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Classification of Small Spin Models

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Abstract

Symmetric spin models were introduced by Jones to provide invariants of links. In his paper he proposed to obtain the classification of models (X, w_+, w_-) with $|X| = 4, 5, 6$ and 7. In the present paper we complete this task by showing that the only spin models of these sizes are the Potts models and the ones coming from cyclic groups for $5 \leq n \leq 7$. For $n = 4$ we have some other models by the product construction of de la Harpe. Furthermore, we classify the non-symmetric Jones-type spin models introduced by Munemasa and Watatani, for $n = 4, 5$, as well.

1 Introduction

Symmetric spin models were introduced by Jones[6] to obtain link invariants for non-oriented links. Later Jaeger[5] and de la Harpe[4] developed the connection between these models and association schemes. Recently several constructions were given; see Bannai and Bannai[1], Nomura[8] and Munemasa and Watatani[7]. Even more recently, Bannai and Bannai have found far reaching generalization of the concept[2].

Definition 1.1 *Let a be a non-zero complex number, n be a positive integer, and D be one of its square roots. A spin model with loop variable D and modulus a is a triple (X, w_+, w_-), where X is a finite set of size $n = D^2$ and w_+, w_- are complex-valued functions on $X \times X$ which satisfy the following properties for all α, β, γ in X :*

1. $w_+(\alpha, \alpha) = a,\ w_-(\alpha, \alpha) = a^{-1}$.
2. $\sum_{x \in X} w_+(\alpha, x) = Da^{-1},\ \sum_{x \in X} w_-(\alpha, x) = Da$.
3. $w_+(\alpha, \beta) w_-(\beta, \alpha) = 1$.
4. $\sum_{x \in X} w_+(\alpha, x) w_-(x, \beta) = n\delta_{\alpha,\beta}$ *(where δ is the Kronecker symbol),*
5. $\sum_{x \in X} w_+(\alpha, x) w_+(\beta, x) w_-(\gamma, x) = D w_+(\alpha, \beta) w_-(\beta, \gamma) w_-(\gamma, \alpha)$.

The spin model is called symmetric *if $w_+(\alpha, \beta) = w_+(\beta, \alpha)$, $w_-(\alpha, \beta) = w_-(\beta, \alpha)$. holds, as well.*

The above definition can be reformulated using $n \times n$ matrices W_+ and W_- see [5]. Let $W_\pm = (w_\pm(\alpha, \beta))_{\alpha \in X, \beta \in X}$ and let $\circ$ denote the Hadamard product of matrices (i.e., the entry-wise product of two matrices of the same size). Furthermore, let us define for $(\beta, \gamma) \in X \times X$ the column vector $\mathbb{Y}_{\beta\gamma}$ indexed by X as

$$\mathbb{Y}_{\beta\gamma}(x) = w_+(\beta, x) w_-(\gamma, x) \qquad \forall x \in X.$$

Proposition 1.2 *(X, w_+, w_-) is a spin model with loop variable D and modulus a if and only if the following properties hold:*

1. $\boldsymbol{I} \circ W_+ = a\boldsymbol{I},\ \boldsymbol{I} \circ W_- = a^{-1}\boldsymbol{I}$.

2. $\boldsymbol{J}W_{+} = Da^{-1}\boldsymbol{J}$, $\boldsymbol{J}W_{-} = Da\boldsymbol{J}$.

3. $W_{+} \circ W_{-}^{\top} = n\boldsymbol{J}$.

4. $W_{+}W_{-} = n\boldsymbol{I}$.

5. *For every* $(\beta, \gamma) \in X \times X$, $W_{+}\mathrm{Y}_{\beta\gamma} = DW_{-}(\beta,\gamma)\mathrm{Y}_{\beta\gamma}$.

Here $\boldsymbol{I}$ denotes the identity matrix and $\boldsymbol{J}$ denotes the matrix whose entries are all 1's. ∎

For a symmetric spin model we have to require W_{+} and W_{-} to be symmetric. We shall use an other interpretation, too. Namely, we can write

$$W_{+} = \sum_{i=0}^{t} a_i A_i,$$

where a_0 is the modulus of the model, $A_0 = \boldsymbol{I}$, $a_i \neq a_j$ for $1 \leq i < j \leq t$ and A_i's are adjacency matrices of edge-disjoint simple digraphs on vertex set X. Thus, $A_i \circ A_j = \delta_{ij} A_i$. We denote the graph whose adjacency matrix is A_i by G_i. The above detremined t is called the *degree* of the model (X, W_{+}, W_{-}). Let us mention that the case when $(X, A_i{:}\, 0 \leq i \leq t)$ is an association scheme is interesting for its own sake.

The following is a fundamental result concerning the classification problem.

Lemma 1.3 ([6],Proposition 2.16.) *For each $z \in \mathbb{C}$ let k_z be the number of ordered pairs (α, β) for which $w_{+}(\alpha, \beta) = z$. Then k_z is a multiple of n.* ∎

As an immediate consequence of this lemma we obtain that the degree of a spin model on a n element set is at most $n-1$. However, we shall need a stronger result in order to reduce the number of cases to be checked. This is Theorem 2.1 of the next section.

