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Localizations of a class of strongly hyperbolic systems

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1. Introduction

In this note we are concerned with strongly hyperbolic systems in an open set Ω in $\mathbb{R}^{n+1}$ with involutive characteristics. We introduce, in section 3, localizations of systems at a multiple characteristic where the dimension of the kernel of the principal symbol is equal to the order of the characteristic. There we also give the definition of non degenerate characteristics (Definition 3.3). Then we study how the localization of systems inherits strong hyperbolicity of the original system. To do so, in section 2, we first study two kinds of second order localizations. The first one is the usual one and obtained by successive localizations but provides less precise informations on the original symbol. The second one, which provides more detailed informations than the first one, is rather complicated and the invariant meaning is less clear. However see Lemma 2.7 below.

In general the localization is not strongly hyperbolic system even if the original system is strongly hyperbolic and the characteristic is involutive, in contrast with the scalar case. Our first result is concerned with a strongly hyperbolic system with an involutive characteristic of order r and hence the localization is a $r \times r$ system. Then we prove that every $(r-1)$-th minor of the localization vanishes of order $s-2$ at every characteristic of order s of the localization (Theorem 4.1). This means that the localization must satisfy a same necessary condition which is verified by the original strongly hyperbolic system (see Theorem 1.1 in [7]).

If the characteristic is involutive and of order r then every $(m-1)$-th minor of $m \times m$ strongly hyperbolic system vanishes of order $r-1$ at the reference characteristic (see Theorem 1.3 in [7]). Let z^0, z^1 be characteristics of the original system and its localization at z^0 of order r and s respectively. Then, assuming that the characteristic set is an involutive C^∞ manifold, we show that, under some restrictions, every $(r-1)$-th minor of the localization vanishes of order $s-1$ at z^1 if (z^0, z^1) is *involutive* (Theorems 5.1 and 5.2). In particular the localization is diagonalizable at this characteristic. If we further assume that the characteristic is non degenerate, refering to our previous results in [5], we can show that the localization is strongly hyperbolic, more precisely the coefficient matrices of the localization are simultaneously symmetrizable (Proposition 5.4). We also show that the same result holds for a larger class of strongly hyperbolic systems which are not coordinate free though (Proposition 5.3). In particular this gives a generalization of Theorem 1 in [9].

2. Higher order localizations

Let $h(x)$ be a monic polynomial in x_1 of degree m:

$$h(x) = x_1^m + \sum_{j=1}^{m} a_j(x')x_1^{m-j}$$

where $a_j(x') \in C^\infty(U)$, $x' = (x_2, ..., x_n)$ and U is an open neighborhood of the origin of $\mathbb{R}^{n-1}$. We assume that $h(x)$ is hyperbolic with respect to the x_1 variable, that is the equation $h(x) = 0$ in x_1 has only real roots for every $x' \in U$. Let $x^0 \in \mathbb{R} \times U = \Omega$ be a characteristic of h of order r_0:

$$d^j(x^0) = 0,\ j < r_0,\ \ d^{r_0}h(x^0) \neq 0.$$

We define $h_{x^0}(x)$ as

$$h(x^0 + \mu x) = \mu^{r_0}\left(h_{x^0}(x) + O(\mu)\right),\ \mu \to 0$$

which is a well defined homogeneous polynomial of degree r_0 on $T_{x^0}\Omega$. Moreover $h_{x^0}(x)$ is hyperbolic with respect to the x_1 variable (cf. Lemma 1.3.3 in [3]). We also define the lineality of $h_{x^0}(x)$ as

$$\Lambda_{x^0}(h) = \{x \in T_{x^0}\Omega | h_{x^0}(y + tx) = h_{x^0}(y),\ \forall t \in \mathbb{R},\ \forall y \in T_{x^0}\Omega\}$$

which is a linear subspace in $T_{x^0}\Omega$ (see [1], [2]).

In the following we denote by μ_0, μ_1 two small parameters with $0 < \mu_0 \leq \mu_1 \ll 1$.

Lemma 2.1. *Let x^1 be a characteristic of order r_1 of h_{x^0} and let $y \in \Lambda_{x^0}(h)$. Then we have*

$$h(x^0 + \mu_0(x^1 + y) + \mu_0\mu_1 x) = \mu_0^{r_0}\mu_1^{r_1}\left(h_1(y, x, \mu_0/\mu_1) + \mu_1 g_1(y, x, \mu_1, \mu_0/\mu_1)\right)$$

where $h(y, x, \xi)$ is a polynomial in (y, x, ξ), homogeneous of degree r_1 in (x, ξ) which is hyperbolic with respect to the x_1 variable and $g_1(y, x, \mu_1, \xi)$ is C^∞ in $|\mu_1| + |\mu_0\mu_1 x| + |\mu_0 y| < \epsilon$, $|\xi| < 2$ with sufficiently small $\epsilon > 0$.

Proof: It is clear that we can write

$$h(x^0 + \mu_0 x) = \mu_0^{r_0}\left(h_{x^0}(x) + \mu_0 g_0(x, \mu_0)\right)$$

where $g_0(x, \mu_0)$ is C^∞ in $|\mu_0| + |\mu_0 x| < \epsilon$ with small ϵ. By Rouché's theorem and hyperbolicity of h it follows that

$$h_{x^0}(x^1 + y + x) + \mu_0 g_0(x^1 + y + x, \mu_0) = h_{x^0}(x^1 + x) + \mu_0 g_0(x^1 + y + x, \mu_0) = 0$$

has r_1 real zeros converging to zero with $(x', \mu_0) \to (0,0)$. Applying Lemma 1.3.3 in [3] we obtain

$$(2.1) \qquad h(x^0 + \mu_0(x^1 + y + x)) = \mu_0^{r_0}\left(h_1(y,x,\mu_0) + \tilde{g}_0(y,x,\mu_0)\right)$$

where $h_1(y,x,\mu_0)$ is a polynomial in (y,x,μ_0), homogeneous in (x,μ_0) of degree r_1 which is hyperbolic with respect to the x_1 variable and $\tilde{g}_0(y,x,\mu_0)$ is C^∞ in $|\mu_0| + |\mu_0 x| + |\mu_0 y| < \epsilon$ with small ϵ of the form

$$\tilde{g}_0(y,x,\mu_0) = \sum_{|\alpha|+j=r_1+1} x^\alpha \mu_0^j G_{\alpha j}(y,x,\mu_0).$$

