbol $<<$ means that each entry of the matrix on the right side is a majorant series of the corresponding entry of the matrix on the left side.

We will need the following two estimates. Let $P(x, D_x)$ be a matrix of *differential* operators of order at most l.

Fix $(x_0;\xi_0) \in T^*\mathbb{C}^n$. By using Cauchy inequalities, it is easy to see that one may find:

a conic neighborhood V of $(x_0;\xi_0)$, a constant $M > 0$ and a matrix function $c(s) = \sum a_j s^j$ holomorphic near $s = 0$, where a_j a constant matrix with nonnegative entries,

such that

$$N_l(P, s) << M(1+ \mid \xi \mid^l) c(s), \text{ uniformly in } (x;\xi) \in V. \tag{3}$$

If $a = (a_{ij})$ is a $N \times N$ matrix of complex numbers we use the notation

$$\|a\| = \mathrm{Sup}_{i,j} \mid a_{ij} \mid.$$

Then if P is a matrix of differential operators of order at most l as before, and if $s = r$ with $0 < r < 1$, we get from the definition the obvious estimate

$$\|P(x;\xi)\| \leq \sum_{0 \leq k \leq l} \|P_{l-k}(x;\xi)\| \leq \frac{l!}{2} \frac{(2n)^l}{r^{2l}} N_l(P, r), \tag{4}$$

where $P_{l-k}(x;\xi)$ is the matrix of symbols of order $l-k$ of P and $P(x;\xi) = \sum_k P_{l-k}(x;\xi)$ is the total symbol matrix.

Proof of (i). The matrix $A = A(x, D_x)$ being as in (1), we may write $A = A_0 + B$ where $A_0 = A_0(x)$ is the matrix of holomorphic functions consisting of the diagonal blocks A_{ii} of A, and $B := A - A_0$ consists of the blocks B_{ij} with $B_{ij} = A_{ij}$ if $i < j$ and zero otherwise. Let ν be the least integer $1 \leq \nu \leq N-1$ such that $B_{ij} = 0$ for all i, j such that $i + \nu \leq j$.
For multi-indices $\alpha = (\alpha_1, \alpha_2, \dots)$, $\beta = (\beta_1, \beta_2, \dots) \in \mathbb{N}^{(\mathbb{N})}$, we set

$$\varphi^{\alpha,\beta}(A_0, B) = A_0^{\alpha_1} B^{\beta_1} A_0^{\alpha_2} B^{\beta_2} \cdots \in \mathsf{M}_N(\mathcal{E}_X).$$

Now because of the particuliar forms of A_0 and B we get that $\varphi^{\alpha,\beta}(A_0, B) = 0$ for $|\beta| > \nu$. Hence for any integer $m \geq 0$, we have

$$A^m = (A_0 + B)^m = \sum_{|\alpha+\beta|=m,\ |\beta|\leq\nu} \varphi^{\alpha,\beta}(A_0, B).$$

Let l be the maximum order of the entries of B, then the above formula implies that for any integer $m \geq 0$, the matrix A^m has entries of order $\leq \nu l$, and we have

$$N_{\nu l}(A^m, s) << \sum_{0\leq k\leq\nu} \binom{m}{k} N_0(A_0, s)^{m-k} N_l(B, s)^k.$$

Fixing $(0;\xi_0) \in T^*\mathbb{C}^n$, and making use of (3) and of its notations, one may find $M > 0$ and a matrix function $c(s)$ holomorphic near $s = 0$ such that

$$N_l(B, s)^k << M(1 + |\xi|^l)^\nu c(s)$$

holds for $0 \leq k \leq \nu$ and $(x;\xi)$ in a conic neighborhood V of $(0;\xi_0)$.
Thus

$$N_{\nu l}(A^m, s) << M(1 + |\xi|^l)^\nu c(s)(1 + N_0(A_0, s))^m, \tag{5}$$

for any integer $m \geq 0$ and any $(0;\xi_0) \in V$.
Reducing the conic neighborhood V if necessary, we may choose $0 < r < 1$ and a constant $M_1 > 0$ such that $\|N_0(A_0, r)\| \leq M_1$ uniformly in $(0;\xi_0) \in V$. Then, since

$$\|A^m(x;\xi)\| \leq \frac{(\nu l)!}{2} \frac{(2n)^{\nu l}}{r^{2\nu l}} N_{\nu l}(A^m, r)$$

by (4), we find by (5) that there is a constant $M_2 > 0$ such that

$$\|A^m(x;\xi)\| \leq M_2(1 + |\xi|^l)^\nu (1 + M_1)^m, \tag{6}$$

for any $m \geq 0$ and $(0;\xi_0) \in V$.
Fix the determination of $\log\tau$ in $\mathrm{Re}\tau > \mathrm{Im}\tau$ such that $\log 1 = 0$. We have to estimate the growth of the the matrix-valued holomorphic function $\exp(A(x,\xi)\log\tau)$

as $|\tau| \to \infty$, τ in a conic neighborhood of $dt\infty$. Let $|\tau| > e^{\pi}$, thus $|\log \tau| \leq \sqrt{2}\log|\tau|$. Using this and (6) we get

$$\begin{aligned}
\|\tau^{A(x;\xi)}\| &= \|\sum_{m\geq 0}(A(x;\xi)\log\tau)^m/m!\| \leq \sum_{m\geq 0}\|A^m(x;\xi)\|\,|\log\tau|^m/m! \\
&\leq M_2(1+|\xi|^l)^\nu \sum_{m\geq 0}(1+M_1)^m 2^{m/2}(\log|\tau|)^m/m! \\
&\leq M_2(1+|\xi|^l)^\nu\,|\tau|^{\sqrt{2}(1+M_1)},
\end{aligned}$$

and this holds for $|\tau| > e^{\pi}$, uniformly for $(x;\xi)$ in a small enough conic neighborhood of $(0;\xi_0)$. The choice of ξ_0 having been made arbitrarily, this proves that $\tau^{A(x;\xi)}$ is a well defined symbol of an operator of $\mathcal{E}^{\mathbb{R},f}_{X,(0;dt)}$ that we denote by $D_t^{A(x,D_x)}$ (we omit the notation I_N).

Proof of (ii). Since $[A,D_t]=0$ one has $[(D_t)^A,t]=A(D_t)^{A-I_N}$. Hence $(D_t)^A(tD_t-A)=t(D_t)^{A+I_N}+A(D_t)^A-(D_t)^AA=tD_t(D_t)^A$. □

Remarks.

1- The theorem should be more generally true when the matrix $A(x,D_x)$ is essentially of nonpositive order, but the proof given above breaks down for this more general case.

2- Distribution solutions of regular operators are investigated in [A-M.F]; the above theorem provides an alternate proof of proposition 3.1 of that paper.

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ERRATUM to **"A conjugacy class of regular operators"**
(*E.Andronikof*, "MICROLOCAL GEOMETRY")

In the last part of the proof of (i) of theorem 1 a confusion between $(A(x;\xi))^m$ and $A^m(x;\xi)$ occurs. The proof stands when replacing

$$\exp(A(x;\xi)\log\tau)$$

with the following matrix-valued function

$$p(x;\xi) := \Sigma_{m\geq 0} A^m(x;\xi)(\log\tau)^m/m!\,,$$

then by defining $(D_t)^A$ as the matrix operator with coefficients in $\mathcal{E}_X^{\mathbf{R},f}$ given by the symbol $p(x;\xi)$.