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A link between hypoellipticity and solvability in the Cauchy problem

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§1 Introduction and Statement of result

Let $t \in \mathbf{R}$ be a time variable and $x \in \mathbf{R}^{n+1}$ be a space variable. We consider a system $\{X_1, \ldots, X_{2n}\}$ of $2n$ vector fields on $\mathbf{R}^{n+1}$, given by

$$X_i = D_i = (\sqrt{-1})^{-1}\partial/\partial x_i,\ 1 \leq i \leq n \quad X_{1+n} = x_i D_{n+1},\ 1 \leq i \leq n.$$

The operators with which we are concerned are of the following form

$$P(D_t, X) = D_t^m + \textstyle\sum_{j=0}^{m-1} a_j(X) D_t^j,$$
$$a_j(X) = \textstyle\sum_{|I|=m-j} a_{j,I} X^I,$$

where for $I = (i_1, \ldots, i_p) \in \{1, \ldots, 2n\}^p$,

$$X^I = X_{i_1} \cdots X_{i_p} \text{ and } a_{j,I} \in C.$$

There is a relation between the system of these vector fields and the Lie algebra of Heisenberg group via Schrödinger representation. In this note, we do not enter into this group theoretical aspect for simplicity.

We say that P is strictly hyperbolic of nondegenerate type in generating directions if for any $\Xi \in \mathbf{R}^{2n}\backslash\{0\}$, the roots ζ_j of polynomial in ζ

$$\zeta^m + \sum_{j=0}^{m-1} a_j(\Xi)\zeta^j = 0, \quad a_j(\Xi) = \sum_{|I|=m-j} a_{j,I}\Xi^I$$

are real distinct and moreover $\zeta_j(\Xi) \neq 0$ or $\zeta_j(\Xi) \equiv 0$.

Now, we consider an extra variable $s \in \mathbf{R}$ and introduce the following operator on $\mathbf{R}^2_{t,s} \times \mathbf{R}^{n+1}_x$,

$$\tilde{P} = P(D_t + iD_s, X).$$

Let Γ be the subset consisting of the points $\rho = (0, 0, 0; \sigma, \tau, \xi)$ of the cotangent space $T^*(\mathbf{R}_t \times \mathbf{R}_s \times \mathbf{R}^{n+1}_x)$ such that

$$\sigma < 0,\ (\tau, \xi) \in \mathbf{R}^{n+2}.$$

Then we can state the main result.

Theorem 1 *Suppose that P is strictly hyperbolic of nondegenerate type in generating directions. Then, the Cauchy problem for P is well posed at the origin if and only if $\tilde{P}$ is micro-hypoelliptic at any point $\rho \in \Gamma$.*

Here, the Cauchy problem for P to be well-posed means that there exist a neighborhood $U \subset \mathbf{R}^{n+1}$ of the origin and a positive number T such that for any $f \in C_0^\infty((-T,T) \times U)$ and $g_j \in C_0^\infty(U)$, the Cauchy problem

$$\begin{cases} Pu & = f \text{ in } (0,T) \times U \\ \partial_t^j u|_{t=0} & = g_j(x) \text{ on } U,\ 0 \le j \le m-1 \end{cases}$$

has a solution $u(x,t) \in C^m((-T,T) \times U)$.

Corollary *Under the same condition as theorem 1, if for any non-real complex number z, $P(zD_t, X)$ is hypoelliptic at the origin, then the Cauchy problem for P is well-posed at the origin.*

In this note, we only consider the sufficient part of our theorem. We note that it is well-known that

Proposition *If $\tilde{P}$ is micro-hypoelliptic at the point $\rho = (0,0,0;\sigma,\tau,\xi)$ of the cotangent space $T^*(\mathbf{R}_t \times \mathbf{R}_s \times \mathbf{R}_x^{n+1})$, the operator $\hat{P} = P(\tau + i\sigma, \hat{X})$ is injective on the space $\mathcal{S}(\mathbf{R}^n)$ for $\lambda = \pm 1$, where $\hat{X}_i = X_i$ if $1 \le i \le n$ and $= x_{i-n}\lambda$ if $1 \le i \le n$.*

Hence it suffices to prove that the injectivity of the operator $\hat{P}|_{\lambda=\pm1}$ at the every point $(0,0,\tau,\sigma)$ with $\sigma < 0$ implies the well-posedness. We conclude this section with a example.