Here note that

$$\begin{aligned}\tilde{g}_0(y,\mu_1 x,\mu_0) &= \mu_1^{r_1+1} \sum_{|\alpha|+j=r_1+1} x^\alpha (\mu_0/\mu_1)^j G_{\alpha j}(y,\mu_1 x, \mu_1(\mu_0/\mu_1)) \\ &= \mu_1^{r_1+1}\tilde{g}_1(y,x,\mu_1,\mu_0/\mu_1).\end{aligned}$$

It is clear that $\tilde{g}_1$ is C^∞ in $|\mu_1| + |\mu_0\mu_1 x| + |\mu_0 y| < \epsilon$, $|\mu_0/\mu_1| < 2$ with small ϵ. Then replacing x by $\mu_1 x$ in (2.1) we get the desired result. $\square$

We are interested in the case either $\mu_0 = \mu_1$ or $\mu_0 = O(\mu_1^{m+1})$. In the former case we set

$$h_{\{x^0,x^1\}}(y,x) = h_1(y,x,1),\; g_1(y,x,\mu) = \mu\tilde{g}_1(y,x,\mu,1)$$

so that

$$(2.2) \qquad h(x^0 + \mu(x^1 + y) + \mu^2 x) = \mu^{r_0+r_1}\left(h_{\{x^0,x^1\}}(y,x) + g_1(y,x,\mu)\right)$$

where g_1 is C^∞ in $|\mu| + |\mu^2 x| + |\mu y| < \epsilon$ with small ϵ and $g_1(y,x,0) = 0$. In the latter case we set

$$\begin{aligned} h_{(x^0,x^1)}(y,x) &= h_1(y,x,0), \\ g_1(y,x,\mu_1,\mu_0/\mu_1) &= \mu_1\tilde{g}_1(y,x,\mu_1,\mu_0/\mu_1) + h_1(y,x,\mu_0/\mu_1) - h_1(y,x,0)\end{aligned}$$

so that

$$(2.3) \quad h(x^0 + \mu_0(x^1 + y) + \mu_0\mu_1 x) = \mu_0^{r_0}\mu_1^{r_1}\left(h_{(x^0,x^1)}(y,x) + g_1(y,x,\mu_1,\mu_0/\mu_1)\right)$$

where $g_1(y,x,\mu_1,\mu_0/\mu_1)$ is C^∞ in $|\mu_1 + |\mu_0\mu_1 x| + |\mu_0 y| < \epsilon$ with small $\epsilon > 0$ and $g_1(y,x,0,0) = 0$. Note that by definition we have

$$(2.4) \qquad h_{(x^0,x^1)}(y,x+w) = h_{(x^0,x^1)}(y,x),\; h_{\{x^0,x^1\}}(y,x+w) = h_{\{x^0,x^1\}}(y,x)$$

for every $w \in \Lambda_{x^0}(h)$.

Lemma 2.2. *$h_{(x^0,x^1)}(y,x)$ is independent of $y \in \Lambda_{x^0}(h)$ and we have*

$$h_{(x^0,x^1)}(x) = (h_{x^0})_{x^1}(x).$$

Proof: Since $h(x^0 + \mu_0 x) = \mu_0^{r_0}\left(h_{x^0}(x) + O(\mu_0)\right)$ it follows that

$$h(x^0 + \mu_0(x^1 + y + \mu_1 x)) = \mu_0^{r_0}\left(h_{x^0}(x^1 + \mu_1 x) + O(\mu_0)\right)$$

because $y \in \Lambda_{x^0}(h)$. Since x^1 is a characteristic of h_{x^0} of order r_1 we see that

$$h_{x^0}(x^1 + \mu_1 x) = \mu_1^{r_1}\left((h_{x^0})_{x^1}(x) + O(\mu_1)\right).$$

Noting $\mu_0 = O(\mu_1^{m+1})$ we get

$$h(x^0 + \mu_0(x^1 + y) + \mu_0\mu_1 x) = \mu^{r_0}\mu_1^{r_1}\left((h_{x^0})_{x^1}(x) + O(\mu_1)\right)$$

which shows the assertion. □

In particular $h_{(x^0,x^1)}(x)$ is well defined independent of the choice of parameters μ_j provided if $\mu_0 = O(\mu_1^{m+1})$. Note that Lemma 2.1 shows that

$$h_{\{x^0,x^1\}}(y, \lambda x) = h_1(y, \lambda x, 1) = \lambda^{r_1} h_1(y, x, 1/\lambda)$$

which implies that

$$\lim_{\lambda\to\infty} \lambda^{-r_1} h_{\{x^0,x^1\}}(y, \lambda x) = h_1(y, x, 0) = h_{(x^0,x^1)}(x) \tag{2.5}$$

that is, $h_{(x^0,x^1)}(x)$ is the principal part of $h_{\{x^0,x^1\}}(y,x)$ with respect to x. Denoting by $\Lambda_{(x^0,x^1)}(h)$ the lineality of $h_{(x^0,x^1)}$:

$$\Lambda_{(x^0,x^1)}(h) = \{x \in T_{x^0}\Omega \,|\, h_{(x^0,x^1)}(y + tx) = h_{(x^0,x^1)}(y), \forall t \in \mathbb{R}, \forall y \in T_{x^0}\Omega\}$$

which is a linear subspace in $T_{x^0}\Omega \cong T_{x^1}\Omega$, it follows from (2.4) that

$$\Lambda_{x^0}(h) \subset \Lambda_{(x^0,x^1)}(h). \tag{2.6}$$

If x^1 is a characteristic of h_{x^0} then $x^1 + y$, $y \in \Lambda_{x^0}(h)$ is also a characteristic of h_{x^0} of the same order and hence

$$h_{\{x^0,x^1+y\}}(0,x) = h_{\{x^0,x^1\}}(y,x), \quad y \in \Lambda_{x^0}(h). \tag{2.7}$$