It is known [1], that there exist spin model coming from the cyclic group C_n for any n. The aim of this paper is to prove the following theorems.

Theorem 1.4 *Let (X, W_+, W_-) be an $n \times n$ symmetric spin model. Furthermore, let us assume that it is of degree $t > 1$, i.e.,*

$$W_+ = a_0 \boldsymbol{I} + a_1 A_1 + \ldots, a_t A_t.$$

Then $(X, \{A_i\}_{0 \leq i \leq t})$ is the (symmetric) association scheme coming from the cyclic group C_n for $5 \leq n \leq 7$. In particular, we have $t = \lfloor \frac{n}{2} \rfloor$. If $n = 4$ the model is either coming from the cyclic group or it is a product of two Potts models on 2 spins.

Theorem 1.5 *Let (X, W_+, W_-) be an $n \times n$ non-symmetric spin model of Jones type. Then it can be written as*

$$W_+ = a_0 \boldsymbol{I} + a_1 A_1 + \ldots, a_n A_n,$$

where $(X, \{A_i\}_{0 \leq i \leq n})$ is the association scheme coming from the cyclic group C_n for $n = 4, 5$.

In Theorem 1.5 we do not state that a_i's are all different. All possible solutions will be given in a subsequent paper by Bannai, Bannai and Jaeger.

In Section 2 we formulate general results and in Section 3 we turn to the symmetric case, while Section 4 deals with non-symmetric models. We will use the three possible interpretations simultaneously, always switching to the one which is most convenient to formulate the statement in question..

We consider two spin models equivalent if one can be obtained from the other by simultaneous permutation of rows and columns (i.e. keeping diagonal elements in the diagonal). In the graph representation this means simultaneous renumbering of the vertices of each graph. Furthermore, we allow renumbering the G_i's when we use the graph interpretation.

2 General results

In this section we prove a strengthening of Lemma 1.3 as follows.

Theorem 2.1 *Let (X, W_+, W_-) be a spin model of degree t. Then each point in G_i has in-degree and out-degree k_i for $i = 1, 2, \ldots, t$ where $k_i \in \mathbb{N}$. In other words, $\boldsymbol{J} A_i = A_i \boldsymbol{J} = k_i \boldsymbol{J}$.*

Proof of Theorem 2.1
Let $X = \{1, 2, \ldots, n\}$ and let $\Delta_\gamma = \mathrm{diag}(w_-(1,\gamma), w_-(2,\gamma), \ldots, w_-(n,\gamma))$ for $\gamma \in X$. We claim that

$$W_+ \, \Delta_\gamma \, W_+ = D \, \Delta_\gamma \, W_+ \, \Delta_\gamma, \quad \forall \gamma \in X. \tag{1}$$

Indeed, the (α, β)-entry of the left hand side of (1) is

$$\sum_{x \in X} w_+(\alpha, x) \, [w_-(x, \gamma) \, w_+(x, \beta)] \tag{2}$$

and the (α, β)-entry of the right hand side of (1) is

$$D \, w_-(\alpha, \gamma) \, w_+(\alpha, \beta) \, w_-(\beta, \gamma). \tag{3}$$

The equality of (3) and (2) is equivalent to (5) of Definition 1.1. Note that Δ_γ is invertible by (3) of Proposition 1.2. Now (1) is equivalent to

$$\Delta_\gamma^{-1} \, W_+ \, \Delta_\gamma \, W_+ \, W_- = D W_+ \, \Delta_\gamma \, W_- \tag{4}$$

i.e.,

$$D \, \Delta_\gamma^{-1} \, W_+ \, \Delta_\gamma = W_+ \, \Delta_\gamma \, W_-. \tag{5}$$

Hence, Δ_γ and $D^{-1} \, W_+$ are conjugate.

It follows that the spectrum of Δ_γ does not depend on the choice of $\gamma \in X$. Equivalently, the columns of W_- are permutations of each other. Using (3) of Proposition 1.2 we obtain the same for W_+.

Using (3JT) of [2] we obtain in a similar way that the rows of W_+ are permutations of each other. Then simple counting shows that the in- and out-degrees of G_i's must coincide. ∎

The following theorem was independently proved by Jaeger [5] and de la Harpe [4], but this proof is simpler.

Theorem 2.2 *Let (X, W_+, W_-) be a symmetric spin model of degree 2. Then G_1 (and consequently G_2) is a strongly regular graph.*

Proof of Theorem 2.2
We have to establish the existence of k, λ and μ, that is we have to prove that G_1 is k-regular and the number of common neighbors of any pair of connected (non-connected) vertices is λ (μ).

