数理解析研究所講究録
第 846 巻 1993 年 90-100

Hadamard Matrices and Generalized Spin Models

九州大（理）　山田美枝子　(Mieko Yamada)

Abstract

The concept of spin models was introduced by V.F. Jones in 1989. K.Kawagoe, A.Munemasa and Y.Watatani generalized it by removing the condition of symmetry. Recently E.Bannai and E.Bannai further generalized the concept of spin models which is called 4-weight spin models or generalized generalized spin models. On the otherhand, A.A. Ivanov and I.V. Chuvaeva showed that symmetric amorphous association schemes of class 4 obtained from Hadamard matrices. An infinite family of Hadamard matrices and of complex Hadamard matrices can be constructed by fusing the relations of these amorphous association schemes.

We show the necessary and sufficient condition that these Hadamard matrices give generalized spin models of symmetric Hadamard type and of pseudo-Jones type. A special class of Hadamard matrices satisfies this necessary and sufficient condition. Furthermore Hadamard matrices constructed by fusing amorphous association schemes are also contained in the special class if Hadamard matrices giving these amorphous assoication schemes are contained in the special class. It means that there exist infinite families of generalized spin models of symmetric Hadamard type and of pseudo-Jones type.

1 Introduction

The concept of spin models was introduced by V.F. Jones [5] in 1989 to give the Link invariant. K.Kawagoe, A.Munemasa and Y.Watatani [6] generalized it by removing the condition of symmetry. Recently E.Bannai and E.Bannai [1] further gneralized the concept of spin models which is called 4-weight spin models or generalized generalized spin models.

Definition 1 [E.Bannai-E.Bannai, [1]] Let X be a finite set and $w_i, (i = 1,2,3,4)$ be functions on $X \times X$ to C. Then (X, w_1, w_2, w_3, w_4) is 4-weight spin model of loop variable D if the following conditions are satisfied for any α, β and $\gamma \in X$:

(1) $w_1(\alpha,\beta)w_3(\beta,\alpha) = 1, w_2(\alpha,\beta)w_4(\beta,\alpha) = 1,$

(2) $\sum_{x\in X} w_1(\alpha,x)w_3(x,\beta) = n\delta_{\alpha,\beta}$, $\sum_{x\in X} w_2(\alpha,x)w_4(x,\beta) = n\delta_{\alpha,\beta}$,

(3a) $\sum_{x\in X} w_1(\alpha,x)w_1(x,\beta)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_4(\gamma,\alpha)w_4(\gamma,\beta)$,

(3b) $\sum_{x\in X} w_1(x,\alpha)w_1(\beta,x)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_4(\alpha,\gamma)w_4(\beta,\gamma)$,
where $D^2 = n = |X|$.

Let L be a diagram of an oriented link. We color the regions of L in black and white so that the unbounded region is colored white and adjacent regions have different colors. We construct an oriented graph assigning a black region to a vertex and a crossing to an edge. We get exactly four kinds of crossings according to the colors of the regions and the orientations of links. Then we attach four weights, 1,2,3,4 to four kinds of edges, namely to four kinds of crossings, respectively. Then we get an oriented graph with four kinds of weights.

Denote the weight n for an edge $\alpha \to \beta$ by $n(\alpha \to \beta)$. Let X be a finite set with $|X| = n = D^2$. Let w_1, w_2, w_3 and w_4 be complex valued functions defined on $X \times X$. Under these assumptions, the partition function Z_L is defined by

$$Z_L = D^{-v(L)} \sum_{\sigma} \prod_{\alpha\to\beta} w_{n(\alpha\to\beta)}(\sigma(\alpha),\sigma(\beta))$$

where a state σ is a map from the vertices of the graph to X and $v(L)$ is a number of vertices of the graph.

If (X, w_1, w_2, w_3, w_4) is a 4-weight spin models with loop variable D, then the partition function Z_L is invariant under the Reidemeister moves of types II and III. See the details in [1].