Lemma 2.3. *We have*

$$h_{\{x^0,x^1\}}(y, x + w) = h_{\{x^0,x^1\}}(y,x), \quad \forall w \in \Lambda_{(x^0,x^1)}(h).$$

Proof: Since $h_{(x^0,x^1)}(x)$ is the principal part of $h_{\{x^0,x^1\}}(y,x)$ with respect to x and $h_{\{x^0,x^1\}}(y,x)$ is hyperbolic with respect to the x_1 variable the assertion follows from Corollary 12.4.8 in [2]. □

Lemma 2.4. *Let $x = (x_a, x_b)$ be a partition of the variable x and assume that $x^0 = (x_a^0, 0) \in \Lambda_{x^0}(h)$ is a characteristic of h and*

$$h(\lambda x_a, x_b) = \lambda^m h(x_a, x_b), \quad \forall \lambda \in \mathbb{R}.$$

Then we have

$$h_{\{x^0,x^1\}}(tx^0, x) = h_{\{x^0,x^1\}}(0, x - t(x_a^1, 0)), \quad \forall t \in \mathbb{R}.$$

Proof: Set $y = x^0 + \mu(x^1 + tx^0) + \mu^2 x$. Then we have

$$y_a = (1 + \mu t)(x_a^0 + \mu x_a^1 + \mu^2((x_a - tx_a^1) + O(\mu))), \; y_b = \mu x_b^1 + \mu^2 x_b.$$

From the assumption it follows that

$$h(x^0 + \mu(x^1 + tx^0) + \mu^2 x) = (1 + \mu t)^m h(x^0 + \mu x^1 + \mu^2(x - t(x_a^1, 0) + O(\mu)))$$

which proves the assertion. □

Set

$$H_l(x^0; x) = \sum_{|\beta|=l} h^{(\beta)}(x^0) x^\beta / \beta!$$

where $h^{(\beta)}(x^0) = \partial^{(\beta)} h(x^0)/\partial x^\beta$. Then

Lemma 2.5. *Let x^0, x^1 be characteristics of h, h_{x^0} of order r and s respectively. Assume that $\Lambda_{(x^0,x^1)}(h)$ is given by $x_a = 0$ where $x = (x_a, x_b)$ is a partition of the variable x. Then we have*

$$H_l^{(\alpha)}(x^0; x + y) = 0, \quad \forall y \in \Lambda_{x^0}(h), \; l + |\alpha| = r + s$$

unless $\alpha = (\alpha_a, 0)$.

Proof: By definition we see easily that

$$h_{\{x^0,x^1\}}(y, x) = \sum_{l+|\beta|=r+s} H_l^{(\beta)}(x^0; x^1 + y) x^\beta / \beta!.$$

Since Lemma 2.3 shows that $h_{\{x^0,x^1\}}(y, x)$ is a polynomial in (y, x_a) we obtain the desired result. □

We now study how $h_{\{x^0,x^1\}}(y,x)$ depends on $y \in \Lambda_{x^0}(h)$ assuming that

$$\Sigma = \{x \in \Omega | d^j h(x) = 0, j < r, d^r h(x) \neq 0\}$$

is a C^∞ manifold through x^0. For $y \in \Sigma$ and $x \in N_y\Sigma$, a normal of Σ at y, we define $h_\Sigma(y,x)$ as

$$h_\Sigma(y,x) = \lim_{\mu \to 0} \mu^{-r} h(y + \mu x)$$

which is well defined on the normal bundle $N\Sigma$ of Σ. Let Ω_Σ be the blow up of Ω along Σ, that is

$$\Omega_\Sigma = (\Omega \setminus \Sigma) \sqcup SN\Sigma$$

where $SN\Sigma$ is the sphere normal bundle of Σ. We have the canonical projection $\pi : \Omega_\Sigma \longrightarrow \Omega$ and remark that $\pi^{-1}\Sigma$ is a submanifold in Ω_Σ of codimention 1. Take the local coordinates $x = (x_a, x_b)$ such that Σ is defined by $x_a = 0$. Recall that for $\overline{p} \in SN\Sigma$ we can choose as a chart near $\overline{p}$, for example,

$$\phi(p) = (x_b, x_a, \rho), \rho = |x_a|, \omega_a = x_a/\rho \in S^{k-1} \subset \mathbb{R}^k \text{ if } p \notin SN\Sigma,$$
$$\phi(p) = (x_b, dx_a(p)/|dx_a(p)|, 0) \text{ if } p \in SN\Sigma.$$

Let $\pi^* h$ be the pull back of h by π. In our coordinates h and h_Σ are given by

$$h(x_a, x_b) = \sum_{|\alpha|=r} C_\alpha(x_b, x_a) x_a^\alpha, \; h_\Sigma(x_a, x_b) = \sum_{|\alpha|=r} C_\alpha(x_b, 0) x_a^\alpha,$$
$$\pi^* h(x_b, \omega_a, \rho) = \sum_{|\alpha|=r} \rho^r C_\alpha(x_b, \rho\omega_a)\omega_a^\alpha$$

where $C_\alpha(x_b, x_a)$ are C^∞. This shows that $\pi^* h$ vanishes of order r on $\pi^{-1}\Sigma$. Let $\tilde{\rho} \in C^\infty(\Omega_\Sigma)$ be a defining function of $\pi^{-1}\Sigma$, that is $\pi^{-1} = \{\tilde{\rho} = 0\}$. Then it is clear that $h^* = \tilde{\rho}^{-r}\pi^* h$ is in $C^\infty(\Omega_\Sigma)$. Let $x^0 \in \Sigma$, $x^1 \in N_{x^0}\Sigma \setminus 0$ be characteristics of h, h_{x^0} of order r and s respectively. In our coordinates $x^0 = (x_b^0, 0)$, $x^1 = (0, x_a^1)$ and $(x_b^0, x_a^1) \in N\Sigma \setminus \Sigma$. Remark that (x_b^0, x_a^1) is a characteristic of order s of h_Σ because

