数理解析研究所講究録
第 846 巻 1993 年 101-125



# On Strongly Closed Subgraphs of Highly Regular Graphs

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**Abstract**

A geodetically closed induced subgraph $\Delta$ of a graph $\Gamma$ is defined to be strongly closed if $\Gamma_i(\alpha) \cap \Gamma_1(\beta)$ stays in $\Delta$ for every $i$ and $\alpha$, $\beta \in \Delta$ with $\partial(\alpha, \beta) = i$. We study the existence conditions of strongly closed subgraphs in highly regular graphs such as distance-regular graphs or distance-biregular graphs.

## 1 Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph. For a subset $\Delta \subset V(\Gamma)$, we identify $\Delta$ with the induced subgraph on $\Delta$. In particular, $\Gamma = V(\Gamma)$.

For two vertices $\alpha$, $\beta$ in $\Gamma$, let $\partial_\Gamma(\alpha, \beta)$ denote the distance between $\alpha$ and $\beta$ in $\Gamma$, i.e., the length of a shortest path connecting $\alpha$ and $\beta$ in $\Gamma$. We also write $\partial(\alpha, \beta)$, when no confusion occurs. Let

$$\Gamma_i(\alpha) = \{\beta \in \Gamma | \partial(\alpha, \beta) = i\} \text{ and } \Gamma(\alpha) = \Gamma_1(\alpha).$$

For vertices $\alpha$, $\beta$ in $\Gamma$ with $\partial(\alpha, \beta) = i$, let

$$\begin{aligned}
C(\alpha,\beta) &= C_i(\alpha,\beta) = \Gamma_{i-1}(\alpha) \cap \Gamma(\beta), \\
A(\alpha,\beta) &= A_i(\alpha,\beta) = \Gamma_i(\alpha) \cap \Gamma(\beta), \\
B(\alpha,\beta) &= B_i(\alpha,\beta) = \Gamma_{i+1}(\alpha) \cap \Gamma(\beta), \text{ and} \\
G(\alpha,\beta) &= \bigcup_{j=0}^{i} \Gamma_j(\alpha) \cap \Gamma_{i-j}(\beta) \\
&= \{\gamma \in \Gamma | \partial(\alpha,\gamma) + \partial(\gamma,\beta) = \partial(\alpha,\beta)\}.
\end{aligned}$$

$G(\alpha, \beta)$ is the set of vertices which lie on a geodesic between $\alpha$ and $\beta$. For the cardinalities, we use lower case letters, i.e.,

$$c_i(\alpha,\beta) = |C_i(\alpha,\beta)|, \ a_i(\alpha,\beta) = |A_i(\alpha,\beta)|, \text{ and } b_i(\alpha,\beta) = |B_i(\alpha,\beta)|.$$

We also write $c_i(\alpha)$ [resp. $a_i(\alpha)$, $b_i(\alpha)$] if the number $c_i(\alpha, \beta)$ [resp. $a_i(\alpha, \beta)$, $b_i(\alpha, \beta)$] does not depend on the choice of $\beta$ under the condition $\partial(\alpha, \beta) = i$, and $c_i$ [resp. $a_i$, $b_i$] if the number $c_i(\alpha, \beta)$ [resp. $a_i(\alpha, \beta)$, $b_i(\alpha, \beta)$] does not depend on the choices of $\alpha$ and $\beta$

under the condition $\partial(\alpha,\beta) = i$. In these cases we say for example that $c_i(\alpha)$ exists or $c_i$ exists.

A connected graph $\Gamma$ is said to be *distance-regular* if $c_i$, $a_i$, $b_i$ exist for all $i$.

A connected bipartite graph $\Gamma$ with a bipartition $P \cup L$ is said to be *distance-biregular* if $c_i(\alpha)$, $b_i(\alpha)$ exist for all $i$ and these numbers depend only on the part $\alpha$ belongs to.

For convienience, if $\Gamma = P \cup L$ is a bipartite graph, we also use notations like $c_i^P$, $b_i^P$, $c_i^L$, $b_i^L$, when the corresponding numbers depend only on the part the base point belongs to.

A subset $\Delta$ of a graph $\Gamma$ is said to be $C_i$*-closed* [resp. $A_i$*-closed*] if $C_i(\alpha,\beta) \subset \Delta$ [resp. $A_i(\alpha,\beta) \subset \Delta$] for every pair of vertices $\alpha$, $\beta$ in $\Delta$ with $\partial_\Gamma(\alpha,\beta) = i$.

A subset $\Delta$ of $\Gamma$ is said to be *geodetically closed* if $C(\alpha,\beta) \subset \Delta$ for every pair of vertices $\alpha$, $\beta$ in $\Delta$, i.e., $\Delta$ is $C_i$-closed for every $i$. In this case, we have $\partial_\Gamma(\alpha,\beta) = \partial_\Delta(\alpha,\beta)$ for all $\alpha$, $\beta \in \Delta$. It is clear that $\Delta$ is geodetically closed if and only if $G(\alpha,\beta) \subset \Delta$ for every pair of vertices $\alpha$, $\beta$ in $\Delta$.

A subset $\Delta$ of $\Gamma$ is said to be *strongly closed* if $C(\alpha,\beta) \subset \Delta$ and $A(\alpha,\beta) \subset \Delta$ for every pair of vertices $\alpha$, $\beta$ in $\Delta$, i.e., $\Delta$ is both $C_i$-closed and $A_i$-closed for every $i$.

We call the induced subgraph on $\Delta$ a geodetically [resp. strongly] closed subgraph when $\Delta$ is a geodetically [resp. strongly] closed subset.

By definition, every strongly closed subgraph is geodetically closed, in particular connected if $\Gamma$ is connected. When $\Gamma$ is bipartite, every geodetically closed subgraph is strongly closed and we do not need to distinguish these notions.

In most known distance-regular graphs, there are many nontrivial geodetically closed subgraphs and in many cases they are even strongly closed. In some cases we can guarantee the existence of strongly [or geodetically] closed subgraphs if we know a part of the parameters $c_i$, $a_i$. See [6, 18, 19, 21, 24], and [5, Section 4.3]. We believe that the investigation of strongly [or geodetically] closed subgraphs is a key in the study of distance-regular graphs.

The first question is the following:

> *Is a strongly closed subgraph $\Delta$ of a distance-regular graph $\Gamma$ always distance-regular?*

By definition, the answer is 'yes' if $\Delta$ is regular. On the contrary, we can find counter examples easily. For example, if the girth of $\Gamma$ is large, we can construct a strongly closed subgraph isomorphic to a tree.

Are there any other types of non-regular strongly closed subgraphs of distance-regular graphs? Theorem 1.1 gives a solution to this problem.

We need a few more definitions to state the theorem.

Let $l(c,a,b) = |\{i|(c_i,a_i,b_i) = (c,a,b)\}|$ and $r(\Gamma) = l(c_1,a_1,b_1)$.

Let $d(\Gamma) = \max\{\partial(\alpha,\beta)|\alpha,\beta \in \Gamma\}$, and $k(\alpha) = |\Gamma(\alpha)| = b_0(\alpha,\alpha)$.

Let $K_{k+1}$ denote the complete graph of valency $k$, and $M_k$ denote a Moore graph of valency $k$, which is known to be of diameter 2 and $k \in \{2, 3, 7, 57\}$.

For a graph $\Gamma$, ${}^t\Gamma$ denotes a subdivision graph obtained by replacing each edge by a path of length $t$.

$K_4$ $\qquad$ ${}^2K_4$ $\qquad$ ${}^3K_4$

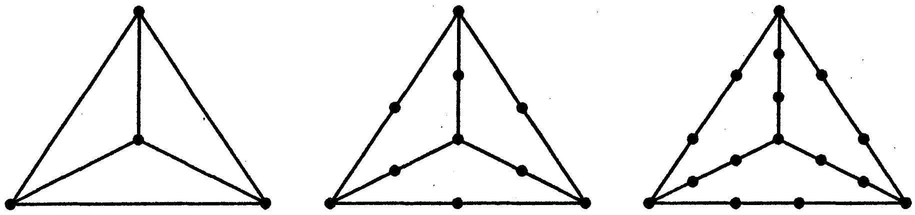

Figure 1.

**Theorem 1.1** *Let $\Delta$ be a strongly closed subgraph of a distance-regular graph $\Gamma$. Then one of the following holds.*

**(i)** *$\Delta$ is a distance-regular graph,*

**(ii)** $2 \leq d(\Delta) \leq r(\Gamma)$,

**(iii)** *$\Delta$ is a distance-biregular graph with $c_{2i-1} = c_{2i}$ for all $i$ with $2i \leq d(\Delta)$. In particular, $r(\Gamma) \equiv d(\Delta) \equiv 0 \pmod 2$; or*

**(iv)** *$\Delta$ is a subdivision graph of a complete graph or a Moore graph obtained by replacing each edge by a path of length three, i.e., $\Delta \simeq {}^3K_{l+1}$ or ${}^3M_l$. In particular, $d(\Delta) = r(\Gamma) + 2 = 5$ or $8$, and $a_1 = 0$, $c_{r+1} = c_{r+2} = a_{r+1} = a_{r+2} = 1$, where $r = r(\Gamma)$.*

*In particular, $(c_{m-1}, a_{m-1}, b_{m-1}) = (c_m, a_m, b_m)$ with $d(\Delta) = m$, except the case (i).*

For the corresponding result when $\Gamma$ is a distance-biregular graph, see the following section.

It follows easily from Theorem 1.1 that if $c_2 \neq 1$, then every strongly closed subgraph in a distance-regular graph is distance-regular. Using this fact, one can prove the following theorem without difficulty, and it is useful when one wants to characterize a distance-regular graph $\Gamma$ by the structure of its antipode $\Gamma_d(\alpha)$.

**Theorem 1.2** **([28])** *Let $\Gamma$ be a distance-regular graph of diameter $d = d(\Gamma)$. If $\Gamma_d(\alpha)$ is strongly closed for some $\alpha \in \Gamma$, then $\Gamma_d(\beta)$ is a clique for every vertex $\beta \in \Gamma$.*

The second question is the following:

*Can we find parametrical conditions for distance-regular graphs to have strongly closed subgraphs?*

In this paper, we shall discuss this problem for the cases (iii) and (iv). Note that if $2 \le m \le r(\Gamma)$, then we can find a strongly closed subgraph $\Delta$ in $\Gamma$ of diameter $m$ which is, roughly speaking, isomorphic to a graph obtained by replacing each edge of a tree by a clique.