We consider the special case of 4-weight spin models. Let $W_i = (w_i(\alpha,\beta))_{\alpha,\beta\in X}$ for i=1,2,3,4. Let ϵ and ϵ' be from $\{+,-\}$. A 4-weghit spin models with $W_1, W_2 \in \{W_\epsilon, W_\epsilon^t\}$ and $W_3, W_4 \in \{W_{\epsilon'}, W_{\epsilon'}^t\}$ is called a generalized spin model of Jones type. A 4-weight spin models with $W_1, W_4 \in \{W_\epsilon, W_\epsilon^t\}$ and $W_2, W_3 \in \{W_{\epsilon'}, W_{\epsilon'}^t\}$ is called a generalized spin model of pseudo-Jones type. Further, a 4-weight spin models with $W_1, W_3 \in \{W_\epsilon, W_\epsilon^t\}$ and $W_2, W_4 \in \{W_{\epsilon'}, W_{\epsilon'}^t\}$ is called a generalized spin model of Hadamard type.

K. Nomura [7] constructed a family of symmetric spin models of Jones type of loop variable $4\sqrt{n}$ from Hadamard matrices of order $4n$. M. Wakimoto [8] showed that spin models of Jones type and 4-weight spin models can be constructed by using Lie algebra.

We treat here generalized spin models of pseudo-Jones type and of symmetric Hadamard type. First we give the definitions.

Definition 2 [E.Bannai-E.Bannai, [1]] (X, w_+, w_-) is a generalized spin model of pseudo-Jones type if the following conditions are satisfied for any α, β and $\gamma \in X$.

(0) $w_+(\alpha,\beta) = w_+(\beta,\alpha), w_-(\alpha,\beta) = w_-(\beta,\alpha)$,

(1J) $w_+(\alpha,\beta)w_-(\alpha,\beta) = 1$,

(2J) $\sum_{x\in X} w_+(\alpha,x)w_-(x,\beta) = n\delta_{\alpha,\beta}$,

(3P) $\sum_{x\in X} w_+(\alpha,x)w_+(x,\beta)w_+(\gamma,x) = Dw_+(\alpha,\beta)w_+(\gamma,\alpha)w_+(\gamma,\beta)$.

Definition 3 [E.Bannai-E.Bannai, [1]] (X, w_+, w_-) is a generalized spin model of symmetric Hadamard type if the following conditions are satisfied for any α, β and $\gamma \in X$:

(0) $w_+(\alpha,\beta) = w_+(\beta,\alpha), w_-(\alpha,\beta) = w_-(\beta,\alpha)$,

(1H) $W_+ \circ W+ = J, W_- \circ W_- = J$,

(2H) $W_+^2 = nI, W_-^2 = nI$,

(3H) $\sum_{x\in X} w_{\epsilon'}(\alpha,x)w_{\epsilon'}(x,\beta)w_\epsilon(\gamma,x) = Dw_{\epsilon'}(\alpha,\beta)w_\epsilon(\gamma,\alpha)w_\epsilon(\gamma,\beta)$,

where $\circ$ is an Hadamard product, J is the matrix whose entries are all 1 and I is the unit matrix.

2 Amorphous association schemes and Hadamard matrices

Theorem 1 [A.A. Ivanov-I.V. Chuvaeva, [3]] *Let $H = (h_{i,j})$ be an Hadamard matrix of order $4n$ and $\Omega = \{0, 1, 2, ..., 4n-1\}$. Put $X = \Omega \times \Omega$. The subsets R_i, $(0 \le i \le 4)$ of $X \times X$ are defined by*

$$R_0 = \{(x,x) | x \in X\},$$

$$R_1 = \{((x_1,x_2),(y_1,y_2)) | x_1 = y_1\},$$

$$R_2 = \{((x_1,x_2),(y_1,y_2)) | x_2 = y_2\},$$

$$R_3 = \{((x_1,x_2),(y_1,y_2)) | h_{x_1x_2}h_{y_1y_2}h_{x_1y_2}h_{y_1x_2} = 1\},$$

$$R_4 = \{((x_1,x_2),(y_1,y_2)) | h_{x_1x_2}h_{y_1y_2}h_{x_1y_2}h_{y_1x_2} = -1\}.$$

Then $(X, R_0, R_1, R_2, R_3, R_4)$ is an amorphous association scheme of class 4.