$$h_{x^0}(x_a) = \sum_{|\alpha|=r} C_\alpha(x_b^0, 0) x_a^\alpha = h_\Sigma(x_b^0, x_a)$$

and hence $h_\Sigma(x_b^0, x_a) = 0$ has the zero $x_1 = 0$ of order s when $x_{a'} = 0$ with $x_a = (x_1, x_{a'})$ and $h_\Sigma(x_b, x_a)$ is hyperbolic with respect to x_1 variable. Note that $(x_b^0, \lambda x_a^1)$, $\lambda \in \mathbb{R} \setminus 0$ is also a characteristic of h_Σ of order s because of the homogeneity with respect to x_a. Since $N\Sigma \setminus \Sigma$ is canonically identified with a subset of Ω_Σ we may consider $X = (x_b^0, \overline{\omega}_a, 0)$, $\overline{\omega}_a = x_a^1/|x_a^1|$ as a point of Ω_Σ.

Lemma 2.6. *Set*

$$\tilde{h}^*(x_b, x_a, \rho) = \sum_{|\alpha|=r} C_\alpha(x_b, \rho x_a)x_a^\alpha.$$

Then we have

$$h_X^*(x_b, \omega_a, \rho) = c\tilde{h}_X^*(x_b, \omega_a, \rho), \ (x_b, \omega_a, \rho) \in T_X\Omega_\Sigma$$

where $c = (\tilde{\rho}^{-1}\rho)(X)^r \neq 0$.

Proof: Recall that $\tilde{h}^*(x_b^0 + x_b, \overline{\omega}_a + x_a, \rho) = 0$ has the zero $x_1 = 0$ of order s precisely when $(x_a, x_{a'}, \rho) = (0,0,0)$. Since $\tilde{h}^*(x_b^0 + x_b, \overline{\omega}_a + x_a, \rho)$ is hyperbolic with respect to the x_1 variable we can write

$$\tilde{h}^*(x_b^0 + x_b, \overline{\omega}_a + x_a, \rho) = \tilde{h}_X^*(x_b, x_a, \rho) + O(|x_a| + |x_b| + |\rho|)^{s+1}$$

where $\tilde{h}_X^*(x_b, x_a, \rho)$ is a homogeneous polynomial in (x_b, x_a, ρ) which is hyperbolic with respect to the x_1 variable. Let $\omega_a(\mu) = \overline{\omega}_a + \mu\omega_a + O(\mu^2) \in S^{k-1} \subset \mathbb{R}^k$, $\omega_a \in T_{\overline{\omega}_a}S^{k-1} \subset \mathbb{R}^k$ and observe

$$h^*(x_b^0 + \mu x_b, \omega_a(\mu), \mu\rho) = (\tilde{\rho}^{-1}\rho)^r\tilde{h}^*(x_b^0 + \mu x_b, \omega_a(\mu), \mu\rho)$$

which is equal to $c\mu^s\left(\tilde{h}_X^*(x_b, \omega_a, \rho) + O(\mu)\right)$ and hence the conclusion. □

Let $\Lambda_X(h^*)$ be the lineality of h_X^* which is a linear subspace in $T_X\Omega_\Sigma$. Here note that $\Lambda_X(h^*)$ is independent of the choice of $\tilde{\rho}$, a defining function of $\pi^{-1}\Sigma$, and hence we may write $\Lambda_X(\pi^*h)$ for $\Lambda_X(h^*)$ without ambiguity.

Lemma 2.7. *Assume that* $\Lambda_X(\pi^*h)$ *is transversal to* $T_X(\pi^{-1}\Sigma)$, *that is* $\Lambda_X(\pi^*h) + T_X(\pi^{-1}\Sigma) = T_X\Omega_\Sigma$. *Then we have*

$$h_{\{x^0,x^1\}}(x_b, x_a) = h_{\Sigma(x_b^0,x_a^1)}(x_b + |x_a^1|\tilde{x}_b, x_a + |x_a^1|^2\tilde{x}_a)$$

with some fixed $\tilde{x}_b$, $\tilde{x}_a$ *where* $h_{\Sigma(x_b^0,x_a^1)} = h_{\Sigma X}$ *is the localization of* h_Σ *at* (x_b^0, x_a^1).

Proof: We first recall that

$$h(x^0 + \mu(x^1 + x_b) + \mu^2 x_a) = \mu^r\tilde{h}^*(x_b^0 + \mu x_b, x_a^1 + \mu x_a, \mu)$$

which gives that

$$\tilde{h}_X^*(x_b, \omega_a, 1) = h_{\{x_b^0, \overline{\omega}_a\}}(x_b, x_a).$$

Noting the following

$$h_{\{x^0,x^1\}}(x_b, x_a + \lambda x_a^1) = h_{\{x^0,x^1\}}(x_b, x_a), \ \forall\lambda \in \mathbb{R},$$
$$h_{\{x^0,x^1\}}(x_b, x_a) = \lambda^{r+s}h_{\{x^0,x^1/\lambda\}}(x_b/\lambda, x_a/\lambda^2), \ \forall\lambda \in \mathbb{R}\setminus 0$$

we easily see that

$$h_{\{x^0,x^1\}}(x_b,x_a) = |x_a^1|^{r+s} h_{\{x_b^0,\overline{\omega}_a\}}(x_b/|x_a^1|, \hat{x}_a^1/|x_a^1|^2)$$

where $x_a = cx_a^1 + \hat{x}_a$, $\hat{x}_a \in T_{\overline{\omega}_a} S^{k-1} \subset \mathbb{R}^k$ and $c \in \mathbb{R}$. Thus we obtain

$$h_{\{x^0,x^1\}}(x_b,x_a) = |x_a^1|^{r+s} \tilde{h}_X^*(x_b/|x_a^1|, \hat{x}_a^1/|x_a^1|^2, 1). \tag{2.8}$$