Case (iii) is treated in Sections 3 and 4. In this case we have $a_i = 0$ for $i \le d(\Delta)$. Though we discuss in full generality, it seems more natural to state the results on bipartite graphs. The first result in this case is an improvement of a result of Ray-Chaudhuri and Sprague on pseudo-projective incidence systems.

Let $q = p^e$ be a prime power and $V$ be a $d$-dimensional vector space over $GF(q)$ when $q \neq 1$, and a $d$-element set when $q = 1$. Let $\begin{bmatrix} V \\ i \end{bmatrix}_q$ denote the collection of $i$-dimensional subspaces of $V$ when $q \neq 1$, and the collection of $i$-subsets when $q = 1$.

Let $J_q(d, s, s-1)$ denote a bipartite graph with a bipartition

$$\begin{bmatrix} V \\ s-1 \end{bmatrix}_q \cup \begin{bmatrix} V \\ s \end{bmatrix}_q,$$

where $x \in \begin{bmatrix} V \\ s-1 \end{bmatrix}_q$, $l \in \begin{bmatrix} V \\ s \end{bmatrix}_q$ is adjacent if and only if $x \subset l$. $J_q(d, s, s-1)$ is a distance-biregular graph and is called an $(s, q, d)$-projective incidence structure in [24].

Throughout this paper, we make a convention that $(q^m - 1)/(q-1) = m$, when $q = 1$.

**Theorem 1.3** *Let $\Gamma$ be a connected bipartite graph of diameter at least five with a bipartition $P \cup L$. Suppose $c_2(x) = 1, c_3(x) = c_4(x) = q+1$ for every $x \in P$, where $q$ is a fixed positive integer. Then $\Gamma$ is a biregular graph of valencies $k^P = k(x)$, and $k^L = k(l)$, where $x \in P$, $l \in L$. If $c_5(x)$ exists for every $x \in P$ and does not depend on the choice of $x \in P$, then one of the following holds.*

**(i)** $\Gamma \simeq J_q(d, s, s-1)$, *where* $k^L = (q^s - 1)/(q-1)$, $k^P = (q^{d-s+1} - 1)/(q-1)$, *or*

**(ii)** $d(\Gamma) \le 7$, $q \neq 1$, $k^P$, $k^L \le 3q - 1$.

*In particular, $q$ is a power of a prime if $k^P \ge 3q$ or $k^L \ge 3q$.*

In [20] Koolen conjectured that under the hypothesis slightly stronger than that of Theorem 1.3, (i) or $d(\Gamma) \le 4$ holds. Hence Theorem 1.3 gives an affirmative (but not complete) solution to the conjecture. For the detailed information on the case (ii), see Section 3.

Ray-Chaudhuri and Sprague obtained only the case (i) under an additional hypothesis $q^2 + q + 1 \le k^L$. So in this paper we shall treat the case when the valency is not so large compared with $c_3^P$, using a result of Terwilliger in [30]. In any case, as we can guess from the conclusion, one of the keys is to show that every pair of vertices of distance four determines a geodetically closed (hence, strongly closed but not regular) subgraph of diameter four assuming that the valency $k^L$ is not so small. See Section 3.

Let $\Gamma$ be a distance-biregular graph with a bipartition $P \cup L$. Assume $r$ is even and

$$c_r^P = 1 < c_{r+1}^P = c_{r+2}^P.$$

This is one of the typical cases corresponding to Theorem 1.1.(iii). By Theorem 1.3, if $r = 2$ and $d(\Gamma) \geq 8$, then $\Gamma$ contains a strongly closed subgraph, which is distance-biregular of diameter four. It seems unlikely to have $r > 4$ and $r = 4$ is rare. We do not have a proof, but we can prove that $r \leq 4$ if $\Gamma$ contains a strongly closed subgraph of diameter $r + 2$. See Section 3. In Section 4, we treat the case $c_{r+1}^P = 2$ with $r = 4$ and prove the following.

**Theorem 1.4** *Let $\Gamma$ be a connected bipartite graph with a bipartition $P \cup L$. Suppose $c_2(x) = c_3(x) = c_4(x) = 1$, $c_5(x) = c_6(x) = 2$ for every $x \in P$. Then $\Gamma$ is a biregular graph of valencies $k^P$ and $k^L$. If $\alpha$, $\beta$ be vertices in $\Gamma$ with $\partial(\alpha,\beta) = 5$, then there is a strongly closed subgraph $\Delta$ containing $\alpha$ and $\beta$ isomorphic to ${}^2M_{k^P}$. In particular, $k^P \in \{2,3,7,57\}$, if $d(\Gamma) \geq 5$.*

We can show under the hypothesis in Theorem 1.4 that $c_i^L$ exists for $i = 1,2,3,4,5,6$, $c_1^L = \cdots c_4^L = 1$ and $c_5^L = c_6^L = 2$. Hence Theorem 1.4 implies that $k^L \in \{2,3,7,57\}$ as well. When $k^P = 2$ or $k^L = 2$, $\Gamma$ itself is a subdivision graph of a Moore graph isomorphic to ${}^2M_k$ for some $k$. When $k^P = k^L = 3$, Foster graph is an example. We do not know any other examples. It may be possible to classify $\Gamma$ satisfying the condition of Theorem 1.4.

Case(iv) in Theorem 1.1 is treated in Section 5, under an additional condition $c_{r+3} = 1$.

**Theorem 1.5** *Let $\Gamma$ be a distance-regular graph of valency $k > 2$ satisfying the following.*

$$\begin{aligned}(c_r, a_r, b_r) &= (1, 0, k-1),\\ (c_{r+1}, a_{r+1}, b_{r+1}) &= (c_{r+2}, a_{r+2}, b_{r+2}) = (1, 1, k-2),\end{aligned}$$

*$r \geq 1$ and $c_{r+3} = 1$. Then $r \equiv 0 \pmod 3$, and the following holds.*

*(1) If $r = 3$, then for every $\alpha$, $\beta \in \Gamma$ with $\partial(\alpha,\beta) = 3$, there is a strongly closed subgraph $\Delta$ containing $\alpha$, $\beta$ isomorphic to ${}^3K_{k+1}$.*

*(2) If $r = 6$, then for every $\alpha$, $\beta \in \Gamma$ with $\partial(\alpha,\beta) = 6$, there is a strongly closed subgraph $\Delta$ containing $\alpha$, $\beta$ isomorphic to ${}^3M_k$. In particular $k \in \{3,7,57\}$.*

The first part $r \equiv 0 \pmod 3$ is due to Boshier-Nomura [4]. It is known that if $l(1,0,k-1) = r \geq 1$, then $l(1,1,k-2) \leq 3$ and if $l(1,1,k-2) = 3$, then $c_{r+4} > 1$ [4, 13].

It is worth mentioning that both results Theorem 1.4 and 1.5 are related to circuit chasing technique. See [26] for a result related to Theorem 1.4.

We use intersection diagrams as our tools. We refer those who are not familiar with them to [4, 13, 14, 16, 23, 25, 26] and [5, Section 5.10] for example.

For subsets $A$, $B$ of $\Gamma$ let $e(A, B)$ denote the number of edges between $A$ and $B$, and $e(x, A) = e(\{x\}, A)$.

$\Gamma^{(i)}$ will denote the distance-$i$-graph on $\Gamma$, i.e., the graph defined on the vertex set $V(\Gamma)$ of $\Gamma$ such that $\alpha$ and $\beta$ are adjacent if and only if $\partial_\Gamma(\alpha, \beta) = i$.

We write $\alpha \sim \beta$ when $\alpha \in \Gamma(\beta)$.

# 2 Strongly Closed Subgraphs

We shall prove Theorem 1.1 and related results in this section. The key of the proof is the determination of graphs such that $c_i$'s and $a_i$'s exist. Problems in similar settings are discussed in [12, 30, 20].

**Proposition 2.1** *Let $\Gamma$ be a connected bipartite graph with a bipartition $P \cup L$. Suppose $c_i^P$ exists for $i = 1, \ldots, m$ with $m \leq d(\Gamma)$. If $c_1^P = \cdots = c_r^P = 1 < c_{r+1}^P$, with $r + 1 \leq m$, then the following hold.*

(1) *If $c_i^P = c_{i-1}^L$ for some $i \leq m$, then $c_i^L$ exists and $c_{i-1}^P = c_i^L$. In particular, $c_1^L, \ldots, c_r^L$ exist and $c_1^L = \cdots = c_r^L = 1$.*

(2) *If $c_1^L, \ldots, c_{2i}^L$ exist and $2i + 1 \leq m$, then $c_{2i+1}^L$ exists and $c_{2j}^P c_{2j+1}^P = c_{2j}^L c_{2j+1}^L$ for all $j \leq i$.*

(3) *If $r$ is even, then $\Gamma$ is biregular of valencies $b_0^P$ and $b_0^L$. Moreover $c_{r+1}^L$ exists and $c_{r+1}^P = c_{r+1}^L$.*

(4) *If $r$ is odd, and $c_{r+1}^L$ exists, then $\Gamma$ is biregular of valencies $b_0^P$ and $b_0^L$. Moreover,*

$$(c_{r+1}^P - 1)(b_0^L - 1) = (c_{r+1}^L - 1)(b_0^P - 1).$$

(5) *Suppose $\Gamma$ is biregular of valencies $k^P = b_0^P$ and $k^L = b_0^L$. Then $|P|k^P = |L|k^L$. Moreover, if $c_1^L, \ldots, c_{2i}^L$ exist with $2i \leq m$, then $b_s^P$, $b_t^L$ exist for $s \leq m$, $t \leq 2i$ and $b_{2j-1}^P b_{2j}^P = b_{2j-1}^L b_{2j}^L$, for all $j \leq i$.*

We can obtain the following theorem as a direct corollary by applying Proposition 2.1 to $\Delta$.