The regularity follows from Theorem 2.1. Now we prove the existence of λ and μ. Let α_{ij} be the number of common a_1 entries of rows i and j. Furthemore, suppose that (i,j) and (r,s) are edges of graph G_1. We have to prove that $\alpha_{ij} = \alpha_{rs}$. Using that the (i,j) (resp. (r,s)) entry of $W_+ W_-$ is 0, we obtain

$$a_0 a_1^{-1} + a_0^{-1} a_1 + n - 2k + 2\alpha_{ij} + (k - 1 - \alpha_{ij})(a_1 a_2^{-1} + a_1^{-1} a_2) = 0$$

and

$$a_0 a_1^{-1} + a_0^{-1} a_1 + n - 2k + 2\alpha_{rs} + (k - 1 - \alpha_{rs})(a_1 a_2^{-1} + a_1^{-1} a_2) = 0.$$

Taking the difference of these two equations we obtain

$$(\alpha_{ij} - \alpha_{rs})(2 - a_1 a_2^{-1} - a_1^{-1} a_2) = 0.$$

If $\alpha_{ij} \neq \alpha_{rs}$, then $a_1 a_2^{-1} + a_1^{-1} a_2 = 2$, which implies $a_1 = a_2$, a contradiction. The existence of μ can be proved exactly the same way. ∎

The next theorem is a natural extension of Theorem 2.2.

Theorem 2.3 *Let (X, W_+, W_-) be a symmetric spin model of degree 3. Then $\{A_0, A_1, A_2, A_3\}$ are the adjacency matrices of a symmetric class 3 association scheme.*

Proof of Theorem 2.3

Let $\mathcal{M}$ be the algebra generated by $\{J, W_+\}$ with respect to the ordinary matrix product, and let $\mathcal{H}$ be the algebra generated by $\{I, W_-\}$ with respect to the Hadamard product, as introduced in [5]. Furthermore, let $\mathcal{A}$ be the algebra generated by $\{A_0, A_1, A_2, A_3\}$ with respect to the Hadamard product. Because $A_i \circ A_j = \delta_{ij} A_i$, we have that $\{A_0, A_1, A_2, A_3\}$ is a basis of $\mathcal{A}$. It is clear that $\mathcal{H} \subseteq \mathcal{A}$. Now, I, J, W_+ and W_- are in $\mathcal{H}$ by [5]. The transition matrix that takes $\{A_0, A_1, A_2, A_3\}$ into $\{I, J, W_+, W_-\}$ is of Vandermonde type. It's determinant is non-zero, because t_i's are distinct for $i = 1, 2, 3$. Thus, $\{I, J, W_+, W_-\}$ is also a basis of $\mathcal{A}$, i.e. $\mathcal{H} = \mathcal{A}$. Consequently, $\mathcal{H}$ is of dimension 4. Using that $\mathcal{H} \cong \mathcal{M}$, we obtain that $\mathcal{M}$ is of dimension four, too. However, $\{I, J, W_+, W_-\} \subset \mathcal{M}$ that yields $\mathcal{M} = \mathcal{H}$. Now, applying Proposition 3 of [5], we obtain that $\mathcal{M}$ is the Bose-Mesner algebra of an association scheme. ∎

For non-symmetric spin models we have the following analogous theorem to Theorem 2.3.

Theorem 2.4 *Let (X, W_+, W_-) be a non-symmetric spin model of degree 2. Furthermore, let us assume that $A_1 \circ A_1^\top = 0$. Then $(X, A_0 = \boldsymbol{I}, A_1, A_1^\top, A_2 - A_1^\top)$ is a non-symmetric class 3 association scheme provided $A_1^\top \neq A_2$. If $A_2 = A_1^\top$, then $(X, A_0, A_1, A_1^\top)$ is a non-symmetric class two association scheme.*

Proof of Theorem 2.4
Let us assume first $A_1^\top \neq A_2$. Let $\mathcal{M}$ be the algebra generated by $(\boldsymbol{J}, W_-, W_-^\top)$ with respect to the ordinary matrix product and let $\mathcal{H}$ be generated by $(\boldsymbol{I}, W_+, W_+^\top)$ with respect to Hadamard product. We shall prove that $\mathcal{M} = \mathcal{H}$, which implies by Theorem 2.4 of [3] that $\mathcal{M}$ is the Bose-mesner algebra of a self-dual association scheme.

Let $\mathcal{A}$ be the algebra generated by $(A_0 = \boldsymbol{I}, A_1, A_1^\top, A_2 - A_1^\top)$ with respect to the Hadamard product. It is easy to see that $\dim(\mathcal{A}) = 4$ and that $\mathcal{H} \subseteq \mathcal{A}$. $\boldsymbol{J}$ is clearly in $\mathcal{H}$. We claim, that $\boldsymbol{I}, \boldsymbol{J}, W_+$ and $W_+^\top$ are linearly independent. Indeed, the transition matrix from the basis $(A_0 = \boldsymbol{I}, A_1, A_1^\top, A_2 - A_1^\top)$ to $(\boldsymbol{I}, \boldsymbol{J}, W_+, W_+^\top)$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ t_0 & t_1 & t_2 & t_2 \\ t_0 & t_2 & t_1 & t_2 \end{bmatrix}.$$

Its determinant is $-(t_1 - t_2)^2 \neq 0$. Thus, $\mathcal{H} = \mathcal{A}$. By Theorem 2.3 of [3] we have that $\mathcal{M}$ and $\mathcal{H}$ are isomorphic. Furthermore, $\boldsymbol{I}, \boldsymbol{J}, W_+$ and $W_+^\top$ are all in $\mathcal{M}$, thus $\mathcal{M} = \mathcal{H}$.