Let $Y = (X, \{R_i\}_{0\le i\le d})$ be a commutative association scheme. A partition $\Lambda_0, \Lambda_1, ..., \Lambda_e$ of the index set is said to be *admissible* if $\Lambda_0 = \{0\}, \Lambda_i \ne \emptyset \quad (1 \le i \le e)$ and $\Lambda_i' = \Lambda_j$ for some $j, (1 \le i, j \le e)$ where $\Lambda_i' = \{\alpha' | \alpha \in \Lambda_i\}$, $R_{\alpha'} = \{(x,y) | (y,x) \in R_\alpha\}$. Let $R_{\Lambda_i} = \bigcup_{\alpha\in\Lambda_i} R_\alpha$. If $(X, \{R_{\Lambda_i}\}_{0\le i\le e})$ becomes an association scheme for every admissble partition, then Y is defined to be *amorphous*.

Corollary 1 [2] *The valencies and intersection numbers of an amorphous association scheme mentioned in Theorem 1 are given as follows:*

(1) $k_1 = k_2 = 4n-1, k_3 = (2n-1)(4n-1), k_4 = 2n(4n-1)$

(2) $p^i_{ii} = g_i^2 - 3g_i + 4n, p^j_{ii} = g_i(g_i - 1), p^i_{ij} = \frac{k_j}{k_i} g_i(g_i-1), p^l_{ij} = g_i g_j$
for $i \neq j \neq l$, $0 \leq i,j,l \leq 4$, *where* $g_1 = g_2 = 1$, $g_3 = 2n-1, g_4 = 2n$.

We have the following theorem by using these amorphous association schemes.

Theorem 2 *Let* $A_i, (0 \leq i \leq 4)$ *be adjacency matrices of an amorphous association scheme obtained from an Hadanard matrix of order* $4n$ *by Theorem 1. Then*

(1) $M_1 = A_0+A_1+A_2+A_3-A_4$, $M_2 = A_0+A_1-A_2-A_3+A_4$, $M_3 = A_0-A_1+A_2-A_3+A_4$ *are regular symmetric Hadamard matrices of order* $(4n)^2$,

(2) $L_1 = A_0 + A_1 + A_2 i + A_3 i - A_4 i$ *and* $L_2 = A_0 + A_1 i + A_2 + A_3 i - A_4 i$ *are regular symmetric complex Hadamard matrices of* $(4n)^2$,

where i *is a primitive* 4^{th} *root of unity.*

By using Theorems 1 and 2 repeatedly and by using the tensor products of matrices, we have,

Corollary 2 *(1) There exists an infinite family of regular symmetaric Hadamard matrices of order* $(4n)^{2l}$, l *: a positive integer,*

(2) there exists an infinite family of regular symmetaric complex Hadamard matrices of order $(4n)^{2l}$, l *: a positive integer.*

3 Classes of Hadamard matrices

Definition 4 Two Hadamard matrices are said to be *equivalent* if one can be obtained from the other by a sequence of the following operations:

(1) Permute rows(or columns),

(2) multiply any row(or column) by -1.

Let $H = (h_{i,j})$ be an Hadamard matrix. Let $I = (i_1, i_2, i_3, i_4)$ be a 4-subset of the index set $\Omega = \{0, 1, ..., 4n-1\}$. We define

$$N_I = N_{(i_1,i_2,i_3,i_4)} = \sum_{j=0}^{4n-1} h_{i_1 j} h_{i_2 j} h_{i_3 j} h_{i_4 j}.$$

Then N_I is invariant under Hadamard transformation for columns. If we define

$$S_k = \sharp\{(i_1, i_2, i_3, i_4) | N_{(i_1,i_2,i_3,i_4)} = k\},$$

$$C_k = S_k + S_{-k},$$

then C_k is invariant under Hadamard transformation (1) and (2) in the above definition. If C_k's of two Hadamard matrices are different, they are inequivalent.

Lemma 1 $N_I \equiv 0$ *(mod 4).*

Corollary 3 *Let H_1 and H_2 be equivalent Hadamard matrices. Let A_i and A_i', i = 0,1,2,3,4 be adjacency matrices obtained from H_1 and H_2 respectively. Then there exists a permutation matrix P such that*

$$A_i' = PA_iP$$

for $0 \leq i \leq 4$.