**Theorem 2.2** *Let $\Gamma$ be a connected bipartite graph with a bipartition $P \cup L$. Suppose $c_i^P$, $c_i^L$ exist for $i = 1, \ldots, m$. Let $c_1^P = \cdots = c_r^P = 1 < c_{r+1}^P$ with $r + 1 \leq m$. If $\Delta$ is a geodetically closed subgraph of $\Gamma$ of diameter $m$, then $\Delta$ is a distance-biregular graph.*

**Remark.** For a diatance-biregular graph $\Gamma = P \cup L$, let $d^P = \max\{\partial(x,\alpha)|\alpha \in \Gamma\}$, where $x \in P$, and $d^L = \max\{\partial(l,\alpha)|\alpha \in \Gamma\}$, where $l \in L$. In Theorem 2.2, if $d^{P\cap\Delta} \geq d^{L\cap\Delta}$, then $k^{P\cap\Delta} = c^P_m$. But we cannot determine the other valency when $d^{P\cap\Delta} > d^{L\cap\Delta}$.

**Proposition 2.3** *Let $\Gamma$ be a connected graph. Suppose $c_i$ exists for $i = 1,\ldots,m$ with $m \leq d(\Gamma)$. Suppose $c_1 = \cdots = c_r = 1$, $a_1,\ldots,a_r$ exist and $a_1 = \cdots = a_r$ and either $c_{r+1} > 1$ or $c_{r+1} = 1$ and $a_{r+1}$ exists with $a_{r+1} \neq a_1$, where $2 \leq r+1 \leq m$. Then one of the following holds.*

**(i)** *$\Gamma$ is regular.*

**(ii)** *$\Gamma$ is a bipartite biregular graph such that $r \equiv 0 \pmod 2$ and $c_{2i-1} = c_{2i}$ for all $i$ with $2i \leq m$.*

**(iii)** *$\Gamma \simeq {}^3K_{k+1}$ or ${}^3M_k$, where $k$ is the largest valency of a vertex in $\Gamma$. In particular, $r = 3$ or $6$.*

**Lemma 2.4** *Let $\Gamma$ be a connected graph of diameter $d = d(\Gamma)$. Suppose $c_d$, $c_{d-1}$, $a_d$, $a_{d-1}$ exist. Then $\Gamma$ is regular of valency $c_d + a_d$ if and only if $(c_{d-1}, a_{d-1}) \neq (c_d, a_d)$.*

**Lemma 2.5** *Let $\Gamma$ be a distance-regular graph of diameter $d = d(\Gamma)$ and $m < d$. Suppose $\Gamma$ has a strongly closed subgraph of diameter $m$ containing $\alpha$ and $\beta$ for every pair of vertices $\alpha$, $\beta$ with $\partial(\alpha,\beta) = m$. Then for all $\gamma$, $\delta \in \Gamma$ with $\partial(\gamma,\delta) \leq m+1$, $C(\gamma,\delta)$ is a coclique.*

Now we prove Theorem 1.1 under weaker conditions.

**Theorem 2.6** *Let $\Gamma$ be a connected graph of diameter $d = d(\Gamma)$. Suppose $c_i$'s and $a_i$'s exist for all $i = 1,\ldots,m$, where $m \leq d$. Let*

$$r = r(\Gamma) = \max\{i|(c_1,a_1) = (c_2,a_2) = \cdots = (c_i,a_i)\}.$$

*If $\Gamma$ contains a strongly closed subgraph $\Delta$ of diameter $m$, then one of the following holds.*

**(i)** *$\Delta$ is a distance-regular graph,*

**(ii)** *$2 \leq m \leq r$,*

**(iii)** *$\Delta$ is a distance-biregular graph and that $r \equiv m \equiv 0 \pmod 2$ and $c_{2i-1} = c_{2i}$ for all $i$ with $2i \leq m$, or*

**(iv)** *$\Delta \simeq {}^3K_{l+1}$ or ${}^3M_l$ and $m = r+2 = 5$ or $8$, $a_1 = \cdots = a_r = 0$, $c_1 = \cdots = c_{r+2} = a_{r+1} = a_{r+2} = 1$.*

*Proof.* Since $\Delta$ is a strongly closed subgraph of $\Gamma$, we can apply Proposition 2.3 to the subgraph $\Delta$. If $r \geq m$, then (i) or (ii) holds.

Assume $r+1 \leq m$. Then $\Delta$ is one of the types in Proposition 2.3. If $\Delta$ is regular, then $\Delta$ is distance-regular as $c_i$'s and $a_i$'s exist for $i \leq d(\Delta) = m$. Suppose $\Delta$ is not regular. Since $\Delta$ is strongly closed, $k(\alpha) = k(\beta)$ if $\alpha$, $\beta \in \Delta$ and $\partial(\alpha, \beta) = m$. So if $\Delta$ is a bipartite biregular graph, $\Delta$ is distance-biregular and $m \equiv 0 \pmod 2$. Hence we have (iii). Suppose $\Delta \simeq {}^3K_{l+1}$ or ${}^3M_l$. Then $r = 3$ or $6$ and $m = r+2$, $c_1 = \cdots = c_m = 1$, $a_1 = \cdots = a_r = 0$, $a_{r+1} = a_{r+2} = 1$ easily follow from the structure of $\Delta$.

**Lemma 2.7** *Let $\Gamma$ be a distance-biregular graph with a bipartition $P \cup L$. Suppose $k^P$, $k^L \geq 2$. Let $d = d(\Gamma)$,*

$$d^P = \max\{\partial(x,\alpha) | x \in P,\ \alpha \in \Gamma\}, \quad d^L = \max\{\partial(l,\alpha) | l \in L,\ \alpha \in \Gamma\},$$

*and $r(\Gamma) = \max\{i | c_i^P = 1\}$. Then $r(\Gamma) = \max\{i | c_i^L = 1\}$ and the following are equivalent.*

**(i)** $r(\Gamma) + 2 = d^P + 1 = d^L = d$.

**(ii)** $d = d^L = r(\Gamma) + 2$, $c_{d-1}^L = c_d^L$ *with $d$ even.*

*In this case $\Gamma$ is a Moore geometry and $d = 4$ or $6$. If $d = 4$, $\Gamma$ is nothing but a nonsymmetric 2-$(|P|, k^L, 1)$ design. If $d = 6$, then the incidence graph on $P$ is a strongly regular graph with parameters $(v,k,\lambda,\mu) = (|P|, k^P(k^L-1), k^L-2, 1)$.*

For the diameter bound of Moore geometries, see [8, 7, 10, 11] and [5, Section 6.8]

**Remark.** In the case Theorem 2.6.(iii), the smallest possible value for $m$ is $r+2$ if the minimum valency is at least 2. By the previous lemma, we have $r = 2$ or $4$. We treat these cases in the following sections. But it may be possible to give a bound of $r = r(\Gamma)$ of distance-regular graphs satisfying $a_1 = 0$, $c_{r+1} = c_{r+2}$ with $r$ even, by showing the existence of geodetically closed subgraphs of diameter $r+2$, i.e., graphs discussed in the previous lemma.

# 3 A Refinement of a Theorem of Ray-Chaudhuri and Sprague

In [24], Ray-Chaudhuri and Sprague proved the following theorem in the context of incidence systems.

**Theorem 3.1** *Let $\Gamma$ be a connected bipartite graph with a bipartition $P \cup L$. For some positive integer $q$, suppose $c_2(x) = 1$, $c_3(x) = c_4(x) = q+1$ for every $x \in P$. Then $\Gamma$ is* *biregular of valencies $k^P$ and $k^L$. If $k^P > q+1$ and $k^L \geq q^2+q+1$, then $\Gamma \simeq J_q(d,s,s-1)$, where $s$ and $d$ are real numbers defined by*

$$k^L = (q^s - 1)/(q-1), \quad k^P = (q^{d-s+1} - 1)/(q-1).$$

*In particular, $q$ is a power of a prime number and both $s$ and $d$ are integers.*

The first part of this section is the following: By reviewing the proof of Ray-Chaudhuri and Sprague, we show that we can conclude either $d(\Gamma) \leq 4$ or $\Gamma \simeq J_q(d,s,s-1)$ if we can construct a geodetically closed subgraph of diameter 4 having vertices of valency $q+1$ and that such a subgraph exists if one of the valencies $k^P$ or $k^L$ is at least $3q$. Roughly speaking, we want to decrease the lower bound of the condition on the valencies in the hypothesis from $q^2+q+1$ to $3q$.

Before we start, we prepare a proposition.

**Proposition 3.2** *Let $\Gamma$ be a connected regular graph of valency $k$ and diameter $d$. Suppose the distance-2-graph $\Delta = \Gamma^{(2)}$ is distance-regular of diameter $\tilde{d}$. If each pair of vertices $\alpha$, $\beta$ at distance three in $\Gamma$ is contained in a shortest circuit of odd length $2m+1$, then $\tilde{d} = m$ and a connected component of $\Delta_{\tilde{d}}(\alpha)$ is a clique of size $k$. Moreover, $\Delta_{\tilde{d}}(\alpha)$ is connected if and only if $d = \tilde{d}$ and $\Gamma$ is a generalized Odd graph, i.e., a distance-regular graph such that $a_i = 0$, $i = 1, \ldots, d-1$ and $a_d \neq 0$.*

*Proof.* Firstly, we have $a_1 = \cdots = a_{m-1} = 0$, $m \geq 3$. And we have the following.

$$\Delta_1(\alpha) = \Gamma_2(\alpha),\ \Delta_{m-1}(\alpha) \supset \Gamma_3(\alpha),\ \Delta_m(\alpha) \supset \Gamma_1(\alpha).$$

Let $\beta \in \Gamma_1(\alpha)$. Then $\Delta_{m+1}(\alpha) \cap \Delta_1(\beta) = \emptyset$, $\tilde{d} = m$. Moreover,

$$\Gamma_1(\alpha) \setminus \{\beta\} \subset \Delta_1(\beta) \cap \Delta_{\tilde{d}}(\alpha) \subset \Gamma_1(\alpha) \setminus \{\beta\}.$$

Hence $\tilde{a}_{\tilde{d}} = k-1$ and a connected component of $\Delta_{\tilde{d}}(\alpha)$ containing $\beta$ is a clique of size $k$.