The case of $A_2 = A_1^\top$ is similar and left to the reader. ∎

3 Symmetric models

In this section we turn to the classification of small symmetric spin models. It is easy to see that for any n, the only $n \times n$ spin model of degree 1 is the Potts model [6]. So we shall always assume that the degree of the model is at least 2. For the sake of completeness we begin with the case $n = 4$.

3.1 $n = 4$

There can be models of degree 2 and 3 besides the Potts model.

Degree 2 G_1 and G_2 are 1 and 2-regular graphs, respectively by Theorem 2.1. It is obvious that the two-regular graph must be the four-cycle, so we obtained the cyclic group case.

Degree 3 Now G_1, G_2 and G_3 are all perfect matchings. Because two of the matchings together form a 4-cycle, we may assume by renumbering the vertices and the G_i's that W_+ is as follows

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{bmatrix}.$$

Writing $x = a_0/a_1$, $y = a_0/a_2$ and $z = a_0/a_3$ we obtain the following set of three equations from the condition $W_+W_- = n\boldsymbol{I}$:

$$\begin{aligned} x + 1/x + y/z + z/y &= 0 \\ y + 1/y + x/z + z/x &= 0 \\ z + 1/z + x/y + y/x &= 0. \end{aligned}$$

The only solution of this system is that one of the variables is equal to 1 and the other two are negatives of each other. Thus, we may assume that W_+ looks like

$$W_+ = \begin{bmatrix} a & a & b & -b \\ a & a & -b & b \\ b & -b & a & a \\ -b & b & a & a \end{bmatrix}.$$

Now taking into account the various equations coming from the star-triangle equality we obtain that both a and b must be fourth roots of unity. All these cases are covered by the direct product construction of de la Harpe [4].

3.2 $n = 5$

By Theorem 2.1 the model is either of degree 1 or degree 2. Thus, if (X, W_+, W_-) is not the Potts model, then we have that G_1 and G_2 are both 5-cycles so that their union is the complete graph K_5 by Theorem 2.2. In this case W_+ looks like

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_2 & a_1 \\ a_1 & a_0 & a_1 & a_2 & a_2 \\ a_2 & a_1 & a_0 & a_1 & a_2 \\ a_2 & a_2 & a_1 & a_0 & a_1 \\ a_1 & a_2 & a_2 & a_1 & a_0 \end{bmatrix}$$

This is the well-studied case of the pentagon [4, 5].

3.3 $n = 6$

In this case a model could be of degree 2,3,...,5.

3.3.1 Degree 2

By Theorem 2.2 the graph G_1 has to be strongly regular. Furthermore, we may assume that it is 1- or 2-regular, otherwise we just have to switch between G_1 and G_2. However, in any case G_1 is disconnected that contradicts to Jaeger's conditions [5, 4].

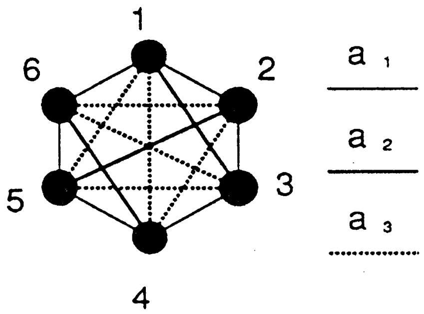
3.3.2 Degree 3

G_1, G_2, and G_3 are regular graphs by Theorem 2.1. There are two possibilities, namely two matching and a three regular graph, or one matching and two 2-regular graphs. In the first case the two matchings together form a 6-cycle, so W_+ is as follows.

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_3 & a_3 & a_3 & a_2 \\ a_1 & a_0 & a_2 & a_3 & a_3 & a_3 \\ a_3 & a_2 & a_0 & a_1 & a_3 & a_3 \\ a_3 & a_3 & a_1 & a_0 & a_2 & a_3 \\ a_3 & a_3 & a_3 & a_2 & a_0 & a_1 \\ a_2 & a_3 & a_3 & a_3 & a_1 & a_0 \end{bmatrix}$$

Taking the $(1,3)$ and $(3,1)$ entries of W_+W_-, we obtain the following two equations:

$$\begin{aligned} a_0a_3^{-1} + a_0^{-1}a_3 + a_1a_2^{-1} + a_3a_1^{-1} + a_2a_3^{-1} + 1 &= 0 \\ a_0a_3^{-1} + a_0^{-1}a_3 + a_1^{-1}a_2 + a_3^{-1}a_1 + a_2^{-1}a_3 + 1 &= 0 \end{aligned}$$



$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_3 & a_1 \\ a_1 & a_0 & a_1 & a_3 & a_2 & a_3 \\ a_2 & a_1 & a_0 & a_1 & a_3 & a_3 \\ a_3 & a_3 & a_1 & a_0 & a_1 & a_2 \\ a_3 & a_2 & a_3 & a_1 & a_0 & a_1 \\ a_1 & a_3 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}$$

Figure 1: Decomposition of K_6 into two C_6's and a matching.