4 Generalized spin models of symmeteric Hadamard type

We give a necessary and sufficient condition that Hadamard matrices M_1, M_2 and M_3 in Theorem 2 give generalized spin models of symmetric Hadamard type.

Theorem 3 *Let H be a normalized Hadamard matrix of order $4n$ and $A_i, (0 \leq i \leq 4)$ be adjacency matrices obtained from H.*

(1) $W_+ = W_- = M_1 = A_0 + A_1 + A_2 + A_3 - A_4$ gives a generalized spin model of symmetric Hadamard type if and only if the following conditions (a),(b) are satisfied for any $\beta_1, \beta_2, \gamma_1$ and $\gamma_2 \in \Omega^* = \{1, 2, ..., 4n-1\}$:

(a) when $(h_{\beta_1\beta_2}, h_{\gamma_1\gamma_2}, h_{\beta_1\gamma_2}, h_{\gamma_1\beta_2})$ = (1,1,1,1),(1,1,-1,-1),(1,-1,-1,1), (1,-1,1,-1),(-1,-1,-1,-1),

$$\sum_{l=-n}^{n} \theta_l l = n,$$

(b) *when* $(h_{\beta_1\beta_2}, h_{\gamma_1\gamma_2}, h_{\beta_1\gamma_2}, h_{\gamma_1\beta_2})$ = *(1,1,-1,),(1,-1,-1,-1),*

$$\sum_{l=-n}^{n} \theta_l l = 0$$

where $\theta_l = \sharp\{x_1 | h_{x_1\beta_2} h_{x_1\gamma_2} = 1,\ N_{(0,\beta_1,\gamma_1,x_1)} = 4l\}$.

(2) $W_+ = W_- = M_2 = A_0 + A_1 - A_2 - A_3 + A_4$ *gives a generalized spin model of symmetric Hadamard type if and only if the above conditions (a) and (b) are satisfied for any* $\beta_1, \beta_2, \gamma_1$ *and* $\gamma_2 \in \Omega^*$.

(3) $W_+ = W_- = M_3 = A_0 - A_1 + A_2 - A_3 + A_4$ *gives a generalized spin model of symmetric Hadamard type if and only if the transpose matrix* H^t *satisfies the above conditions (a) and (b) for any* $\beta_1, \beta_2, \gamma_1$ *and* $\gamma_2 \in \Omega^*$.

To prove the Theorem 3, the following lemma is useful.

Lemma 2 *Let* H *be a normalized Hadamard matrix of order* $4n$. *Choose three rows* $\alpha_1 = 0, \beta_1$ *and* γ_1. *Then*

$$\sum_{l=-n}^{n} \xi_l l = n$$

where $\xi_l = \sharp\{x_1 | N_{(\alpha_1,\beta_1,\gamma_1,x_1)} = 4l\}$. *It is also true for columns.*

Proof of Theorem 3. (1) Since Hadamard matrices $M_i, 0 \le i \le 3$, are regular symmetric, the conditions (0),(1H),and (2H) hold. Therefore we get a necessary and sufficient condition by verifying the condition (3H).

When we choose three rows α, β and γ of M_1, we may assume one of them, say α, is equal to 0=(0,0). We obtain only one inequivalent normalized Hadamard matrix on whichever entry we normalize an Hadamard matrix. Assume that $\alpha = (\alpha_1, \alpha_2) \neq 0 = (0,0)$. The row α_1 and the column α_2 can be transformed into the row and the column with all 1 entries by multiplying some rows and columns by -1. Denote this Hadamard matrix by H'. Then we get the normalized Hadamard matrix H by permuting rows and columns of H';

$$H = QH'R,$$

where Q and R are permutation matrices. Hence if the permutations Q and R act on the rows and columns of M_1 simultaneously, we obtain the same matrix M_1. Namely there exists a permutation matrix P such that

$$M_1 = PM_1P.$$

Put $M_1 = (m(\alpha,\beta))$ where $m(\alpha,\beta) = h_{\alpha_1\alpha_2}h_{\beta_1\beta_2}h_{\alpha_1\beta_2}h_{\beta_1\alpha_2}$, $\alpha = (\alpha_1,\alpha_2), \beta = (\beta_1,\beta_2)$. The left-hand side of the star triangle relation (3H)

$$S(\alpha,\beta,\gamma) = \sum_{x\in X} m(\alpha,x)m(\beta,x)m(\gamma,x)$$

is invariant under the column permutation of M_1. Since β and γ run over X, we may put $\alpha = 0$.