If $\Delta_{\tilde{d}}(\alpha)$ is connected, as $\Delta$ is distance-regular, $\Delta_{\tilde{d}}(\gamma) = \Gamma_1(\gamma)$ is a clique of size $k$ in $\Delta$ for every $\gamma \in \Gamma$. Hence $\Gamma$ is a generalized Odd graph. See [1], [2, Section III.4], and [5, Section 4.2].

In the following we also treat the case when $\Gamma$ is a $k$-regular with the same conditions on $c_i$'s as those in Theorem 3.1.

Let $q$ be a positive integer and $r$ a positive even integer. A connected graph $\Gamma$ is said to be a *$P(r,q)$-graph* if $c_i$, $a_j$ exist for $1 \leq i \leq r+2$, $1 \leq j \leq r+1$ and they satisfy

$$c_1 = \cdots = c_r = 1,\ a_1 = \cdots = a_{r+1} = 0,\ c_{r+1} = c_{r+2} = q+1.$$

**Lemma 3.3** *Let $q$ be a positive integer and $r$ an even positive integer. The following hold.*

**(1)** *Let $\Gamma$ be a connected bipartite graph of diameter at least $r+1$ with a bipartition $P \cup L$. If $c_i^P$ exists for $1 \le i \le r+2$, and $c_1^P = \cdots c_r^P = 1$, $c_{r+1}^P = c_{r+2}^P = q+1$ , then $\Gamma$ is a $P(r,q)$-graph.*

**(2)** *Let $\Gamma$ be a $P(r,q)$-graph. Then one of the following holds.*

- **(i)** $\Gamma$ *is a bipartite biregular (possibly regular) graph; or*
- **(ii)** $\Gamma$ *is a nonbipartite regular graph, i.e., a regular graph containing a circuit of odd length.*

*Proof.* (1) This follows from Proposition 2.1.(1), (2).
(2) This follows from Proposition 2.3.

Let $\Gamma$ be a $P(r,q)$-graph of diameter at least $r+1$. According to the previous lemma, there are two possibilities.

**(i)** $\Gamma$ is a bipartite graph with a bipartition $P \cup L$ and biregular of valencies $k^P$ and $k^L$.

**(ii)** $\Gamma$ is a nonbipartite graph and regular of valency $k$. In this case, let $\Gamma = P = L$.

We give a list of known $P(r,q)$-graphs, which is not a polygon. $r = 2$ for the first three examples and $r = 4$ for the rest.

1. $J_q(d, s, s-1)$.
2. $O_k$, the Odd graph of valency $k$, (nonbipartite).
3. $2M_7$, the doubled Hoffman-Singleton graph, ($d = 5$, $q = 5$).
4. ${}^2M_k$, $k = 3,\ 7$, ($d = 6$, $q = 1$).
5. Foster graph, that is the three fold cover of the incidence graph of $GQ(2,2)$, the generalized quadrangle of order $(2,2)$, ($d = 8$, $q = 1$).

In this section we study $P(2,q)$-graphs. Let $\Gamma$ be a $P(2,q)$-graph of diameter at least five.

For $\alpha,\ \beta \in \Gamma$ with $\partial(\alpha,\beta) = 2$ and $\gamma \in C(\alpha,\beta)$, let

$$T(\alpha,\beta) = \Gamma_2(\alpha) \cap \Gamma_2(\beta) \cap \Gamma_3(\gamma).$$

We say $\Gamma$ satisfies the condition $\#^L$ [resp. $\#^P$], if $\delta,\ \eta \in T(\alpha,\beta)$ implies $\partial(\delta,\eta) \le 2$ for all $\alpha,\ \beta \in L$ [resp. $P$] with $\partial(\alpha,\beta) = 2$.

The condition above is called 'Pasch's axiom' in [24].

**Lemma 3.4** **(1)** *If $k^L \ge 3q$ or $q = 1$, then $\Gamma$ satisfies the condition $\#^L$.*

**(2)** *If $k^P \geq 3q$ or $q = 1$, then $\Gamma$ satisfies the condition $\#^P$.*

*Proof.* By symmetry it suffices to prove (1).
Let $m_1$, $m_2 \in L$ with $\partial(m_1, m_2) = 2$ and $\{x\} = C(m_1, m_2)$. Let $T = T(m_1, m_2)$.
If $l \in T$, then $C(m_2, l) \subset \Gamma_3(m_1)$. Hence

$$|T| = |T(m_1, m_2)| = b_2^L(c_3^L - 1) = (k^L - 1)q.$$

Suppose the condition $\#^L$ fails. Then there exist $l$, $l' \in T$ with $\partial(l, l') = 4$. Let $\{x_i\} = C(l, m_i)$, $\{x_i'\} = C(l', m_i)$, $i = 1, 2$. Since $c_3 = c_4 = q+1$, for $i$, $j = 1, 2$,

$$x_1' \in C(l, l') = C(x_j, l'), \text{ or } \partial(x_i', x_j) = 2.$$

So we have that

$$x_1' \in C(x_2, m_1) \setminus \{x, x_1\},\ x_2' \in C(x_1, m_2) \setminus \{x, x_2\}.$$

Hence $|T \cap \Gamma_4(l)| \leq (q-1)^2$. Similarly, $|T \cap \Gamma_4(l')| \leq (q-1)^2$. In particular, $q \neq 1$. Thus

$$\begin{aligned}
(q+1)^2 &= |\Gamma_2(l) \cap \Gamma_2(l')| \\
&\geq |T \cap \Gamma_2(l) \cap \Gamma_2(l')| + |\{m_1, m_2\}| \\
&\geq |T| + |\{m_1, m_2\}| - |T \cap \Gamma_4(l)| - |T \cap \Gamma_4(l')| \\
&\geq (k^L - 1)q + 2 - 2(q-1)^2.
\end{aligned}$$

So $3q^2 - 2q + 1 \geq (k^L - 1)q$ or $k^L \leq 3q - 1 + \frac{1}{q}$. Since $q \neq 1$, $k^L \leq 3q - 1$, as desired.

For $m_1$, $m_2 \in \Gamma_2(l)$ with $m_1 \neq m_2$, we write $m_1 \approx m_2$ if $\partial(m_1, m_2) = 2$ and $C(m_1, m_2) \subset \Gamma_3(l)$, or equivalently if $m_2 \in T(l, m_1)$. Since the relation $\approx$ is symmetric, it defines a graph on $\Gamma_2(l)$.

Let $L_1(l, m)$ be a connected component in $\Gamma_2(l)$ containing $m$ with respect to $\approx$. Let

$$L(l, m) = \{l\} \cup L_1(l, m),\ P(l, m) = \bigcup_{n \in L(l,m)} \Gamma(n),\ \Delta(l, m) = P(l, m) \cup L(l, m).$$

**Lemma 3.5** *Suppose $\Gamma$ satisfies the condition $\#^L$. Then for $l$, $m \in L$ with $\partial(l, m) = 2$, $\Delta = \Delta(l, m)$ is a geodetically closed subgraph of $\Gamma$ of diameter 4.*

*Proof.* Since $\Gamma$ satisfies the condition $\#^L$, we have $\partial(m_1, m_2) \leq 2$, if $m_1$, $m_2 \in T(l, m)$. Hence we can prove the assertion without difficulty.

Let $D = \{\Delta(l, m) | \partial(l, m) = 2,\ l,\ m \in L\}$.

**Corollary 3.6** *If $\Gamma$ satisfies the condition $\#^L$, then the following hold.*

**(1)** *$L(l, m)$ is a maximal clique in $\Gamma^{(2)}$.*

(2) *If* $l,\ m \in \Delta_1 \cap \Delta_2 \cap L$, *then* $\Delta_1 = \Delta_2$ *or* $l = m$.

(3) $\Delta$ *is a bipartite biregular graph of valencies* $q+1$ *on* $P(l,m)$ *and* $k^L$ *on* $L(l,m)$.

(4) $|L(l,m)| = qk^L + 1$.

(5) $|\{\Delta \in D | l \in \Delta\}| = (k^P - 1)/q$ *for every* $l \in L$.

Let $\Pi$ be a bipartite graph on $L \cup D$ with adjacency defined as follows: For $l \in L,\ \Delta \in D,\ l \in \Delta$ and the valency of $l$ in $\Delta$ is $k^L$. Note that $k^L > q+1$ as $d(\Gamma) \geq 5$.

**Lemma 3.7** *If* $\Gamma$ *satisfies the condition* $\#^L$, *then* $\Pi$ *is a* $P(2,q)$*-graph of valencies* $(k^P - 1)/q$ *on* $L$ *and* $qk^L + 1$ *on* $D$.

**Proposition 3.8** *Let* $\Gamma$ *be a* $P(2,q)$*-graph of diameter at least five satisfying the condition* $\#^L$. *Then one of the following holds.*

(i) $\Gamma \simeq J_q(d,s,s-1)$, *where* $k^L = (q^s - 1)/(q-1)$, $k^P = (q^{d-s+1} - 1)/(q-1)$, *or*

(ii) $\Gamma$ *is a regular nonbipartite graph of valency* $k$ *and* $\Gamma^{(2)}$ *is isomorphic to a connected component of the distance-2-graph of* $J_q(2s-3, s-2, s-3)$, *where* $k = (q^{s-1} - 1)/(q-1)$. *Moreover, if each pair of vertices of* $\Gamma$ *at distance three is contained in a shortest circuit of odd length, then* $q = 1$ *and* $\Gamma$ *is isomorphic to an Odd graph.*

*Proof.* Firstly, note that $J_q(d,s,s-1) \simeq J_q(d, d-s+1, d-s)$, if we take the dual interchanging $P$ and $L$.

Suppose $\Gamma$ is bipartite. Since $d(\Gamma) \geq 5$, $k^P,\ k^L > q+1$. By Theorem 3.1, (i) holds if $k^P \geq q^2 + q + 1$, using the first remark above.