Example 1

$$P = D_t^2 - (D_1^2 + x_1^2 D_2^2 + \alpha D_2),$$

where $\alpha \in \mathbf{C}$. Since Hermite operator on $\mathbf{R}$

$$(-(d/dx)^2 + x^2)$$

has a eigenvalue 1 with eigenfunction $\exp(-x^2/2)$, a simple calculation can show that the ordinary differential equation

$$\{(\tau + i\sigma)^2 - (D_1^2 + x_1^2 + \varepsilon\alpha)\}u(x_1) = 0$$

has no non-trivial solution in $\mathcal{S}(\mathbf{R}^n)$ for any $\sigma < 0,\ \varepsilon = \pm 1,\ \tau \in \mathbf{R}$ if and only if

$$\alpha \in \mathbf{R} \text{ and } |\alpha| \leq 1.$$

This is a necessary and sufficient condition for the Cauchy problem for P to be well-posed. Ivrii-Petkov-Hörmander have shown the similar result for more general operators with double characteristics using Melin's inequality. It seems that their method can not be applied to the higher order case. One of disadvantages of our method is that it is difficult to state our condition more explicitly. In the final section, we shall give some remarks on this problem.

§2 Outline of proof

It suffice to consider the Cauchy problem with zero Cauchy data. Then taking Laplace transform with respect to the time variable, we have the operator P_ζ given by

$$\zeta^m + \sum a_j(X)\zeta^j.$$

If we can show that for some appropriate space W_j, there are some positive constants C, p and R such that P_ζ is a invertible operator from W_1 to W_2 with its norm less than $C(1+|\zeta|)^p$ for $\Im\zeta < -R$, then the solution is given by the inverse Laplace transform

$$u = (2\pi i)^{-1} \int_{\Im\zeta=-R} e^{i\zeta t} P_\zeta^{-1} \tilde{f} d\zeta,$$

where $\tilde{f}$ is the Laplace transform of the right hand side of the equation in the Cauchy problem.

The important property of P is its quasi-homogeneity, which means

$$P_\zeta(\hat{X}) = |\lambda|^{m/2} P_{\zeta/|\lambda|^{1/2}}(X^\pm),$$

where $X^\pm = \hat{X}|_{\lambda=\pm 1}$. We may consider the operator

$$\mathcal{P}_{\zeta,\pm} = \zeta^m + \sum a_j(X^\pm)\zeta^j$$

in the domain $\{\zeta \in \mathbf{C}; \Im\zeta < 0\}$.

We shall use a calculus of a class of ψ d.op. introduced by Shubin ([S]):

$$A^m = \{a(x,\xi,y) \in C^\infty(\mathbf{R}^{3n}); |D_x^\alpha D_\xi^\beta D_y^\gamma a| \leq C_{\alpha\beta\gamma}\langle x,\xi,y\rangle^{m-|\alpha+\beta+\gamma|}\},$$

where $\langle z\rangle = (1+|z|^2)^{1/2}$. The Hilbert spaces attached to this calculus are the following.

$$B^s = \{u \in \mathcal{S}'(\mathbf{R}^n); (D_x^2 + |x|^2)^{s/2} u \in L^2(\mathbf{R}^n)\}$$

and

$$B_\lambda^s = \{u \in \mathcal{S}'(\mathbf{R}^n); (D_x^2 + |x|^2\lambda^2)^{s/2} u \in L^2(\mathbf{R}^n\}.$$

We denote the natural norms by $\|\cdot\|_s$, and $\|\cdot\|_{s,\lambda}$, respectively.

Our claim is the following.