When $\alpha = \beta = \gamma$ or $\alpha = \beta$ or $\beta = \gamma$ or $\gamma = \alpha$, the condition (3H) is satisfied from the regularity of M_1. We may assume that $\alpha \neq \beta \neq \gamma$. ¿From $h_{00} = h_{0x_2} = h_{x_10} = 0$,

$$S(0,\beta,\gamma) = \sum_{x\in X} m(0,x)m(\beta,x)m(\gamma,x) = h_{\beta_1\beta_2}h_{\gamma_1\gamma_2}\sum_{x_1} h_{x_1\beta_2}h_{x_1\gamma_2}\sum_{x_2} h_{x_1x_2}h_{\beta_1x_2}h_{\gamma_1x_2}$$

Put $N_{x_1} = N_{(0,\beta_1,\gamma_1,x_1)} = \sum_{x_2} h_{x_1x_2}h_{\beta_1x_2}h_{\gamma_1x_2}$. Since $N_0 = N_{\beta_1} = N_{\gamma_1} = 0$,

$$S(0,\beta,\gamma) = h_{\beta_1\beta_2}h_{\gamma_1\gamma_2}\sum_{x_1\neq 0,\beta_1,\gamma_1} h_{x_1\beta_2}h_{x_1\gamma_2}N_{x_1}.$$

We define $\theta_l = \sharp\{x_1|N_{x_1} = 4l,\ h_{x_1\beta_2}h_{x_1\gamma_2} = 1\}$ and $\eta_l = \sharp\{x_1|N_{x_1} = 4l,\ h_{x_1\beta_2}h_{x_1\gamma_2} = -1\}$. Then

$$S(0,\beta,\gamma) = 4h_{\beta_1\beta_2}h_{\gamma_1\gamma_2}\sum_{l=-n}^{n}(\theta_l - \eta_l)l.$$

Next we consider the right-hand side of the star triangle relation (3H).

$$4n\cdot m(\alpha,\beta)m(\beta,\gamma)m(\gamma,\alpha) = 4nh_{\beta_1\gamma_2}h_{\gamma_1\beta_2}.$$

Hence we have

$$\sum_{l=-n}^{n}(\theta_l - \eta_l)l = n\cdot m(\beta,\gamma).$$

Since from Lemma 2,

$$\sum_{l=-n}^{n}\xi_l l = \sum_{l=-n}^{n}(\theta_l + \eta_l)l = n,$$

the necessary and sufficient condition is given by

$$\sum_{l=-n}^{n}\theta_l l = n(m(\beta,\gamma)+1)/2.$$

Notice that we may exchange the rows β_1 and γ_1, and the columns β_2 and γ_2 to each other. Hence the values $(h_{\beta_1\beta_2}, h_{\gamma_1\gamma_2}, h_{\beta_1\gamma_2}, h_{\gamma_1\beta_2})$ reduce to the 7 cases in Theorem 3.

(2) We can prove in the same way as the following.

(3) Similarly to the case (1), we verify the condition (3H) for $\alpha \neq \beta \neq \gamma$. We may put $\alpha = 0$. Let $W_+ = W_- = (w(x,y))_{x,y\in X}$ and $x = (x_1,x_2), y = (y_1,y_2)$. Then

$$w(x,y) = \begin{cases} 1 & (x,y)\in R_0 \quad\text{or}\quad R_2, \\ -h_{x1x2}h_{y_1y_2}hx1y2h_{y_1x_2} & (x,y)\in R_1, R_3 \quad\text{or}\quad R_4, \end{cases}$$

where $R_i, 0 \le i \le 4$, are defined in Theorem 1.