Assume $k^P < q^2 + q + 1$. Since $\Gamma$ satisfies the condition $\#^L$, $\Pi$ is a $P(2,q)$-graph of valencies $(k^P - 1)/q$ on L. Since $(k^P - 1)/q < q+1$, $\partial_\Pi(l,m) \leq 2$ for all $l,\ m \in L$. Hence $\partial_\Gamma(l,m) \leq 2$ for all $l,\ m \in L$, which is not the case.

Suppose $\Gamma$ is not bipartite. By the previous lemma, $\Pi$ is a bipartite $P(2,q)$-graph of valencies $(k-1)/q$ on $L$ and $qk+1$ on $D$.

Suppose $(k-1)/q \leq q+1$. Since $d(\Gamma) \geq 5$, there are vertices $l_0,\ l_1,\ l_2,\ l_3$ such that

$$\partial(l_0,l_1) = \partial(l_1,l_2) = \partial(l_2,l_3) = 2, \quad \partial(l_0,l_2) = 4.$$

Since $|\Pi_3(l_0) \cap \Pi(l_2)| = q+1$, $(k-1)/q = q+1$ and $\Delta(l_2,l_3) \in \Pi_3(l_0) \cap \Pi(l_2)$. So there is a vertex $l \in \Delta(l_2,l_3)$ such that $\partial(l,l_3) = \partial(l_0,l) = 2$. Hence $\partial(l_3,l_0) \leq 4$. In particular $d(\Gamma) = 5$, $a_5$ exists and $a_5 = 0$. Since $\Gamma$ is not bipartite, we may assume that $\partial(l_0,l_3) = 3$. Then $|\Gamma_2(l_3) \cap \Gamma_2(l_0)| = 0$. This is a contradiction.

Thus $(k-1)/q > q+1$, $qk+1 > q^2+q+1$. Hence by Theorem 3.1, $\Pi \simeq J_q(d,s,s-1)$, where $qk+1 = (q^s - 1)/(q-1)$, $(k-1)/q = (q^{d-s+1} - 1)/(q-1)$.

Therefore $k = (q^{s-1} - 1)/(q-1)$ and $d = 2s - 3$. Since $\partial_\Gamma(l,m) = 2$ if and only if $\partial_\Pi(l,m) = 2$, $\Gamma^{(2)}$ is isomorphic to a connected component of the distance-2-graph of $\Pi$ on $L$.

If $\Gamma$ satisfies the additional condition in (ii), we can apply Proposition 2.2. If $q \neq 1$, then $\Gamma^{(2)}$ is a Grassman graph, which is also called a $q$-analogue of Johnson graph. But in this case it is easy to check that the antipode is connected, while it is not a clique. Hence $q = 1$ and $\Gamma^{(2)} \simeq J(2s-3, s-2)$. Thus $\Gamma$ is an Odd graph.

In the following, we investigate the case when $\Gamma$ does not satisfy $\#^L$. By symmetry proved in Lemma 3.3, we may assume that $\Gamma$ does not satisfy $\#^P$ either. Hence by Lemma 3.4, we need only to consider the case $k^P$, $k^L \leq 3q - 1$.

The key to analize this case is the following proposition proved by Terwilliger. We kept the notations in [30], where $M_i$ is no longer a Moore graph.

**Proposition 3.9** ([30]) *Let integers $c$, $p$ and $s$ all be at least 2. Suppose the vertices of some graph $\Gamma$ can be partitioned into $s+1$ disjoint sets $V\Gamma = \bigcup_{i=0}^{s} M_i$, where for any $u$, $v \in V\Gamma$, $u \in M_i$, $v \in M_j$ and $(u,v) \in E\Gamma$ implies $|i-j| \leq 1$. For $i = 1$ or $s$, let $l_i$ and $L_i$ denote the minimum and maximum number of vertices in $M_{i-1}$ any vertex in $M_i$ is adjacent to, and for $i = 0$ or $s-1$, let $r_i$ and $R_i$ denote the minimum and maximum number of vertices in $M_{i+1}$ any vertex in $M_i$ is adjacent to. Also assume*

**(i)** *$\partial(u,v) = s$ for some $u \in M_0$ and $v \in M_s$,*

**(ii)** *for integers $0 \leq i$, $j \leq s$ and for any $u \in M_i$ and $v \in M_j$, there are either $c$ or 0 paths of length $s$ connecting them if $|j-i| = s$, and either 0 or 1 paths of length $|j-i|$ connecting them if $1 \leq |j-i| \leq s-1$, and*

**(iii)** *for any $u$, $v \in V\Gamma$ with $u \in M_1$, $v \in M_{s-1}$, and $\partial(u,v) > s-2$, there are at most $p$ paths $\{u = v_0, v_1, \ldots, v_{s-1}, v_s = v\}$, where either $v_1 \in M_0$ or $v_{s-1} \in M_s$.*

*Then*
$$\frac{p}{c-1} \geq \frac{r_{s-1}}{R_0 - 1} + \frac{l_1}{L_s - 1}.$$

**Proposition 3.10** *Let $\Gamma$ be a $P(2,q)$-graph of diameter at least five. If $c_5^P$ exists, then $c_5$ exists, i.e., $c_5^L$ exists and $c_5^P = c_5^L$, $c_5 > q+1$ and the following hold.*

**(1)** *If $d(\Gamma) \geq 7$, then $c_5 \geq 2q+1$.*

**(2)** *If $\alpha$, $\beta$, $\gamma \in \Gamma$ with $\partial(\alpha,\beta) = 8$, $\partial(\alpha,\gamma) = 3$, $\partial(\gamma,\beta) = 5$, then $k(\gamma) \geq 3q+2$.*

**(3)** *For $\alpha \in \Gamma$ let $j = k(\alpha) - c_5$. If $a_4 = 0$, then*
$$k(\alpha) \geq \frac{2q + j + 3 + \sqrt{4jq^2 + (j-1)^2}}{2}.$$
*In particular, if $j \geq 4$, then $k(\alpha) \geq 3q + 4$.*

*Proof.* It follows from Proposition 2.1.(2) that $c_5$ exists.

(1) Let $\alpha$, $\beta \in \Gamma$ with $\partial(\alpha, \beta) = 7$. Let

$$M_i = \Gamma_{2+i}(\alpha) \cap \Gamma_{5-i}(\beta), \ i = 0, 1, 2, 3.$$

Apply Proposition 3.9.

(2) Since $d \geq 8$, we can apply (1). We have

$$k(\gamma) \geq c_3(\alpha, \gamma) + c_5(\beta, \gamma) \geq 3q + 2.$$

(3) Let $\alpha \in \Gamma$ and $M_i = \Gamma_{i+2}(\alpha)$, $i = 0, 1, 2, 3$. Apply Proposition 3.8.

We now summarize our results in this section, from which we have Theorem 1.3 as a corollary.

**Theorem 3.11** *Let $\Gamma$ be a $P(2,q)$-graph of diameter at least five. Suppose $c_5$ exists. Then $\Gamma$ is a bipartite biregular graph of valencies $k^P$ and $k^L$, or a regular graph of valency $k = k^P = k^L$ and one of the following holds.*

**(i)** $\Gamma \simeq J_q(d, s, s-1)$, *where* $k^L = (q^s - 1)/(q - 1)$, $k^P = (q^{d-s+1} - 1)/(q - 1)$,

**(ii)** *$\Gamma$ is a regular nonbipartite graph of valency $k$ and the distance-2-graph $\Gamma^{(2)}$ is isomorphic to a connected component of the distance-2-graph of $J_q(2s-3, s-2, s-3)$, where $k = (q^{s-1} - 1)/(q - 1)$. Moreover, if each pair of vertices of $\Gamma$ at distance three is contained in a shortest circuit of odd length, then $q = 1$ and $\Gamma$ is isomorphic to an Odd graph; or*

**(iii)** *$d(\Gamma) \leq 7$ and $k^P$, $k^L \leq 3q - 1$, $q \neq 1$. Moreover if $a_4 = 0$, then $\Gamma$ is bipartite and $k^P - c_5$, $k^L - c_5 \leq 3$. In particular, if $\Gamma$ is not bipartite and $a_4$ exists, then $d(\Gamma) \leq 6$.*

**Corollary 3.12** *Let $\Gamma$ be a distance-regular graph of valency $k$. Suppose $c_2 = 1$, $c_3 = c_4 = q + 1$ and $a_1 = a_2 = a_3 = 0$ for some positive integer $q$. Then one of the following holds.*

**(i)** $\Gamma \simeq J_q(2s-1, s-2, s-3)$, *where* $k = (q^s - 1)/(q - 1)$.

**(ii)** $\Gamma \simeq O_k$, *an Odd graph of valency $k$; or*

**(iii)** *$d(\Gamma) \leq 7$, and the equality holds only if $\Gamma$ is bipartite.*

Koolen [20] conjectured the following:

> If $\Gamma$ is a distance-biregular graph of diameter at least 5 such that $c_i$ exists for all $i$, and $c_2 = 1$, $c_3 = c_4 > 2$, then $\Gamma \simeq J_q(d, s, s-1)$.

Our results asserts that $d(\Gamma) \leq 7$ and the parameters are restricted very much. It is known that if $d(\Gamma) = 5$ or 7, then $\Gamma$ is distance-regular, under the assumption of the conjecture above. See [9, 20].

We also note that for $d(\Gamma) = 5$, the doubled Moore graph satisfy the hypothesis with $c_5 = q+2$. Moreover if it's valency is not 3, say 7, then it does not come from $J_q(d, s, s-1)$. So this gives a counter example to the conjecture above.

# 4 $P(r, 1)$-graphs

According to the remark following Lemma 3.3, a $P(r, 1)$-graph is a connected graph $\Gamma$, which is either a bipartite biregular graph with a bipartition $P \cup L$ or a nonbipartite regular graph such that

$$c_1 = \cdots = c_r = 1,\ a_1 = \cdots = a_{r+1} = 0,\ c_{r+1} = c_{r+2} = 2,$$

where $r$ is an even positive integer. In this section we study $P(r, 1)$-graphs and we show the following when $r = 4$. We do not know any $P(r, 1)$-graphs with $r > 4$.