Substracting the second one from the first and multiplying by $a_1a_2a_3$ we obtain

$$a_1^2a_3 - a_2^2a_3 + a_3^2a_2 - a_1^2a_2 + a_2^2a_1 - a_3^2a_1 = 0$$

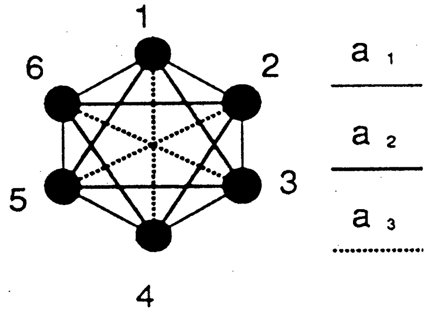
that is equivalent to

$$(a_2 - a_1)(a_3 - a_1)(a_3 - a_2) = 0.$$

However, this contradicts to the assumption of degree 3. We call the above type equation pair *the cyclic permutation equation.*

In the second case, we still have two choices. The first one is that G_1 is a matching and G_2 and G_3 are 6-cycles. There is only one way to decompose K_6 into one matching and two 6-cycles (see Figure 1). Indeed, we may assume that one of the 6-cycles is (123456). Then in the other cycle we have to have a pair of adjacent edges so that the difference of the end points of one of them is 2, and the same for the other one is 3. By renumbering the vertices cyclically and possibly reversing the cyclic order we may assume that $(1,4)$ and $(2,4)$ are edges of the other cycle. Now if $(2,5)$ were an edge of the second cycle then the other edge from vertex 5 should go to vertex 3, but that would imply $(3,6)$ is an edge of the second cycle and $(6,1)$, too. But that is a contradiction. Thus, the other edge from vertex 2 in the second cycle must be $(2,6)$. The resulting decomposition is shown on Figure 1. Taking the difference of $(1,2)$ and $(2,1)$ entries of W_+W_- we obtain the cyclic permutation equation as before, hence a contradiction.

The other possibility is the decomposition $K_6 = M_6 \cup 2\,C_3 \cup C_6$, where M_6 is a matching on six points (see Figure 2). This corresponds to the Cyclic Group C_6.



$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_2 & a_1 \\ a_1 & a_0 & a_1 & a_2 & a_3 & a_2 \\ a_2 & a_1 & a_0 & a_1 & a_2 & a_3 \\ a_3 & a_2 & a_1 & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_2 & a_1 & a_0 & a_1 \\ a_1 & a_2 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}$$

Figure 2: The Cyclic Group decomposition of K_6

3.3.3 Degree 4

According to Theorem 2.1 there are two possible cases.

Case 1. G_1, G_2 and G_3 are matchings and G_4 is a 6-cycle.There is only one such decomposition, because two of the matchings together form another 6-cycle and we can apply the argument of Section 3.2.2. The corresponding W_+ is

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_1 \\ a_1 & a_0 & a_1 & a_4 & a_2 & a_3 \\ a_2 & a_1 & a_0 & a_1 & a_3 & a_4 \\ a_3 & a_4 & a_1 & a_0 & a_1 & a_2 \\ a_4 & a_2 & a_3 & a_1 & a_0 & a_1 \\ a_1 & a_3 & a_4 & a_2 & a_1 & a_0 \end{bmatrix}.$$

Taking the $(1,5)$ and $(5,1)$ entries of W_+W_- we obtain the cyclic permutation equation as before, hence a contradiction.

Case 2. G_1, G_2 and G_3 are matchings and G_4 is a union of two triangles. It is easy to see that if we assume that the two triangles are on vertices {1,2,3} and {4,5,6}, respectively, then taking the $(1,2)$ and $(2,1)$ entries of W_+W_- we obtain the cyclic permutation equation, hence a contradiction.

3.3.4 Degree 5

Now we have a decomposition of K_6 into five matchings. There is only one way to decompose K_6 into five matchings up to permutation of the matchings, because two pairs of these matchings form two 6-cycles and we can apply the argument of Section 3.2.2. The resulting W_+ is as follows.