When $(\alpha,\beta),(\beta,\gamma),(\alpha,\gamma) \in R_2$, it is easy to prove that the condition (3H) holds.

We distinguish 2 cases.

Case 1. Exactly one of $(\alpha,\beta),(\beta,\gamma),(\alpha,\gamma)$ is contained in R_2.

First we suppose $(\alpha,\beta) \in R_2$, namely $\alpha_2 = \beta_2$, and $(\beta,\gamma),(\alpha,\gamma) \notin R_2$. We get the right-hand side of the condition(3H) is $4nh_{\beta_1\gamma_2}$. Now we verify the left-hand side $S(0,\beta,\gamma)$ of the condition.

$$\begin{aligned}
S(0,\beta,\gamma) &= \sum_{(\alpha,x),(\beta,x)\in R_2,(\gamma,x)\notin R_2} w(\alpha,x)w(\beta,x)w(\gamma,x) \\
&\quad + \sum_{(\alpha,x),(\beta,x)\notin R_2,(\gamma,x)\in R_2} w(\alpha,x)w(\beta,x)w(\gamma,x) \\
&\quad + \sum_{(\alpha,x),(\beta,x)\notin R_2,(\gamma,x)\notin R_2} w(\alpha,x)w(\beta,x)w(\gamma,x) \\
&= -\sum_{x_1} h_{\gamma_1\gamma_1}h_{x_1\alpha_2}h_{\gamma_1\alpha_2}h_{x_1\gamma_2} + \sum_{x_1} h_{x_1\gamma_2}h_{\beta_1\beta_2}h_{x_1\gamma_2}h_{\beta_1\gamma_2}h_{x_1\beta_2} \\
&\quad -h_{\beta_1\alpha_2}h_{\gamma_1\gamma_2}\sum_{x_2} h_{\beta_1x_2}h_{\gamma_1x_2}\sum_{x_1} h_{x_1x_2}h_{x_1\alpha_2}h_{x_1\gamma_2} \\
&= h_{\beta_1\beta_2}h_{\beta_1\gamma_2}\sum_{x_1} h_{x_1\alpha_2} \\
&= 4nh_{\beta_1\gamma_2}
\end{aligned}$$

Hence the condition (3H) holds. We can prove that the condition (3H) holds for the cases $(\beta,\gamma) \in R_2, (\alpha,\beta), (\alpha,\gamma) \notin R_2$ and $(\alpha,\gamma) \in R_2, (\alpha,\beta),(\beta,\gamma) \notin R_2$ in the same way.

Case 2. $(\alpha,\beta),(\beta,\gamma),(\alpha,\gamma) \notin R_2$.

The right-hand side of (3H) is $-4nh_{\beta_1\gamma_2}h_{\gamma_1\beta_2}$.

$$\begin{aligned}
S(0,\beta,\gamma) &= \sum_{(\alpha,x)\in R_2,(\beta,x),(\gamma,x)\notin R_2} w(\alpha,x)w(\beta,x)w(\gamma,x) \\
&\quad + \sum_{(\beta,x)\in R_2,(\alpha,x),(\gamma,x)\notin R_2} w(\alpha,x)w(\beta,x)w(\gamma,x) \\
&\quad + \sum_{(\gamma,x)\in R_2,(\alpha,x),(\beta,x)\notin R_2} w(\alpha,x)w(\beta,x)w(\gamma,x) \\
&\quad + \sum_{(\alpha,x),(\beta,x),(\gamma,x)\notin R_2} w(\alpha,x)w(\beta,x)w(\gamma,x) \\
&= h_{\beta_1\beta_2}h_{\gamma_1\gamma_2}\sum_{x_1} h_{x_1\beta_2}h_{x_1\gamma_2} + h_{\gamma_1\gamma_2}h_{\gamma_1\beta_2}\sum_{x_1} h_{x_1\gamma_2} + h_{\beta_1\beta_2}h_{\beta_1\gamma_2}\sum_{x_1} h_{x_1\beta_2} \\
&\quad -h_{\beta_1\beta_2}h_{\gamma_1\gamma_2}\sum_{x_2} h_{\beta_1x_2}h_{\gamma_1x_2}\sum_{x_1} h_{x_1x_2}h_{x_1\beta_2}h_{x_1\gamma_2} \\
&= -h_{\beta_1\beta_2}h_{\gamma_1\gamma_2}\sum_{x_2} h_{\beta_1x_2}h_{\gamma_1x_2}\sum_{x_1} h_{x_1x_2}h_{x_1\beta_2}h_{x_1\gamma_2}
\end{aligned}$$