**Theorem 4.1** *Let $\Gamma$ be a $P(4, 1)$-graph of diameter at least four and $\alpha$, $\gamma \in \Gamma$ with $\partial(\alpha, \gamma) = 4$. Then there is a geodetically closed subgraph $\Delta$ containing $\alpha$, $\gamma$ isomorphic to ${}^2M_{k(\alpha)}$. Here $k(\alpha)$ denotes the valency of $\alpha$ in $\Gamma$. In particular, $k(\alpha) \in \{2, 3, 7, 57\}$.*

Let $\Gamma$ be a $P(r, 1)$-graph with $r \geq 4$.

Fix a vertex $\alpha \in \Gamma$. For $\gamma$, $\delta \in \Gamma_r(\alpha)$, we write $\gamma \approx \delta$ if $\partial(\gamma, \delta) = 2$ and $C(\gamma, \delta) \subset \Gamma_{r+1}(\alpha)$. For $\gamma \in \Gamma_r(\alpha)$, let $C = C_\gamma$ be the connected component in $\Gamma_r(\alpha)$ containing $\gamma$ with respect to the relation $\approx$. Let $\Pi = \Pi_\gamma$ be a graph on $C_\gamma$ defined by the relation $\approx$. For $\gamma$, $\delta \in \Gamma$ with $\partial(\gamma, \delta) = r$, and $0 \leq i \leq r$, let

$$\{g_i(\gamma, \delta)\} = \Gamma_{r-i}(\gamma) \cap \Gamma_i(\delta).$$

For $\delta \in \Gamma_r(\alpha)$, let

$$\alpha(\delta) = g_1(\delta, \alpha),\ \beta(\delta) = g_2(\delta, \alpha),\ \text{ and }\ \gamma(\delta) = g_4(\delta, \alpha).$$

Firstly we note that the intersection diagram with respect to $x$, $l$ with $\partial(x, l) = 1$ has the following shape, where $D^i_j = \Gamma_i(x) \cap \Gamma_j(l)$. See the properties (a) $\sim$ (e) below.

$$\begin{tikzcd}
\{x\} = D^0_1 \arrow[r, dash] \arrow[dd, dash] & \cdots\cdots \arrow[r, dash] & D^{r-1}_r \arrow[r, dash] & D^r_{r+1} \arrow[r, dash] \arrow[dd, dash] & D^{r+1}_{r+2} \arrow[r, dash] \arrow[dr, dash] & {} \arrow[d, dash] \\
 & & & & & D^{r+2}_{r+2} \arrow[r, dash] \arrow[dl] \arrow[d, dash] & {} \\
\{l\} = D^1_0 \arrow[r, dash] & \cdots\cdots \arrow[r, dash] & D^r_{r-1} \arrow[r, dash] & D^{r+1}_r \arrow[r, dash] & D^{r+2}_{r+1} \arrow[r, dash] & {}
\end{tikzcd}$$

Figure 2.

(a) $D^i_i = \emptyset$, for $1 \le i \le r+1$.

(b) For $y \in D^{i+1}_i$, $z \in D^i_{i+1}$, $e(y, D^i_{i-1}) = e(z, D^{i-1}_i) = 1$, $1 \le i \le r$.

(c) For $y \in D^{r+2}_{r+1}$, $z \in D^{r+1}_{r+2}$, $e(y, D^{r+1}_r) = e(z, D^r_{r+1}) = 2$.

(d) For $y \in D^{r+1}_r$, $z \in D^r_{r+1}$, $e(y, D^r_{r+1}) = e(z, D^{r+1}_r) = 1$.

(e) $e(D^{i+1}_i, D^i_{i+1}) = 0$, $1 \le i \le r-1$ and $i = r+1$.

The following two lemmas are related to circuit chasing technique. See [4, 13, 14] and [5, Section 5.10].

**Lemma 4.2** *Let $x_0 \sim x_1 \sim \cdots \sim x_{2r+2t} = x_0$ be a circuit of length $2r+2t$. i.e., a closed path and $x_{i-1} \ne x_{i+1}$, $i = 1, \ldots, 2r+2t-1$ and $x_{2r+2t-1} \ne x_1$. Suppose*

$$x_r,\ x_{r+2}, \ldots,\ x_{r+2t} \in \Gamma_r(x_0),\ x_{r+1},\ x_{r+3},\ \ldots,\ x_{r+2t-1} \in \Gamma_{r+1}(x_0).$$

*Set $D^i_j = \Gamma_i(x_0) \cap \Gamma_j(x_1)$. Then the following hold.*

**(1)** *$t \ge 1$ and $x_r \in D^r_{r-1}$, $x_{r+1} \in D^{r+1}_r$, $x_{r+2} \in D^r_{r+1}$.*

**(2)** *If $t \ge 2$, then $x_{r+3} \in D^{r+1}_{r+2}$ and $x_{r+4} \in D^r_{r+1}$.*

**(3)** *If $t = 2$, then the mutual distance of the vertices in the circuit is uniquely determined. In particular,*

$$\partial(x_2, x_{r+2}) = \partial(x_2, x_{r+4}) = r,\ \partial(x_2, x_{r+5}) = r+1.$$

**(4)** *If $t = 3$, then $x_{r+5} \in D^{r+1}_{r+2}$, $x_{r+6} \in D^r_{r+1}$ and*

$$\partial(x_2, x_{r+4}) = \partial(x_2, x_{r+6}) = \partial(x_4, x_{r+6}) = r,\ \partial(x_4, x_{r+5}) = \partial(x_4, x_{r+7}) = r+1.$$

*Proof.* In the following, we use (a) $\sim$ (e) to determine the locations of $x_j$'s in the diagram with respect to an edge $x_{i-1} \sim x_i$, using the information on the distances from $x_{i-1}$.

(1) Since $x_{i-1} \ne x_{i+1}$, for all $i$, and $c_1 = \cdots = c_r = 1$, $t \ge 1$. It is clear that $x_r \in D^r_{r-1}$. Since $x_{r+1} \in \Gamma_{r+1}(x_0) \cap \Gamma(x_r)$, $x_{r+1} \in D^{r+1}_r$. $x_r \ne x_{r+2} \in \Gamma_r(x_0) \cap \Gamma(x_{r+1})$ implies that $x_{r+2} \in D^r_{r+1}$.

(2) Since $x_{r+2} \in D^r_{r+1}$ and $e(x_{r+2}, D^{r+1}_r) = 1$ with $x_{r+1} \in D^{r+1}_r \cap \Gamma(x_{r+2})$, $x_{r+3} \in D^{r+1}_{r+2}$, $x_{r+4} \in D^r_{r+1}$.

(3) It is easy to determine the mutual distances as follows.

| | $x_r$ | $x_{r+1}$ | $x_{r+2}$ | $x_{r+3}$ | $x_{r+4}$ | $x_{r+5}$ |
|---|---|---|---|---|---|---|
| $x_0$ | $r$ | $r+1$ | $r$ | $r+1$ | $r$ | $r-1$ |
| $x_1$ | $r-1$ | $r$ | $r+1$ | $r+2$ | $r+1$ | $r$ |
| $x_2$ | $r-2$ | $r-1$ | $r$ | $r+1$ | $r$ | $r+1$ |

Now the distance pattern with respect to $x_2$ is the same as that with respect to $x_0$, the mutual distance of the vertices in the circuit is uniquely determined and the assertion follows.

(4) We do the same as in (3).

| | $x_r$ | $x_{r+1}$ | $x_{r+2}$ | $x_{r+3}$ | $x_{r+4}$ | $x_{r+5}$ | $x_{r+6}$ | $x_{r+7}$ | $x_{r+8}$ | $x_{r+9}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $x_0$ | $r$ | $r+1$ | $r$ | $r+1$ | $r$ | $r+1$ | $r$ | $r-1$ | $r-2$ | $r-3$ |
| $x_1$ | $r-1$ | $r$ | $r+1$ | $r+2$ | $r+1$ | $r+2$ | $r+1$ | $r$ | $r-1$ | $r-2$ |
| $x_2$ | $r-2$ | $r-1$ | $r$ | $r+1$ | $r$ | $r+1$ | $r$ | $r+1$ | $r$ | $r-1$ |
| $x_3$ | $r-3$ | $r-2$ | $r-1$ | $r$ | $r+1$ | $r+2$ | $r+1$ | $r+2$ | $r+1$ | $r$ |
| $x_4$ | $r-4$ | $r-3$ | $r-2$ | $r-1$ | $r$ | $r+1$ | $r$ | $r+1$ | $r$ | $r+1$ |

Note that since $x_{r+7} \in D_r^{r-1}$, $x_{r+5}$ cannot be in $D_r^{r+1}$.

**Lemma 4.3** *Let $y_0 \sim y_1 \sim y_2 \sim y_3 \sim y_4$ be a path of length four such that $y_{i-1} \neq y_{i+1}$, $i = 1, \ldots, 3$. Suppose $y_0$, $y_4 \in \Gamma_r(\alpha)$. Then one of the following holds.*

**(i)** $y_2 \in \Gamma_{r-2}(\alpha)$,

**(ii)** $y_1 \in \Gamma_{r-1}(\alpha)$ *or* $y_3 \in \Gamma_{r-1}(\alpha)$ *and* $\alpha(y_0) \neq \alpha(y_4)$,

**(iii)** $y_1$, $y_3 \in \Gamma_{r+1}(\alpha)$, $y_2 \in \Gamma_r(\alpha)$ *and* $\alpha(y_0) \neq \alpha(y_4)$,

**(iv)** $y_2 \in \Gamma_{r+2}(\alpha)$ *and* $\alpha(y_0) = \alpha(y_4)$, *while* $\beta(y_0) \neq \beta(y_4)$, *or*

**(v)** $y_2 \in \Gamma_{r+2}(\alpha)$ *and* $\alpha(y_0) \neq \alpha(y_4)$, $\partial(\beta(y_0), y_4) = r+2$.

By Lemma 4.2 and 4.3, we can prove the following concerning the connected component in $\Gamma_r(\alpha)$ with respect to $\approx$.