The same argument with $\rho = 0$ shows that

$$h_{\Sigma(x_b^0,x_a^1)}(x_b,x_a) = |x_a^1|^{r+s} \tilde{h}_X^*(x_b/|x_a^1|, \hat{x}_a^1/|x_a^1|^2, 0). \tag{2.9}$$

From hypotheses there is $(x_b', \omega_a', \rho') \in T_X\Omega_\Sigma$ with $\rho' \neq 0$ such that

$$h_X^*((x_b,\omega_a,\rho) + t(x_b',\omega_a',\rho')) = h_X^*(x_b,\omega_a,\rho)$$

for $\forall(x_b,\omega_a,\rho) \in T_X\Omega_\Sigma$ and $t \in \mathbb{R}$. Taking $\rho = 1$, $t = -1/\rho'$ we get

$$h_X^*(x_b,\omega_a,0) = h_X^*(x_b + x_b'/\rho', \omega_a + \omega_a'/\rho', 1). \tag{2.10}$$

Now it is clear that

$$h_{\{x^0,x^1\}}(x_b,x_a) = h_{\Sigma(x_b^0,x_a^1)}(x_b + |x_a^1|\tilde{x}_b, x_a + |x_a^1|^2\tilde{x}_a)$$

with $\tilde{x}_b = -x_b'/\rho'$, $\tilde{x}_a = -\omega_a'/\rho'$. This is the desired assertion. □

Let β be the canonical projection $\beta : N\Sigma \longrightarrow \Sigma$ and denote by $d\beta$ the differential of β;

$$d\beta_X : T_X N\Sigma \longrightarrow T_{x^0}\Sigma.$$

Lemma 2.8. *Assume that $\Lambda_X(\pi^*h)$, $T_X(\pi^{-1}\Sigma)$ are transversal to $\Lambda_X(h_\Sigma)$, Ker $d\beta_X$ respectively. Then there is a polynomial Q on $N_{x^0}\Sigma$ such that*

$$h_{\{x^0,x^1\}}(x_b,x_a) = Q(x_a + \tilde{x}_a)$$

with a fixed $\tilde{x}_a$. In particular $h_{\{x^0,x^1\}}(x_b,x_a)$ is independent of x_b.

Proof: In our coordinates β is given by $\beta : (x_b,x_a) \longrightarrow (x_b,0)$ and hence Ker $d\beta_X$ $= \{(0,x_a) | x_a \in N_{x^0}\Sigma\}$. From hypotheses it follows that $\Lambda_X(h_\Sigma)$ contains the set $\{(x_b,0) | x_b \in T_{x^0}\Sigma\}$. Then we see that

$$h_{\Sigma(x_b^0,x_a^1)}(x_b + |x_a^1|\tilde{x}_b, x_a + |x_a^1|^2\tilde{x}_a) = h_{\Sigma(x_b^0,x_a^1)}(0, x_a + |x_a^1|^2\tilde{x}_a)$$

which proves the assertion noting that $h_{\Sigma(x_b^0,x_a^1)}(0,x_a)$ is a well defined polynomial on $N_{x^0}\Sigma$. □

3. Localizations of system

Let Ω be an open set in $\mathbb{R}^{n+1}$ with local coordinates $x = (x_0, x')$ where $x' = (x_1, ..., x_n)$ and let $T^*\Omega$ be the cotangent bundle over Ω with corresponding coordinates (x, ξ). Let L be a first order differential operator on $C^\infty(\Omega, \mathbb{C}^n)$ with symbol $L(x, \xi) \in C^\infty(T^*\Omega, \mathrm{Hom}\,(\mathbb{C}^m, \mathbb{C}^m))$. We denote by $h(x, \xi)$ the determinant of $L(x, \xi)$. Following Vaillant [9] (see also [1]) we define the localization of $L(x, \xi)$ at a characteristic $z^0 = (x^0, \xi^0) \in T^*\Omega \setminus 0$ of order r of h with

$$\dim \mathrm{Ker} L(z^0) = r.$$

Let π be the natural projection $\pi : \mathbb{C}^m \mapsto \mathbb{C}^m/\mathrm{Im}L(z^0)$ and ι be the inclusion $\iota : \mathrm{Ker}L(z^0) \mapsto \mathbb{C}^m$.

DEFINITION 3.1. We define $L_{z^0}(z)$ by

$$L_{z^0}(z) = \lim_{\mu \to 0} \mu^{-1} \pi L(z^0 + \mu z)\iota, \quad z \in T_{z^0}(T^*\Omega).$$

Taking bases for $\mathbb{C}^m$ and then for $\mathrm{Ker}L(z^0)$, $\mathrm{Ker}\,{}^tL(z^0)$, where ${}^tL(z^0)$ denotes the transposed matrix of $L(z^0)$, we examine the definition. We choose $u_j, v_j \in C^m$ so that

$$\mathrm{Ker}L(z^0) = \mathrm{span} - \{u_1, ..., u_r\}, \ \mathrm{Ker}\,{}^tL(z^0) = \mathrm{span} - \{v_1, ..., v_r\}.$$

With $U = (u_1, ..., u_r)$, $V = (v_1, ..., v_r)$, which are $m \times r$ matrices, we set

$$U_{(U,V)}(z) = {}^t\, V L(z) U.$$

Then in theses bases $L_{z^0}(z)$ is expressed by $L_{(U,V)z^0}(z)$:

$$L_{(U,V)z^0}(z) = \lim_{\mu \to 0} \mu^{-1} L_{(U,V)}(z^0 + \mu z).$$

For another pair of bases $\tilde{U}$, $\tilde{V}$ for $\mathrm{Ker}L(z^0)$, $\mathrm{Ker}\,{}^tL(z^0)$ respectively it is clear that

$$L_{(\tilde{U},\tilde{V})}(z) = M_1 L_{(U,V)}(z) M_2$$

with some non singular $M_i \in M(r, \mathbb{C})$ and hence

$$L_{(\tilde{U},\tilde{V})z^0}(z) = M_1 L_{(U,V)z^0}(z) M_2. \tag{3.1}$$

We next examine the effects of a change of basis for C^m. Let $L^T(z) = T^{-1}L(z)T$ with a non singular $T \in M(m, \mathbb{C})$ and let U_1, V_1 be a pair of bases for $\mathrm{Ker}L^T(z^0)$, $\mathrm{Ker}\,{}^tL^T(z)$. Then it is also clear that

$$L^T_{(U_1,V_1)z^0}(z) = N_1 L_{(U,V)z^0}(z) N_2 \tag{3.2}$$

with non singular $N_i \in M(r, \mathbb{C})$. From (3.1) the determinant of $L_{z^0}(z)$ is well defined up to non zero multiple constant.