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_3 & a_5 & a_4 & a_2 \\ a_1 & a_0 & a_2 & a_4 & a_3 & a_5 \\ a_3 & a_2 & a_0 & a_1 & a_5 & a_4 \\ a_5 & a_4 & a_1 & a_0 & a_2 & a_3 \\ a_4 & a_3 & a_5 & a_2 & a_0 & a_1 \\ a_2 & a_5 & a_4 & a_3 & a_1 & a_0 \end{bmatrix}$$

Taking the difference of the $(1,2)$ entry of W_+W_- with the $(3,4)$, $(4,3)$, $(5,6)$ and $(6,5)$ entries, respectively, we obtain the following four equations.

$$\begin{aligned} a_3a_2^{-1} + a_2a_5^{-1} + a_5a_4^{-1} - a_2a_4^{-1} - a_3a_5^{-1} - a_5a_2^{-1} &= 0 \\ a_3a_2^{-1} + a_4a_3^{-1} + a_5a_4^{-1} - a_3a_4^{-1} - a_4a_2^{-1} - a_5a_3^{-1} &= 0 \\ a_3a_2^{-1} + a_4a_3^{-1} + a_2a_5^{-1} - a_2a_3^{-1} - a_4a_2^{-1} - a_3a_5^{-1} &= 0 \\ a_4a_3^{-1} + a_2a_5^{-1} + a_5a_4^{-1} - a_2a_4^{-1} - a_5a_3^{-1} - a_4a_5^{-1} &= 0 \end{aligned}$$

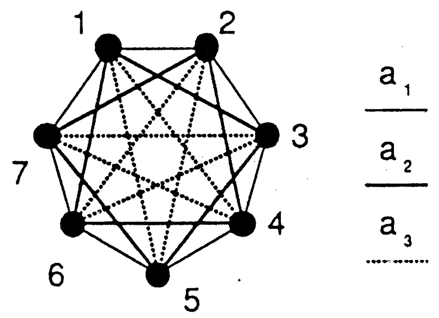
Reducing we obtain

$$\begin{aligned} (a_5 - a_2)(a_3a_4 - a_5a_4 - a_2a_4 + a_5a_2) &= 0 \\ (a_3 - a_4)(a_5a_2 + a_3a_4 - a_3a_2 - a_2a_4) &= 0 \\ (a_2 - a_3)(a_3a_2 + a_5a_4 - a_5a_2 - a_3a_5) &= 0 \\ (a_4 - a_5)(a_3a_2 + a_5a_4 - a_3a_4 - a_3a_5) &= 0. \end{aligned}$$

Using that a_i's are different, it yields $a_3 = -a_5$ and $a_4 = -a_2$ that easily leads to a contradiction. Thus, the case $n = 6$ is finished.

3.4 $n = 7$

Applying again Theorem 2.1 we obtain that the number of different off-diagonal entries of W_+ is at most three. If (X, W_+, W_-) were of degree two, then G_1 should be a strongly regular graph. However, strongly regular graph on 7 vertices does not exist. So, we may assume that the model is of degree 3. Now G_1, G_2 and G_3 are all regular graphs. Thus, they are all 2-regular graphs, i.e. unions of cycles. Furthermore, by Theorem 2.3 we have that A_i's are adjacency matrices of an association scheme. However, it is well known



$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_3 & a_2 & a_1 \\ a_1 & a_0 & a_1 & a_2 & a_3 & a_3 & a_2 \\ a_2 & a_1 & a_0 & a_1 & a_2 & a_3 & a_3 \\ a_3 & a_2 & a_1 & a_0 & a_1 & a_2 & a_3 \\ a_3 & a_3 & a_2 & a_1 & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_3 & a_2 & a_1 & a_0 & a_1 \\ a_1 & a_2 & a_3 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}$$

Figure 3: The Cyclic Group decomposition $K_7 = C_7 \cup C_7 \cup C_7$

folklore that the only association scheme on 7 points with $k_1 = k_2 = k_3 = 2$ is the scheme of the 7-gon. Thus the only case here is the cyclic group case (see Figure 3). The proof of Theorem 1.4 is now completed.

4 Non-symmetric models

In this section G_i's are oriented graphs. Furthermore, we always assume that the models are *really* non-symmetric, i.e., there exists at least one G_i and an edge $(k,l) \in E(G_i)$ such that $(l,k) \notin E(G_i)$. By Thorem 2.1 we have that in-degree=out-degree=k_i for every vertex in G_i $i = 1, 2, \ldots$. If the model is of degree 1, then it is symmetric and it is the Potts model.

4.1 $n = 4$

4.1.1 Degree 2

We may assume that $k_1 = 1$ and $k_2 = 2$. Now G_1 contains either two independent edges directed in both ways, or it is a directed four-cycle. In the first case we obtain a symmetric model. In the second case we may assume that the directed four cycle is (1234), i.e., W_+ is

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_2 \\ a_2 & a_0 & a_1 & a_2 \\ a_2 & a_2 & a_0 & a_1 \\ a_1 & a_2 & a_2 & a_0 \end{bmatrix}.$$

Taking the $(1,2)$ and $(2,1)$ entries of W_+W_- we obtain a cyclic permutation equation in variables a_0, a_1, a_2, i.e., $(a_0 - a_1)(a_0 - a_2)(a_1 - a_2) = 0$. This implies that $a_0 = a_1$ or $a_0 = a_2$. In the first case we obtain

$$W_+ = \begin{bmatrix} a_0 & a_0 & a_2 & a_2 \\ a_2 & a_0 & a_0 & a_2 \\ a_2 & a_2 & a_0 & a_0 \\ a_0 & a_2 & a_2 & a_0 \end{bmatrix}.$$

Taking the $(1,2)$ and $(1,3)$ entries of W_+W_- we obtain

$$a_0a_2^{-1} + a_0^{-1}a_2 + 2 = 0 = 2(a_0a_2^{-1} + a_0^{-1}a_2),$$

a contradiction.