Claim: *There are positive integer p such that for any $s \in \mathbf{R}$ we have*

$$\|u\|_{m-1+s} \leq C_s(1 + |\Im\zeta|^{-p})\|\mathcal{P}_{\zeta,\pm}u\|_s$$

for $\Im\zeta < 0$ and $u \in \mathcal{S}$, and the same estimate for the operator $\mathcal{P}_{\zeta,\pm}^*$ holds.

We may consider the case for $\mathcal{P}_{\zeta,\pm}$ because $\mathcal{P}_{\zeta,\pm}$ is a Fredholm operator with index 0 as proved later. The proof of this claim consists of three parts.

Let $R > 0, r > 0$. We divide the domain $\{\Im\zeta < 0\}$ into three subdomain:

$D_1 = \{\Im\zeta < -R\}, D_2 = \{-R \le \Im\zeta < 0,\ |\Re\zeta| > r\}, D_3 = \{-R \le \Im\zeta < 0,\ |\Re\zeta| \le r\}$.

We can view $\mathcal{P}_{\zeta,\pm}$ as a ψd.op. with parameter ζ. By the strictly hyperbolicity in generating directions, we can see that the principal symbol of this ψd.op. satisfy

$$|\sigma_0(\mathcal{P}_{\zeta,\pm})| \ge C|\Im\zeta|\langle\zeta,\xi\rangle^{m-1}.$$

Therefore, in the first domain D_1, the existence of the parametrix shows that for some large R, the following estimate holds

$$\|u\|_{m-1+s} \le C\|\mathcal{P}_{\zeta,\pm}u\|_s$$

if $\zeta \in D_1$.

In the second domain D_2,by making use of Fourier integral operator attached to the class A^m (c.f. [H]), we can construct a good quasi-inverse of $\mathcal{P}_{\zeta,\pm}$ to obtain

$$\sum_{j=0}^{m-1}\int_0^{T'} \|D_t^j u(t)\|_{m-1-j}^2 dt \le C\{\int_0^{T'} (\|Lu(t)\|_0^2 + \|u(t)\|_0^2)dt\}$$

for any $u \in C_0^\infty((0,T);\mathcal{S}(\mathbf{R}^n))$. Let ρ be a non-negative function on $(0,T)$ with compact support such that

$$\int_0^T \rho(t)dt = 1.$$

Putting

$$u(t) = \rho(t)e^{i\zeta t}v(x)$$

into the above estimate, we can find a large enough r such that if $\zeta \in D_2$, then

$$\|v\|_{m-1+s} \le C\|\mathcal{P}_{\zeta,\pm}v\|_s$$

for any $v \in \mathcal{S}$.

Finally in the third domain D_3, we shall use an elliptic argument. Since the inverse $\mathcal{P}^{-1}$ of the operator $A = \mathcal{P}_{0,\pm}$ is a compact operator and for each ζ, we can write

$$\mathcal{P}_{\zeta,\pm} = A(I + \text{ a compact op.}),$$

$\mathcal{P}_{\zeta,\pm}$ is a Fredholm operator on B^s with index 0. Hence, we know that the inverse of $\mathcal{P}_{\zeta,\pm}$ is a meromorphic function with valued in the space of bounded operator on B^s in $\zeta \in \mathbf{C}$. Then our injectivity assumption imply that its poles are only in the upper half space $\{\Im\zeta \geq 0\}$. Since the number of poles in the $\overline{D_3} \cap \mathbf{R}$ is finite, for $\zeta \in D_3$, we have

$$\|u\|_{m-1+s} \leq C|\Im\zeta|^{-p}\|\mathcal{P}_{\zeta,\pm}u\|_s.$$

Here p represents the maximum of the order of the finitely many poles. Since if $u \in L^2$ and $\mathcal{P}_{\zeta,\pm}u = 0$, then $u \in \mathcal{S}$ for any ζ, we can coclude that p can be chosen independently of s. This completes the proof of our claim.