Lemma 3.1. *We have*

$$(\det h)_{z^0}(z) = \det L_{z^0}(z)$$

up to non zero multiple constant.

Proof: As noted above it is enough to show the assertion with suitably chosen bases U, V for $\mathrm{Ker}L(z^0)$, $\mathrm{Ker}{}^tL(z^0)$ and a basis for $\mathbb{C}^m$. After a change of basis for $\mathbb{C}^m$ we may assume that

$$L(z^0) = G \oplus O$$

where $G \in M(m-r, \mathbb{C})$ is non singular and O denotes the zero matrix of order r. Write

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

where $L_{ij}(z^0) = O$ unless $(i,j) = (1,1)$ and $L_{11}(z^0) = G$. Thus choosing U, V suitably we have

$$L_{(U,V)}(z) = L_{22}(z).$$

Since $L_{11}(z^0 + \mu z) = G + O(\mu)$, $L_{ij}(z^0 + \mu z) = \mu L'_{ij}(z) + O(\mu^2)$ as $\mu \to 0$ we see that

$$\det L(z^0 + \mu z) = \mu^r\{(\det G)\det L'_{22}(z) + O(\mu)\}$$

and hence

$$(\det L)_{z^0}(z) = (\det G)\det L'_{22}(z).$$

On the other hand, by definition, we have

$$L_{(U,V)z^0}(z) = L'_{22}(z)$$

and hence the assertion. □

From (3.1) it is clear that every s-th minor of $L_{(\tilde{U},\tilde{V})z^0}(z)$ is a linear combination of s-th minors of $L_{(U,V)z^0}(z)$ and *vice versa*.

Lemma 3.2. *Every $(r-1)$-th minor of $L_{z^0}(z)$ is a linear combination of $m_{z^0}(z)$'s where $m(z)$ are $(m-1)$-th minor of $L(z)$.*

Proof: It is enough to show the assertion for $L_{(U,V)z^0}(z)$ with suitably chosen U, V and a basis for $\mathbb{C}^m$. As observed in the proof of Lemma 3.1 we may assume that

$$L(z^0 + \mu z) = \begin{pmatrix} G + O(\mu) & O(\mu) \\ O(\mu) & \mu L'_{22}(z) + O(\mu^2) \end{pmatrix}.$$

Let $m(z)$ be the $(m-1)$-th minor of $L(z)$ obtained removing i-th row and j-th colomn of $L(z)$. Similarly we denote by $l(z)$ the thus obtained $(r-1)$-th minor of $L'_{22}(z)$. Then it is clear that

$$m_{z^0}(z) = \mu^{r-1}\{(\det G)l(z) + O(\mu)\}$$

as $\mu \to 0$ and hence $l(z) = (\det G)^{-1} m_{z^0}(z)$ which proves the assertion. □

Recall that $L_{z^0}(z)$ is $\mathrm{Hom}(\mathrm{Ker}L(z^0), \mathbb{C}^m/\mathrm{Im}L(z^0))$ valued linear function in z.

DEFINITION 3.2. Let $L_{z^0}(z) = (\phi^i_j(z))$. We call

$$d(L_{z^0}) = \dim \text{span} - \{\phi^i_j\}$$

the reduced dimension of L_{z^0}.

DEFINITION 3.3. Assume that $L(z)$ is real. Let z^0 be a characteristic of order r of h with $\dim \mathrm{Ker}L(z^0) = r$. We say that z^0 is non degenerate if

$$d(L_{z^0}) \geq r(r+1)/2.$$

4. Necessary conditions (I)

Let

$$L(x, D) = \sum_{j=0}^{n} A_j(x) D_j$$

be a differential operator of order 1 on $C^\infty(\Omega, \mathbb{C}^m)$. We assume that $h(x, \xi)$ is hyperbolic with respect to $t(x) \in C^\infty(\Omega)$, $dt(x) \neq 0$, that is

$$h(x, \xi + \lambda dt(x)) = 0$$

has only real roots for every $x \in \Omega$, $\xi \in T^*_x\Omega$. Let $\sigma = \sum_{j=0}^{n} d\xi_j \wedge dx_j$ be the canonical symplectic two form on $T^*\Omega$ and for $S \subset T_w(T^*\Omega)$ we denote by S^σ the anihilator of S with respect to σ:

$$S^\sigma = \{z \in T_w(T^*\Omega) | \sigma(z, u) = 0, \forall u \in S\}.$$

In what follows we assume that $t(x) = x_0$ and $A_0 = I_m$, the identity matrix of order m without restrictions. Recall that we say that L is strongly hyperbolic near the origin if the Cauchy problem for $L(x, D) + B(x)$ is correctly posed for every $B(x) \in C^\infty(\Omega, M(m, \mathbb{C}))$ in both Ω^t, Ω_t with small t where $\Omega^t = \{x \in \Omega | x_0 < t\}$ and $\Omega_t = \{x \in \Omega | x_0 > t\}$.

In this section we show the following result.

Theorem 4.1. *Assume that $A_j(x)$ are real analytic in Ω contaning the origin. Let $z^0 \in T^*_0\Omega \setminus 0$, $z^1 \in T_{z^0}(T^*\Omega)$ be characteristics of order r and s of h and $h_{z^0} = \det L_{z^0}$ respectively with $\Lambda_{z^0}(h)^\sigma \subset \Lambda_{z^0}(h)$. If L is strongly hyperbolic near the origin then every $(r-1)$-th minor of L_{z^0} vanishes of order $s-2$ at z^1.*

This result is optimal in a sense. We give an example.