In the second case we have

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_0 & a_0 \\ a_0 & a_0 & a_1 & a_0 \\ a_0 & a_0 & a_0 & a_1 \\ a_1 & a_0 & a_0 & a_0 \end{bmatrix}.$$

This is an instance of the cyclic group case.

4.1.2 Degree 3

G_1, G_2 and G_3 all have both in-degree and out-degree 1. If the model is non-symmetric, then we may assume that G_1 is a directed four-cycle. This implies that the other two graphs must be the reverse four-cycle and the diagonals oriented in both ways. Thus,

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_3 & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_1 & a_2 & a_3 & a_0 \end{bmatrix}.$$

This is another instance of the cyclic group case.

4.2 $n = 5$

4.2.1 Degree 2

Now we have two cases to be distinguished: $k_1 = 1$ and $k_1 = 2$.

$k_1 = 1$ In this case G_1 cannot be symmetric. There are two possibilities for G_1. One is that it is a union of a directed triangle and an edge oriented both ways, the other is the directed five-cycle. In the first case assuming that the directed triangle is (123) we have that

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_2 & a_2 \\ a_2 & a_0 & a_1 & a_2 & a_2 \\ a_1 & a_2 & a_0 & a_2 & a_2 \\ a_2 & a_2 & a_2 & a_0 & a_1 \\ a_2 & a_2 & a_2 & a_1 & a_0 \end{bmatrix}.$$

From the $(1,2)$ and $(2,1)$ entries of W_+W_- we obtain the cyclic permutation equation $(a_0 - a_1)(a_0 - a_2)(a_1 - a_2) = 0$. This implies that either $a_0 = a_1$ or $a_0 = a_2$. In the first case the $(4,5)$ entry of W_+W_- would be 5, a contradiction. In the second case we obtain the system of equations

$$\begin{aligned} a_0a_1^{-1} + a_0^{-1}a_1 + 3 &= 0 \\ 4a_0 + a_1 &= \pm\sqrt{5}a_0^{-1} \\ 4a_0^{-1} + a_1^{-1} &= \pm\sqrt{5}a_0 \end{aligned}$$

that has no solution.

If G_1 is a directed five-cycle, then we obtain an instance of the cyclic group model.

$k_1 = 2$ Because the model is non-symmetric, we may assume that $(1,2) \in E(G_1)$ and $(2,1) \in E(G_2)$. Now the first two rows of W_+ look like

$$\begin{matrix} a_0 & a_1 & a_2 & x & y \\ a_2 & a_0 & a_1 & z & u \end{matrix}$$

after suitable rearrangement of the last three rows and columns, where the part $\begin{matrix} x & y \\ z & u \end{matrix}$ stands for either $\begin{matrix} a_1 & a_2 \\ a_2 & a_1 \end{matrix}$ or $\begin{matrix} a_1 & a_2 \\ a_1 & a_2 \end{matrix}$. In any case, from the $(1,2)$ and $(2,1)$ entries of W_+W_- we obtain again the cyclic permutation equation $(a_0 - a_1)(a_0 - a_2)(a_1 - a_2) = 0$, which implies that either $a_0 = a_1$ or $a_0 = a_2$. By symmetry reasons we may assume that $a_0 = a_2$. Now the product of row i of W_+ and column j of W_- for $i \neq j$ is either $2(a_0a_1^{-1} + a_0^{-1}a_1) + 1$ or $a_0a_1^{-1} + a_0^{-1}a_1 + 3$. However, both cannot occur at the same time. If the first two rows of W_+ are $\begin{matrix} a_0 & a_0 & a_1 & a_1 & a_0 \\ a_1 & a_0 & a_0 & a_0 & a_1 \end{matrix}$, then a_0's should stand under the a_1's of the first row, the second row, ... of W_+ otherwise we would get both types of products, a contradiction. However, that would imply four of the a_0's in the third row, also a contradiction.

On the other hand, if the first two rows of W_+ look like $\begin{matrix} a_0 & a_0 & a_1 & a_1 & a_0 \\ a_1 & a_0 & a_0 & a_1 & a_0 \end{matrix}$, then the $(3,5)$ entry of W_+must be a_1, otherwise again both types of product would occur. However, that would imply three a_1's in the fifth column, a contradiction. To finish this case we have to note only that the first two rows of W_+can be assumed of one of the above two forms via suitable rearrangement of the rows and columns.