**Lemma 4.4** *Let $\{\alpha_1, \ldots, \alpha_{k(\alpha)}\} = \Gamma(\alpha)$, $\gamma \in \Gamma_r(\alpha)$, $C = C_\gamma$. Let $S_i = \{\delta \in C | \alpha(\delta) = \alpha_i\}$. Then the following hold.*

**(1)** *For $\delta \in S_i$, $|\Pi(\delta) \cap S_j| = 1 - \delta_{i,j}$ and $S_i \subset \Gamma_{r-2}(\beta(\delta))$. In particular, $\Pi$ is a $k(\alpha)$-partite $(k(\alpha)-1)$-regular graph.*

**(2)** *Let $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ be a path in $\Pi$. If $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists $\delta_4 \in \Pi(\delta_3)$, $\delta_5 \in \Pi(\delta_4)$ such that $\gamma(\delta_0) = \gamma(\delta_5)$.*

If $r = 4$, $\gamma(\delta) = \delta$ for every $\delta \in \Pi$. So by Lemma 4.4, we have the following.

**Lemma 4.5** *If $r = 4$, then the following holds.*

**(1)** *If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ and $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists $\delta_4$ such that $\delta_0 \approx \delta_4 \approx \delta_3$.*

(2) *If* $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ *and* $\alpha(\delta_0) = \alpha(\delta_3)$, *then* $\beta(\delta_0) = \beta(\delta_3)$.

(3) *If* $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3 \approx \delta_4$ *with* $\alpha(\delta_0) = \alpha(\delta_3)$, $\alpha(\delta_1) = \alpha(\delta_4)$, *then there exists* $\delta_5$ *such that* $\delta_0 \approx \delta_5 \approx \delta_4$.

(4) $d(\Pi) \leq 3$ *and if* $\partial_\Pi(\delta, \delta') = 3$, *then* $\beta(\delta) = \beta(\delta')$.

*Proof.* (1) Since $\gamma(\delta) = \delta$ for every $\delta \in \Pi$, (1) is a direct consequence of Lemma 4.4.(2).

(2) This follows from Lemma 4.4.(1).

(3) By (2), $\beta(\delta_0) = \beta(\delta_3) \neq \beta(\delta_1) = \beta(\delta_4)$. Now $\delta_3$, $\beta(\delta_1) \in \Gamma_4(\delta_0)$, and there is a path of length 4,

$$y_0 = \delta_3 \sim y_1 \sim y_2 = \delta_4 \sim y_3 \sim y_4 = \beta(\delta_1),$$

where $y_1 \in C(\delta_3, \delta_4)$, $y_3 = g_1(\alpha, \delta_4)$.

It is easy to check that $y_1$, $y_3 \in \Gamma_5(\delta_0)$ and that $g_1(\delta_3, \delta_0) \neq g_1(\beta(\delta_1), \delta_0)$. Hence by Lemma 4.3.(iii) or (v) occurs.

If (v) occurs, $\partial(\beta(\delta_0), \delta_4) = 6$, which is not the case. Hence $\partial(\delta_0, \delta_4) = 4$.

Let $\delta_0 = z_0 \sim z_1 \sim z_2 \sim z_3 \sim z_4 = \delta_4$ be a path connecting $\delta_0$ and $\delta_4$. Then by Lemma 4.3, we have (iii) as $\partial(\beta(\delta_0), \delta_4) = 4$. Hence we can set $z_2 = \delta_5$.

(4) This follows from (1), (2) and (3).

*Proof of Theorem 4.1.* Let $r = 4$ and

$$\begin{aligned} L(\alpha,\gamma) &= \{\alpha\} \cup \bigcup_{\delta \in C_\gamma} (\Gamma_2(\alpha) \cap \Gamma_2(\delta)) \cup C_\gamma, \\ P(\alpha,\gamma) &= \bigcup_{\delta \in L(\alpha,\gamma)} \Gamma_1(\delta), \\ \Delta &= \Delta(\alpha,\gamma) = P(\alpha,\gamma) \cup L(\alpha,\gamma) \end{aligned}$$

In this definition we also write $P(\Delta) = P(\alpha, \gamma)$, and $L(\Delta) = L(\alpha, \gamma)$.

We shall show in the sequel that $\Delta$ is a geodetically closed subgraph isomorphic to ${}^2M_{k(\alpha)}$.

Let $\gamma = \gamma_1$ and $\{\gamma_2, \ldots, \gamma_{k(\alpha)}\} = \Pi(\gamma)$. Thanks to Lemma 4.4,

$$L(\Delta) = \{\alpha\} \cup \{\beta(\gamma_1), \ldots, \beta(\gamma_{k(\alpha)})\} \cup C_\gamma.$$

By Lemma 4.5, the distance-2-graph induced on $L(\Delta)$ is of diameter 2 and geodetically closed.

If $k(\alpha) = 2$, there is nothing to prove. Assume $k(\alpha) > 2$.

$\partial(\beta(\gamma), \gamma_2) = 4$ and

$$\Pi(\gamma_2) \setminus \{\gamma_1\} = \{\delta_1, \ldots, \delta_{k(\alpha)-1}\} \subset \Gamma_4(\beta(\gamma)),$$

there is a vertex $\delta_i' \in \Pi(\delta_i) \cap \Gamma_2(\beta(\gamma))$ for each $i$. Since the girth of $\Gamma$ is 10, we can conclude that the valenncy of $\beta(\gamma)$ in the distance-2-graph induced on $L(\Delta)$ equals $k(\alpha)$. By Lemma 4.5, this means that the valency of vertex in $P(\Delta)$ is 2.

Now we can conclude that $\Delta$ is geodetically closed subgraph of $\Gamma$ isomotphic to ${}^2M_{k(\alpha)}$ easily.

This completes the proof of Theorem 4.1.

We remark that in the final step, we can also apply [5, Theorem 1.17.1] to determine the regularity of the distance-2-graph induced on $L(\Delta)$. See the proof of [5, Proposition 4.3.11].

## 5 Proof of Theorem 1.5

In this section, we give a proof of Theorem 1.5. We can follow the proof in the previous section step by step, replacing each path of length 2 by a path of length 3.

Let $\Gamma$ be a graph satisfying the hypothesis in Theorem 1.5.

Fix a vertex $\alpha \in \Gamma$. For $\gamma$, $\delta \in \Gamma_r(\alpha)$, we write $\gamma \approx \delta$ if $\partial(\gamma, \delta) = 3$. Then $C(\gamma, \delta) \cup C(\delta, \gamma) \subset \Gamma_{r+1}(\alpha)$. For $\gamma \in \Gamma_r(\alpha)$, let $C = C_\gamma$ be the connected component in $\Gamma_r(\alpha)$ containing $\gamma$ with respect to the relation $\approx$. Let $\Pi = \Pi_\gamma$ be a graph on $C_\gamma$ defined by the relation $\approx$. Hence $C$ is a connected component of the distance-3-graph of $\Gamma$ induced on the set $\Gamma_r(\alpha)$.

For $\gamma$, $\delta \in \Gamma$ with $\partial(\gamma, \delta) = r$, and $0 \leq i \leq r$, let

$$\{g_i(\gamma, \delta)\} = \Gamma_{r-i}(\gamma) \cap \Gamma_i(\delta).$$

For $\delta \in \Gamma_r(\alpha)$, let

$$\alpha(\delta) = g_1(\delta, \alpha),\ \alpha'(\delta) = g_2(\delta, \alpha),\ \beta(\delta) = g_3(\delta, \alpha),\ \text{ and }\ \gamma(\delta) = g_6(\delta, \alpha).$$

Firstly we note that the intersection diagram with respect to $x$, $y$ with $\partial(x, y) = 1$ has the following shape, where $D_j^i = \Gamma_i(x) \cap \Gamma_j(y)$. See the properties (a) $\sim$ (g) below.

$$\begin{array}{ccccccccccc}
\{x\} = D_1^0 - & \cdots\cdots & - D_r^{r-1} & \text{---} & D_{r+1}^r & & \text{------} & D_{r+2}^{r+1} & \text{------} & D_{r+3}^{r+2} & - \\
\Big| & & & & & \diagdown & & & & & \diagdown \\
\Big| & & & & & & D_{r+1}^{r+1} & \text{------} & D_{r+2}^{r+2} & \text{------} & \\
\Big| & & & & & \diagup & & & & & \diagup \\
\{y\} = D_0^1 - & \cdots\cdots & - D_{r-1}^r & \text{---} & D_r^{r+1} & & \text{------} & D_{r+1}^{r+2} & \text{------} & D_{r+2}^{r+3} & -
\end{array}$$

Figure 3.

**(a)** $D_i^i = \emptyset$, for $1 \leq i \leq r$.

**(b)** For $y \in D_i^{i+1}$, $z \in D_{i+1}^i$, $e(y, D_{i-1}^i) = e(z, D_i^{i-1}) = 1$, $1 \leq i \leq r + 2$.

(c) For $y \in D_i^{i+1}$, $z \in D_{i+1}^i$, $e(y, D_i^{i+1}) = e(z, D_{i+1}^i) = 0$, $1 \le i \le r$ and $e(y, D_i^{i+1}) = e(z, D_{i+1}^i) = 1$, $i = r+1$, $r+2$.

(d) For $y \in D_{r+1}^{r+1}$, $e(y, D_r^{r+1}) = e(y, D_{r+1}^r) = 1$ and $e(y, D_{r+1}^{r+1}) = 0$.

(e) For $y \in D_r^{r+1}$, $z \in D_{r+1}^r$, $e(y, D_{r+1}^{r+1}) = e(z, D_{r+1}^{r+1}) = 1$.

(f) For $y \in D_{r+2}^{r+2}$, $e(y, D_{r+1}^{r+1}) = e(y, D_{r+2}^{r+2}) = 1$.

(g) $e(D_i^{i+1}, D_{i+1}^i) = 0$, $1 \le i \le r+2$.

We again apply circuit chasing technique.