EXAMPLE 5.1: Let

$$L(z) = \begin{pmatrix} \xi_0 & \xi_1 & 0 \\ x_0^2\xi_1 & \xi_0 & 0 \\ 0 & 0 & \xi_0 - 2x_0\xi_1 \end{pmatrix}, \; z^0 = (0, e_n), \; n \geq 2.$$

For this $L(z)$ it is not difficult to examine the followings.

1) L is strongly hyperbolic near the otigin (see Example 1.2. in [6]) and z^0 is a characteristic of order 3 of h with $\Lambda_{z^0}(h)^\sigma \subset \Lambda_{z^0}(h)$.

2)

$$L_{z^0}(z) = \begin{pmatrix} \xi_0 & \xi_1 & 0 \\ 0 & \xi_0 & 0 \\ 0 & 0 & \xi_0 \end{pmatrix}$$

and $z^1 = (0, e_1)$ is a characteristic of $\det L_{z^0}(z)$ of order 3.

3) the 2-minor

$$\begin{vmatrix} \xi_1 & 0 \\ 0 & \xi_0 \end{vmatrix}$$

vanishes of order $1 = 3 - 2$ at z^1.

To show the result we first derive an a priori estimate for well posed Cauchy problem which will be needed in the following sections also. Let $\sigma = (\sigma_0, ..., \sigma_n) \in \mathbb{Q}_+^{n+1}$ and set

$$y(\lambda) = y^0 + \sum_{j=1}^{s} y^j \lambda^{-\epsilon_j}, \; \eta(\lambda) = \eta^0 + \sum_{j=1}^{s} \eta^j \lambda^{\epsilon_j} \tag{4.1}$$

where $y^j, \eta^j \in \mathbb{R}^{n+1}$ and $\epsilon_j \in \mathbb{Q}_+$, $0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_s$. For a differential operator P on $C^\infty(\Omega, \mathbb{C}^m)$ with $C^\infty(\Omega)$ coefficients we set with $\kappa \in \mathbb{Q}_+$

$$P_\lambda(y(\lambda), \eta(\lambda); x, \xi) = P(y(\lambda) + \lambda^{-\sigma}x, \lambda^\kappa \eta(\lambda) + \lambda^\sigma \xi) \tag{4.2}$$

where $\lambda^{-\sigma} = (\lambda^{-\sigma_0}x_0, ..., \lambda^{-\sigma_n}x_n)$ etc. Assuming that the Cauchy problem for $P(x, D)$ is correctly posed in both Ω^t and Ω_t for every small t we derive an a priori estimate for $P_\lambda(x, D)$.

Proposition 4.2. *Let $\sigma \in \mathbb{Q}_+^{n+1}$ and $\kappa, \epsilon_j \in \mathbb{Q}_+$. Assume that $0 \in \Omega$, $y^0 = 0$ and the Cauchy problem for $P(x, D)$ is correctly posed in both Ω^t and Ω_t for every small t. Then for every compact set $\tilde{Y}$, $\tilde{H} \subset \mathbb{R}^{(n+1)s}$, $W \subset \mathbb{R}^{n+1}$ and for every positive $T > 0$ we can find $C > 0$, $\overline{\lambda} > 0$ and $p \in \mathbb{N}$ such that*

$$|u|_{C^0(W^t)} \leq C\lambda^{(\overline{\sigma}+\kappa)p} |P_\lambda u|_{C^p(W^t)},$$
$$|u|_{C^0(W_t)} \leq C\lambda^{(\overline{\sigma}+\kappa)p} |P_\lambda u|_{C^p(W_t)}$$

for $\lambda \geq \overline{\lambda}$, $u \in (C_0^\infty(W))^m$, $|t| < T$, $Y = (y^1, ..., y^s) \in \tilde{Y}$, $H = (\eta^1, ..., \eta^s) \in \tilde{H}$ *where* $\overline{\sigma} = \max_j \sigma_j$.

Proof: Set

$$\tilde{P}(x, D) = e^{-i\lambda^\kappa <\eta(\lambda), x>} P(x, D) e^{i\lambda^\kappa <\eta(\lambda), x>}$$

so that $\tilde{P}(x, \xi) = P(x, \lambda^\kappa \eta(\lambda) + \xi)$. Let $K \subset \Omega$ be a compact set and recall Proposition 2.1 in [7]:

$$|v|_{C^0(K^t)} \leq C|Pv|_{C^p(K^t)}, \ |t| < \tau, \ v \in (C_0^\infty(K))^m$$

with an integer $p \in \mathbb{N}$ and a $\tau > 0$. Since

$$|u|_{C^0(K^t)} \leq C_1 \lambda^{\kappa p} |e^{-i\lambda^\kappa <\eta(\lambda), x>} v|_{C^p(K^t)}$$

it follows that

$$|u|_{C^0(K^t)} \leq C_2 \lambda^{\kappa p} |\tilde{P}(x, D) u|_{C^p(K^t)}.$$

Repeating the proof of Proposition 2.2 in [7] we get the desired assertion. □

5. Necessary conditions (II)

Let

$$\Sigma = \{z \in T^*\Omega | d^j h(z) = 0, j < r, d^r h(z) \neq 0\}$$

be the set of characteristics of order r of h. We assume that Σ is an involutive C^∞ manifold through z^0. Denote by p the canonical projection from $T^*\Omega$ onto Ω: $T^*\Omega \mapsto \Omega$ and assume that

(5.1) $$dp_{z^0} : T_{z^0}(T^*\Omega) \mapsto T_{p(z^0)}\Omega \text{ is surjective on } T_{z^0}\Sigma.$$

Let $z^1 \in N_{z^0}\Sigma \setminus 0$ be a multiple characteristic of h_{z^0}. As in section 2 we denote by $(T^*\Omega)_\Sigma$ the blow up of $T^*\Omega$ along Σ and by π the canonical projection from $(T^*\Omega)_\Sigma$ onto $T^*\Omega$. We assume that