4.2.2 Degree 3

We may assume that $k_1 = k_2 = 1$ and $k_3 = 2$. If G_1 is a union of a directed triangle and an edge directed in both ways, then we may assume that the triangle is (123). By symmetry, and the regularity of the G_i's we may assume that the last two rows of W_+ are as follows:

$$\begin{matrix} a_3 & a_3 & a_2 & a_0 & a_1 \\ a_2 & a_3 & a_3 & a_1 & a_0 \end{matrix}.$$

Now G_1 contains the triangle (123) and we can apply the regularity, so W_+ is

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_3 & a_3 & a_2 \\ a_3 & a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_0 & a_3 & a_3 \\ a_3 & a_3 & a_2 & a_0 & a_1 \\ a_2 & a_3 & a_3 & a_1 & a_0 \end{bmatrix}.$$

Taking the $(1,2)$ and $(2,1)$ entries of W_+W_- we obtain a cyclic permutation equation in variables a_0, a_1 and a_3. On the other hand, $(3,5)$ and $(5,3)$ entries give the cyclic permutation equation in variables a_0, a_2 and a_3. This implies that $a_0 = a_3$. However, in this case the $(1,3)$ and $(3,1)$ entries give a cyclic permutation equation in a_1, a_2 and a_3, a contradiction. Thus, we may assume that both G_1 and G_2 are directed five-cycles. Let us denote for a digraph G by $-G$ the graph with edges exactly the reverses of those of G. If $G_1 = -G_2$ or $G_1 \cap -G_2 = \emptyset$, then we have instances of the cyclic group case.

Thus, we may assume that G_1 is the five-cycle (12345) and that $(2,1) \in G_3$ and $(3,2) \in G_2$. Now we have to consider two cases.
Case 1. $(2,5) \in G_2$. It is easy to see that this implies that G_2 is the cycle (13254). So W_+ is

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_3 \\ a_3 & a_0 & a_1 & a_3 & a_2 \\ a_3 & a_2 & a_0 & a_1 & a_3 \\ a_2 & a_3 & a_3 & a_0 & a_1 \\ a_1 & a_3 & a_3 & a_2 & a_0 \end{bmatrix}.$$

Considering again several entries of the product W_+W_-, we have that $(3,4)$ and $(4,3)$ give cyclic permutation equation in variables a_0, a_1 and a_3, while $(4,5)$ and $(5,4)$ give one in variables a_0, a_1 and a_2. This implies $a_0 = a_1$. Substituting that value we get that $(1,2)$ and $(2,1)$ give the cyclic permutation in variables a_1, a_2 and a_3, a contradiction.
Case 2. $(2,4) \in G_2$. This results in that $G_2 = (15324)$, and

$$W_+ = \begin{bmatrix} a_0 & a_1 & a_3 & a_3 & a_2 \\ a_3 & a_0 & a_1 & a_2 & a_3 \\ a_3 & a_2 & a_0 & a_1 & a_3 \\ a_2 & a_3 & a_3 & a_0 & a_1 \\ a_1 & a_3 & a_2 & a_3 & a_0 \end{bmatrix}.$$

In a similar way as in Case 1 we obtain from the cyclic permutation equations given by $(1,2)$ and $(2,1)$ morover $(2,3)$ and $(3,2)$ that $a_0 = a_1$. However, with this value we obtain the cyclic permutation equation from $(4,5)$ and $(5,4)$ in variables a_1, a_2 and a_3, a contradiction.

4.2.3 Degree 4

First we must observe that if $G_i = -G_j$ for some i and j, then we have an instance of the cyclic group case. Next, if G_1 is a union of a directed triangle and an edge directed in both ways, then assuming that the triangle is (123), we have

$$W_+ = \begin{bmatrix} a_0 & a_1 & * & * & * \\ * & a_0 & a_1 & * & * \\ a_3 & * & a_0 & a_1 & * \\ a_{i_1} & a_{i_2} & a_{i_3} & a_0 & a_1 \\ a_{j_1} & a_{j_2} & a_{j_3} & a_3 & a_0 \end{bmatrix}.$$

Here i_1, i_2, i_3 and j_1, j_2, j_3 are permutations of $\{2,3,4\}$ so that $i_k \neq j_k$ for $k = 1,2,3$. This immediately implies a cyclic permutation equation in variables a_2, a_3 and a_4, a contradiction.

Thus, we may assume that all four G_i's are five-cycles. If $G_1 \cap G_2 = \emptyset$, then again we have an instance of the cyclic group case. So we assume that $G_1 = (12345)$, $(2,1) \notin G_2$ and $(3,2) \in G_2$. Similarly to the degree 3 case we obtain that either $G_2 = (13254)$ or $G_2 = (15324)$. However, it is not hard to verify that in neither of these cases can we decompose the remaining edges into two directed five-cycles. This completes the proof of Theorem 1.5. ∎

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