Putting $\tilde{N}_{x_2} = \tilde{N}_{(0,\beta_2,\gamma_2,x_2)} = \sum_{x_1} h_{x_1x_2}h_{x_1\beta_2}h_{x_1\gamma_2}$. Since $\tilde{N}_0 = \tilde{N}_{\beta_2} = \tilde{N}_{\gamma_2} = 0$,

$$S(0,\beta,\gamma) = -h_{\beta_1\beta_2}h_{\gamma_1\gamma_2} \sum_{x_2\neq 0,\beta_2,\gamma_2} h_{\beta_1x_2}h_{\gamma_1x_2}\tilde{N}_{x_2}.$$

Define $\tilde{\theta}_l = \sharp\{x_1|\tilde{N}_{x_1} = 4l,\ h_{\beta_1x_2}h_{\gamma_1x_2} = 1\}$ and $\tilde{\eta}_l = \sharp\{x_1|\tilde{N}_{x_1} = 4l,\ h_{\beta_1x_2}h_{\gamma_1x_2} = -1\}$. Then

$$S(0,\beta,\gamma) = -h_{\beta_1\beta_2}h_{\gamma_1\gamma_2} \sum_{l=-n}^{n} (\tilde{\theta}_l - \tilde{\eta}_l)l.$$

From the Lemma 2, it follows that

$$\sum_{l=-n}^{n} \tilde{\theta}_l l = n(m(\beta,\gamma)+1)/2$$

is a necessary and sufficient condition. It means that the transpose matrix H^t satisfies the conditions (a) and (b). □

5 Generalized spin models of pseudo-Jones type

We give a necessary and sufficient condition that complex Hadamard matrices $L_1, \overline{L_1}$ and $L_2, \overline{L_2}$ in Theorem 2 give generalized spin models of pseudo-Jones type.

Theorem 4 *Let i be a primitive 4^{th} root of unity and H be a normalized Hadamard matrix of order $4n$. Let A_i, $(0 \leq i \leq 4)$ be adjacency matrices obtained from H.*

(1) $W_+ = L_1 = A_0 + A_1 + A_2 i + A_3 i - A_4 i$, $W_- = \overline{L_1}$ gives a generalized spin model of pseudo-Jones type if and only if the conditions (a) and (b) in Theorem 3 are satisfied for any $\beta_1, \beta_2, \gamma_1$ and $\gamma_2 \in \Omega^*$.

(2) $W_+ = L_2 = A_0 + A_1 i + A_2 + A_3 i - A_4 i$, $W_- = \overline{L_2}$ gives a generalized spin model of pseudo-Jones type if and only if the transpose matrix H^t satisfies the conditions (a) and (b) for any $\beta_1, \beta_2, \gamma_1$ and $\gamma_2 \in \Omega^*$.

Proof. (1) Let $W_+ = (w(x,y))_{x,y\in X}$ and $x = (x_1,x_2), y = (y_1,y_2)$. Then the entry $w(x,y)$ is given by

$$w(x,y) = \begin{cases} 1 & (x,y) \in R_0 \quad \text{or} \quad R_1, \\ -i \cdot h_{x_1x_2}h_{y_1y_2}h_{x_1y_2}h_{y_1x_2} & (x,y) \in R_2, R_3 \quad \text{or} \quad R_4. \end{cases}$$

We can prove the theorem similarly to the Theorem 3. □

Remark. The following (1)-(3) are equivalent and (4)-(5) are equivalent:

(1) $W_+ = W_- = M_1$ gives a generalized spin model of symmetric Hadamard type.

(2) $W_+ = W_- = M_2$ gives a generalized spin model of symmetric Hadamard type.