(5.2) $$\Lambda_X(\pi^* h) \text{ is transversal to } T_X(\pi^{-1}\Sigma)$$

where $X = (z^0, z^1) \in N\Sigma \setminus \Sigma$ is considered canonically as a point in $(T^*\Omega)_\Sigma$. We also denote by β the projection from $N\Sigma$ onto Σ off the fibers. Then we have

Theorem 5.1. *Assume that $A_j(x)$ are real analytic in Ω which contains the origin and L is strongly hyperbolic near the origin. Let Σ be the characteristic set of order r which is assumed to be an involutive C^∞ manifold containing $z^0 \in T_0^*\Omega \setminus 0$. Let $z^1 \in N_{z^0}\Sigma \setminus 0$ be a characteristic of order s of $h_{z^0} = \det L_{z^0}$ and with $X = (z^0, z^1)$ we assume (5.1) , (5.2) and that*

(5.3) $$\Lambda_X(h_\Sigma) \text{ is transversal to } \mathrm{Ker} d\beta_X.$$

Then every $(r-1)$-th minor of the localization L_{z^0} vanishes of order $(s-1)$ at z^1. In particular we have

$$\dim \mathrm{Ker} L_{z^0}(z^1) = s.$$

Recall that on $N\Sigma$ we have an invariant two form, denoted by $\tilde{\sigma}$ and called the relative symplectic two form (see [4], [8]) which is given by

$$\tilde{\sigma} = \sum_{j=0}^{k} dx_j^* \wedge dx_j = dx_a^* \wedge dx_a$$

where we have assumed that Σ is defined by $\xi_a = 0$ and $N\Sigma$ is parametrized by $(x_a, x_b, \xi_b; x_a^*)$. We now assume that

$$\Lambda_X(h_\Sigma)^{\tilde{\sigma}} \subset \Lambda_X(h_\Sigma) \tag{5.4}$$

where $\Lambda_X(h_\Sigma)^{\tilde{\sigma}}$ denotes the anihilator of $\Lambda_X(h_\Sigma)$ with respect to the relative symplectic two form $\tilde{\sigma}$. Then

Theorem 5.2. *Replacing (5.3) by (5.4) in Theorem 5.1 we get the same conclusion as in Theorem 5.1.*

We denote by ρ the radial vector field on $T^*\Omega$ and recall that $\Lambda_{(z^0,z^1)}(h)$ is the lineality of $h_{(z^0,z^1)}(h)$ and $\Lambda_{z^0}(h) \subset \Lambda_{(z^0,z^1)}(h)$ hence

$$\Lambda_{(z^0,z^1)}(h)^\sigma \subset \Lambda_{z^0}(h)^\sigma.$$

Proposition 5.3. *Assume that $A_j(x)$ are real analytic in Ω which contains the origin and L is strongly hyperbolic near the origin. Let $z^0 \in T_0^*\Omega\backslash 0$, $z^1 \in T_{z^0}(T^*\Omega)$ be characteristics of order r and s of h and $h_{z^0} = \det L_{z^0}$ respectively with*

$$\rho(z^0) \notin \Lambda_{z^0}(h)^\sigma \subset \Lambda_{z^0}(h). \tag{5.5}$$

Assume that we can find local coordinates x near the origin with $t(x) = x_0$ such that

$$h_{\{z^0,z^1\}}(v,z) = h_{\{z^0,z^1\}}(0,z), \quad \forall v \in \Lambda_{(z^0,z^1)}(h)^\sigma. \tag{5.6}$$

Then every $(r-1)$-th minor of L_{z^0} vanishes of order $s-1$ at z^1. In particular we have

$$\dim \mathrm{Ker} L_{z^0}(z^1) = s.$$

Proposition 5.4. *Assume that $A_j(x)$ are real analytic in Ω which contains the origin and L is strongly hyperbolic near the origin. Let $z^0 \in T_0^*\Omega \setminus 0$ be a non degenerate characteristic of h of order r with (5.5). Assume that for every multiple characteristic $z^1 \in T_{z^0}(T^*\Omega)$ of $h_{z^0} = \det L_{z^0}$ we can find local coordinates x with $t(x) = x_0$ verifying (5.6). Then $L_{z^0}(z)$ is symmetrizable by a non singular constant matrix T;*

$$T^{-1} L_{z^0}(z) T$$

is symmetic for every z. In particular $L_{z^0}(z)$ is strongly hyperbolic.

This result clearly generalizes Theorem 1 in [9].

Corollary 5.5. *Assume that $A_j(x)$ are real analytic in Ω which contains the origin and L is strongly hyperbolic near the origin. Let $z^0 \in T_0^*\Omega \setminus 0$, $z^1 \in T_{z^0}(T^*\Omega)$ be a characteristic of order r of h with (5.5). Assume that for every $z^1 \in \Lambda_{z^0}(h)$ we can find local coordinates x near the origin with $t(x) = x_0$ verifying (5.6). Then we have*

$$\Lambda(\det L_{z^0}) \subset \Lambda(m)$$

for every $(r-1)$-th minor of $L_{z^0}(z)$.

Proof: We first note that $\Lambda(\det L_{z^0}) = \Lambda_{z^0}(h)$ by Lemma 3.1. Let m be a $(r-1)$-th minor of L_{z^0} and let $z^1 \in \Lambda(\det L_{z^0})$. Since $z^1 \in \Lambda_{z^0}(h)$ and hence is a characteristic of order r of h_{z^0} it follows from Proposition 5.3 that

$$d^j m(z^1) = 0, \; j < r-1.$$

On the other hand since $m(z)$ is homogeneous of degree $r-1$ in z it is clear that

$$m(z^1 + z) = m(z)$$

which proves $z^1 \in \Lambda(m)$. □

Corollary 5.6. *Under the same assumptions as in Corollary 5.5 we have*

the reduced dimension of L_{z^0} = the reduced dimension of $\det L_{z^0}$.

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