**Lemma 5.1** *Let $x_0 \sim x_1 \sim \cdots \sim x_{2r+3t} = x_0$ be a circuit of length $2r+3t$. i.e., a closed path and $x_{i-1} \ne x_{i+1}$, $i = 1, \ldots, 2r+3t-1$ and $x_{2r+3t-1} \ne x_1$. Suppose*

$$x_r,\ x_{r+3}, \ldots,\ x_{r+3t} \in \Gamma_r(x_0),\ x_{r+1},\ x_{r+2},\ x_{r+4},\ x_{r+5},\ \ldots,\ x_{x+3t-2},\ x_{r+3t-1} \in \Gamma_{r+1}(x_0).$$

*Set $D_j^i = \Gamma_i(x_0) \cap \Gamma_j(x_1)$. Then the following hold.*

(1) *$t \ge 1$ and $x_r \in D_{r-1}^r$, $x_{r+1} \in D_r^{r+1}$, $x_{r+2} \in D_{r+1}^{r+1}$ and $x_{r+3} \in D_{r+1}^r$.*

(2) *If $t \ge 2$, then $x_{r+4}$, $x_{r+5} \in D_{r+2}^{r+1}$ and $x_{r+6} \in D_{r+1}^r$.*

(3) *If $t = 2$, then the mutual distance of the vertices in the circuit is uniquely determined. In particular, $r \equiv 0 \pmod 3$, and*

$$\partial(x_3, x_{r+3}) = \partial(x_3, x_{r+6}) = r,\ \partial(x_3, x_{r+7}) = r+1.$$

(4) *Suppose $r \ge 6$. If $t = 3$, then $x_{r+7}$, $x_{r+8} \in D_{r+2}^{r+1}$, $x_{r+9} \in D_{r+1}^r$ and*

$$\partial(x_3, x_{r+6}) = \partial(x_3, x_{r+9}) = \partial(x_6, x_{r+9}) = r,\ \partial(x_6, x_{r+8}) = \partial(x_6, x_{r+10}) = r+1.$$

**Lemma 5.2** *Let $y_0 \sim y_1 \sim y_2 \sim y_3 \sim y_4 \sim y_5 \sim y_6$ be a path of length 6 such that $y_{i-1} \ne y_{i+1}$, $i = 1, \ldots, 5$. Suppose $y_0$, $y_6 \in \Gamma_r(\alpha)$. Then one of the following holds.*

(i) *$y_3 \in \Gamma_{r-3}(\alpha)$,*

(ii) *$y_1$, $y_2$, $y_4$, $y_5 \in \Gamma_{r+1}(\alpha)$, $y_3 \in \Gamma_r(\alpha)$ and $\alpha(y_0) \ne \alpha(y_6)$,*

(iii) *$y_3 \in \Gamma_{r+2}(\alpha)$ and $y_5 \in \Gamma_{r+1}(\alpha) \cap \Gamma_{r+1}(\alpha(y_0))$, while $\partial(\beta(y_0), y_5) \ge r+1$.*

**Lemma 5.3** *Let $\{\alpha_1, \ldots, \alpha_k\} = \Gamma(\alpha)$, $\gamma \in \Gamma_r(\alpha)$, $C = C_\gamma$. Let $S_i = \{\delta \in C | \alpha(\delta) = \alpha_i\}$. Then the following hold.*

(1) *For $\delta \in S_i$, $|\Pi(\delta) \cap S_j| = 1 - \delta_{i,j}$ and $S_i \subset \Gamma_{r-3}(\beta(\delta))$. In particular, $\Pi$ is a $k$-partite $(k-1)$-regular graph.*

**(2)** *Let $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ be a path in $\Pi$. If $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists $\delta_4 \in \Pi(\delta_3)$, $\delta_5 \in \Pi(\delta_4)$ such that $\gamma(\delta_0) = \gamma(\delta_5)$.*

**Lemma 5.4** *If $r = 6$, then the following holds.*

**(1)** *If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ and $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists $\delta_4$ such that $\delta_0 \approx \delta_4 \approx \delta_3$.*

**(2)** *If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ and $\alpha(\delta_0) = \alpha(\delta_3)$, then $\beta(\delta_0) = \beta(\delta_3)$.*

**(3)** *If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3 \approx \delta_4$ with $\alpha(\delta_0) = \alpha(\delta_3)$, $\alpha(\delta_1) = \alpha(\delta_4)$, then there exists $\delta_5$ such that $\delta_0 \approx \delta_5 \approx \delta_4$.*

**(4)** *$d(\Pi) \leq 3$ and if $\partial_\Pi(\delta, \delta') = 3$, then $\beta(\delta) = \beta(\delta')$.*

*Proof of Theorem 1.5.* Suppose $r = 3$. Let

$$\begin{aligned} L(\alpha,\gamma) &= \{\alpha\} \cup C_\gamma, \\ P(\alpha,\gamma) &= \bigcup_{\delta \in L(\alpha,\gamma)} \Gamma_1(\delta), \\ \Delta &= \Delta(\alpha,\gamma) = P(\alpha,\gamma) \cup L(\alpha,\gamma) \end{aligned}$$

In this definition we also write $P(\Delta) = P(\alpha,\gamma)$, and $L(\Delta) = L(\alpha,\gamma)$. Clearly $L(\Delta)$ is a maximal clique in the distance-3-graph of $\Gamma$, and the assertion follows easily from Lemma 5.3.

Let $r = 6$ and

$$\begin{aligned} L(\alpha,\gamma) &= \{\alpha\} \cup \bigcup_{\delta \in C_\gamma} (\Gamma_3(\alpha) \cap \Gamma_3(\delta)) \cup C_\gamma, \\ P(\alpha,\gamma) &= \bigcup_{\delta \in L(\alpha,\gamma)} \Gamma_1(\delta), \\ \Delta &= \Delta(\alpha,\gamma) = P(\alpha,\gamma) \cup L(\alpha,\gamma) \end{aligned}$$

In this definition we also write $P(\Delta) = P(\alpha,\gamma)$, and $L(\Delta) = L(\alpha,\gamma)$.

We shall show in the sequel that $\Delta$ is a geodetically closed subgraph isomorphic to ${}^3M_{k(\alpha)}$.

Let $\gamma = \gamma_1$ and $\{\gamma_2, \ldots, \gamma_k\} = \Pi(\gamma)$. Thanks to Lemma 4.4,

$$L(\Delta) = \{\alpha\} \cup \{\beta(\gamma_1), \ldots, \beta(\gamma_k)\} \cup C_\gamma.$$

By Lemma 5.4, the distance-3-graph induced on $L(\Delta)$ is of diameter 2 and geodetically closed.

$\partial(\beta(\gamma), \gamma_2) = 6$ and

$$\Pi(\gamma_2) \setminus \{\gamma_1\} = \{\delta_1, \ldots, \delta_{k-1}\} \subset \Gamma_6(\beta(\gamma)),$$

there is a vertex $\delta_i' \in \Pi(\delta_i) \cap \Gamma_3(\beta(\gamma))$ for each $i$. Since the girth of $\Gamma$ is 15, we can conclude that the valenncy of $\beta(\gamma)$ in the distance-3-graph induced on $L(\Delta)$ equals $k$. By Lemma 5.4, this means that the valency of vertex in $P(\Delta)$ is 2.

Now we can conclude that $\Delta$ is geodetically closed easily.

This completes the proof of Theorem 1.5.

# 6 Concluding Remarks

It may be too optimistic to expect a classification of $P(r,q)$-graphs or the graphs similar to those discussed in the previous section in the near future. But we believe that the investigation of such graphs plays a key role to give an absolute bound of the girth of distance-biregular graphs or distance-regular graphs.

We list several problems, which we want to see solved.

1. Study geodetically closed subgraphs of distance-regular graphs and prove results corresponding to Proposition 2.3 and Theorem 2.6, especially when $a_1 \neq 0$. See [20].

2. Classify $P(r,q)$-graphs.

   a) For $r = 2$, it may be possible to improve Lemma 3.4 to have $2q$ as the lower bound. Then we have $d \leq 5$, by Proposition 3.10.

   b) For $q = 1$, the classification implies a classification of distance-biregular graphs with vertices of valency three, [26]. Hence we can obtain an absolute diameter bound of distance-regular graphs of order $(s,2)$, i.e., those with $\Gamma(x) \simeq 3 \cdot K_s$. See [17, 3, 15, 31].

3. Let $\Gamma$ be a bipartite biregular graph with a bipartition $P \cup L$, or a regular graph with $\Gamma = P = L$. For a positive integer $q$ and a positive odd integer $r$, we call $\Gamma$ a $P(r,q)$-graph, if it is a connected graph such that

$$c_1^P = \cdots = c_r^P = 1,\ a_1 = \cdots = a_{r+1} = 0,\ c_{r+1}^P = q+1 \text{ and } c_{r+1}^L = c_{r+2}^P.$$

   Classify them. If $q = 1$, then $\Gamma$ is a thin generalized polygon by a result in [26].

4. Study a distance-regular graphs $\Gamma$ with $r = r(\Gamma)$, $c_{r+1} = c_{r+2} = 1$, and clarify the correspondence with $P(r,q)$-graphs. In particular, show $r \leq 6$ in Theorem 1.5.

5. Let $\Gamma$ be a connected graph of diameter $d$. For a subset $I \subset \{1,\ldots,d\}$, let $\Gamma^{(I)}$ denote the distance-$I$-graph, i.e., $V(\Gamma^{(I)}) = V(\Gamma)$, and $\alpha$, $\beta$ are adjacent in $\Gamma^{(I)}$ if and only if $\partial(\alpha,\beta) \in I$. Study $\Gamma$ such that at least one of the connected components of $\Gamma^{(I)}$ is distance-regular of diameter at least three. To start with, assume $\Gamma^{(I)}$

is connected. It is not hard to determine parametrical conditions if $\Gamma$ itself is a diatance-regular graph. In particular, classify distance-regular graphs $\Gamma$ such that $\Gamma^{(2)}$ is distance-regular of diameter $d(\Gamma) \neq d(\Gamma^{(2)}) \geq 3$. See Proposition 3.2 and [27, 29].

6. Give a geometrical classification of Moore graphs. One of the reasons, we could not obtain the results for $P(r,1)$-graphs with $r \geq 6$, is a lack of such classification. We believe that this is one of the keys when we develope structure theories of distance-regular graphs just as the group theoretical proof of Burnside's $p^a q^b$ theorem gave a breakthrough to the classification of finite simple groups.

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