(3) $W_+ = L_1, W_- = \overline{L_1}$ gives a generalized spin model of pseudo-Jones type.

(4) $W_+ = W_- = M_3$ gives a generalized spin model of symmetric Hadamard type.

(5) $W_+ = L_2, W_- = \overline{L_2}$ gives a generalized spin model of pseudo-Jones type.

6 A special class of Hadamard matrices and generalized spin models

Theorem 5 *Assume that an Hadamard matrix of order* $4n$ *satisfies*

$$(c)\quad C_{4n} = \frac{1}{4}\binom{4n}{n}, C_0 = \binom{4n}{4} - \frac{1}{4}\binom{4n}{3}, C_l = 0 \quad (l \neq 0, 4n).$$

Then the normalized matrix of H satisfies the necessary and sufficient conditions (a),(b) in Theorem 3. It implies that M_1, M_2 obtained from H give spin models of symmetric Hadamard type and L_1, $\overline{L_1}$ gives a generalized spin model of pseudo-Jones type.

Proof. It turns out that the condition (c) means there exists only one row x such thath $N_{(\alpha,\beta,\gamma,x)} = N_x = \pm 4n$ for fixed three rows α, β and γ. Assume that H satisfies the condition (c). Denote the normalized Hadamard matrix by H'. Assume that the row x_1 satisfies $N_{x_1} = N(0, \beta_1, \gamma_1, x_1) = 4n$ (the case $N_{x_1} = -4n$ does not occur). If $h_{x_1\beta_2}h_{x_1\gamma_2} = 1$, which is equivalent to $m(\beta, \gamma) = 1$, then $\theta_n = 1$ and $\theta_l = 0$ for $l \neq 0, n$. If $h_{x_1\beta_2}h_{x_1\gamma_2} = -1$, which is equivalent to $m(\beta, \gamma) = -1$, then $\theta_l = 0$ for $l \neq 0$. It follows that H' satisfies the necessary and sufficient conditions (a) and (b). □

It is obvious that if the transpose matrix H^t satisfies the above condition (c), then the normalized matrix of H^t satisfies the necessary and sufficient condition (3) in Theorem 3.

Corollary 4 *Assume that both H and H^t satisfy the condition (c) in Theorem 5. H is not necessarily equivalent to H^t.*

Let A_i, A_i', $(0 \leq i \leq 4)$ be adjacency matrices obtained from H, H^t respectively.

Then $M_1 = A_0+A_1+A_2+A_3-A_4$ and $M_1' = A_1'+A_2'+A_3'-A_4'$ also satisfy the condition (c). Namely, infinite families constructed from H and H^t mentioned in Corollay 2 satisfy the condition (c).

Hence there exist infinite families of generalized spin models of pseudo-Jones type and of symmetric Hadamard type with loop variable $(4n)^{2l}$, l :positive integer.

There is only 1 inequivalent class of Hadamard matrices of orders 4 and 8. They satisfy the condition (c). There are 5 inequivalent classes of order 16. Class I according to the classification by M.Hall Jr. satisfies the condition (c) but other classes not.

参考文献

[1] E.Bannai and E.Bannai, Generalized generalized spin models, preprint.

[2] T.Ito, A.Munemasa and M. Yamada, Amorphous association schemes over characteristic 4, *Europ. J. Combinatorics*, 12 (1991), 513-526.

[3] A.A. Ivanov and I.V. Chuvaeva, Action of the group M_{12} on Hadamard matrices, *Investigations in Algebraic Theory of Combinatorial Objects*, VNIISI, Moscow, Institute for System Studies, 1985, 159-169(in Russian).

[4] F.Jaeger, Storongly regular graphs and spin models for the Kauffman polynomial, *Geom. Dedicata*, 44 (1992), 23-52.

[5] V.F.R. Jones, On knot invariants related to some statistical mechanical models, *Pac. J. Math.* 137 (1989), 311-334.

[6] K.Kawagoe, A. Munemasa and Y. Watatani, Generalized spin models, preprint.

[7] K. Nomura, Spin models constructed from Hadamard matrices, *J. Combinatorial Theory* Ser. A (to appear).

[8] M. Wakimoto, Private communication.