Returning to the operator $\mathcal{P}_\zeta(\hat{X})$, we see that for some $q > 0$,

$$\|u\|_{m-1+s,\lambda} \leq C\langle\lambda\rangle^q\|\mathcal{P}_\zeta(\hat{X})u\|_s,$$

$$\|u\|_{m-1+s,\lambda} \leq C\langle\lambda\rangle^q\|\mathcal{P}_\zeta^*(\hat{X})u\|_s.$$

It is not difficult to see that these estimate and a standard argument with slightly modified imply our assertion.

§3 Remarks

Our injectivity condition is satisfied for what operator ?. This is a interesting but difficult problem. Little operator which satified it are known. One of them is the following operator

$$P = \{D_t^2 - a(D_1^2 + D_2^2)\}\{D_t^2 - b(D_1^2 + D_2^2)\} + cD_3^2,$$

where a and b are positive constants and c is a real constant. The Cauchy problem for P is well-posed if and only if

$$-ab \leq c \leq \frac{1}{4}(a-b)^2.$$

For general operators, we can show the following negative result.

Theorem 2 *Let P be a operator in §1. Suppose that m is even and*

$$a_0(\Xi) \neq 0$$

for any $\Xi \in \mathbf{R}^{2n}\backslash 0$. Then, there exists a non-empty open set U of $\mathbf{C}$ such that the Cauchy problem for the operator

$$P + zD_{n+1}^{m/2}$$

is not well-posed at the origin if $z \in U$.

Proof: We suppose that for any $z \in \mathbf{C}$, the Cauchy problem for $Q(z) = P + zD_{n+1}^{m/2}$ is well-posed at the origin. Then, it is easy to see that for any z and ζ, $\Im\zeta < 0$,

$$\ker \mathcal{Q}_{\zeta,+}(z) = \{0\}.$$

Let ζ be fixed. From the argument as in the previous section, it follows that the inverse of $\mathcal{Q}_{\zeta,+}(z)$ is a entire function with valued in $\mathcal{L}(L^2)$. Moreover, from the known result on the spectral analysis, we can obtain the asymptotic behavior of the eigenvalues of the operator $\mathcal{Q}_{0,+}$ and we can conclude that the operator $\mathcal{Q}_{\zeta,+}^{-1}$ belongs to , so called, C_p class. (c.f.[DS]) This fact implies that for any $\varepsilon > 0$ and a constant C,

$$\|\mathcal{Q}_{\zeta,+}^{-1}(z)\| \leq Ce^{|z|^{p+\epsilon}} \quad z \in \mathbf{C}.$$

Since the principal symbol of $a_0(X)$ satisfies

$$\sigma_0(x,\xi) \geq \delta\langle x,\xi\rangle^m$$

with $\delta > 0$, $\mathcal{Q}_{\zeta,+}$ has a minimal growth on any ray contained in $\mathbf{C}\backslash\mathbf{R}_+$. Therefore we can apply the princilple of Phragmen-Lindelöf to our operator, we see that $\|\mathcal{Q}_{\zeta,+}^{-1}(z)\|$ is bounded on $\mathbf{C}$. This is a contradiction. Let (ζ_0, z_0) be a point such that

$$\ker \mathcal{Q}_{\zeta_0,+}(z_0) \neq 0.$$

Then for some positive integer p, we have

$$\int_{\Sigma} \mathcal{Q}_{\zeta,+}^{-1}(z)(\zeta - \zeta_0)^p d\zeta \neq 0$$

at $z = z_0$, where $\Sigma = \{|\zeta - \zeta_0| = \varepsilon\}$ is a contour sufficiently near ζ_0 and contained in $\{\Im\zeta < 0\}$. On the other hand, for any z, $\mathcal{Q}_{\zeta,+}(z)^{-1}$ is also a meromorphic function in ζ. Since on $\zeta \in \Sigma$, $\mathcal{Q}_{\zeta,+}(z)$ is uniformly continuous in z near z_0, we see that

$$\int_{\Sigma} \mathcal{Q}_{\zeta,+}^{-1}(z)(\zeta - \zeta_0)^p d\zeta \neq 0$$

if $|z - z_0|$ is sufficiently small. □

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