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# Computability in Analysis

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## 1. Introduction

In various theories of analysis, transcendental principles such as the strong comprehension axioms and the axioms of choice are prevalent. There have been, however,speculations on elementary methods of analysis, due to the foundational concerns traditionally and due to the modern interests in computability. We list some references which are along this line. [1] introduces constructive analysis, which was founded by Bishop in the 60's, from scratch to advanced topics.[4] and [6] are trials to develop analysis within the framework of "definable" logic. [3] is a comprehensive record of reverse mathematics, whose objective is to see how some mathematical theorems and logical principles imply one another.

Here we give a brief account of the theory of computability structures in analysis, which was designed by M. Pour-El and has been thoroughly presented in [2]. There the authors define computable reals and functions, computability structures in Banach spaces, and investigate their relationship to classical notions and objects in analysis. The characteristic feature of this approach is to regard computable structures as substructures of classical mathematics. This makes the theory more interesting than doing mathematics within a restricted framework. In [5], the idea is generalized to the computability sturucture in Frechet spaces.

We wish to extend such an idea to the theories of measure and integration. As a start, we are working on the computability structure in metric spaces, and hence our exposition is centered in the case of metric spaces.

## 2. Recursive functions

A number-theoretic function is recursive if there is an ideal program to compute it. (There is a strict definition of a function being recursive, but we will not elaborate.) It can be generalized to functions of several variables as well as to relations among natural numbers. Since the rational numbers are regarded as pairs of natural numbers, we can identify them with natural numbers. The popular functions such as addition, subtraction, and multiplication are recursive. Many basic relations among natural numbers (hence among rationals) such as equality and inequality are recursive.

Recursive functions can be described using finitely many symbols, and hence there are only countably many such.

The following is a fact crucial to counter-example making.

Fact. There is a recursive function whose range is not recursive. (That is, it is not decidable whether a natural number belongs to the range or not.)

3. Computable sequences of reals

In the following, we write exp(x,y) to express "x to the y."

A sequence of rationals, say r, is said to be computable if there are recursive functions a,b,c such that r(n)=exp(-1,a(n))b(n)/c(n) for each n. A sequence of rationals r effectively converges to a real number x if it converges to x and the modulus of convergence can be evaluated by a recursive function. That is, there is a recursive a such that

for each n, if k > a(n), then |r(n)-x| < 1/exp(2,n).

A real number is computable if there is a computable sequence of rationals which converges effectively to it. A sequence of reals is computable if there is a computable double sequence of rationals which converges effectively (in two coordinates) to it.

Any popular reals, such as e and pi, are computable.

Some basic facts regarding the computability of reals are stated below.

Fact. Let x be a computable real. If x>0 (as a fact), then there is an effective way which shows it. (There is a program of deciding "y > 0 ?" and the computation halts for x.) This is not the case with "x = 0 ?."

Fact. If a double sequence of reals converges to a sequence of reals effectively, then the limit sequence is computable.

Fact. If a monotone sequence converges to a computable real, then the convergence is effective.

Fact. Computable reals form a subfield with rational coefficients in the field of real numbers.

Hereafter any method of deciding or computing will be said to be effective if it involves only recursive functions.

4. Counterexample technique

Most of the counterexamples (to computability) is based on the following construction. Let a be a one-to-one recursive function whose range is not recursive, and put

s(k) = the sum of exp(2,-a(m)) for m = 0,1,2, · · ,k.

Then s is a computable sequence of rationals converging to a non-computable real.

5. Computable functions

In order to make the discussion simple, we restrict our consideration to the funtions defined on a compact interval I = [u,v], where u and v are computable.

A function f on I is said to be computable if it satisfies the following.

(1) f preserves computability. That is, for every computable sequence of reals in I, the image sequence by f is also copmutable.

(2) f is effectively and uniformly continuous. That is, there is a recursive e such that, for very natural number p

if |x-y|<1/exp(2,e(p)), then |f(x)-f(y)|<1/exp(2,p).

Fact. This definition is equivalent to the effective Weierstrass property. That is, a function on I is computable iff it is the effective limit of rational-coefficient polynomials.

The familiar function such as addition, sin, etc. are computable. A smooth operator such as the integral preserves the computability of a function, while the derivative does not. There is a computable twice differentiable function on [0,1] whose derivative is not computable, while the derivative of a computable C-2 function is computable.

We can define a computable sequence of (computable) functions similarly.

The classical theorems one sees in the calculus books, such as the maximum and the minimum value theorems,the intermediate value theorem and the mean value theorem, hold effectively.

Proving the effective versions of the classical theorems,we notice that with an exception or two, the effective proofs are mere effectivizations of the classical proofs. That is, a classical statement of the form

For every p, there is an n satisfying A(p,n)

be replaced by

There is a recursive a such that, for every p, A(p,a(p)) holds.

This fact suggests the logical structure in some part of classical analysis. It is, however,a theme yet to be worked on.

6. Computability structure in Banach spaces

Consider any Banach space B=(X,|| ||). Let S be a family of sequences from X. S is a computability structure for B if S is closed under effective linear forms with respect to rational coefficients and the effective limits, and if the norms of any sequence belonging to S form a computable sequence of reals.

A sequence of S is called an effective generating set if its linear spans are dense in X.

Any familiar Banach space has an intrinsic computability structure (unique up to the isomorphism). The main theorem for the computability structure stands as follows.

The Main Theorem. Let X and Y be Banach spaces and let T be a closed, linear operator from X to Y. Assume X and Y have computability structures, X has an effective generating set s which is in the domain of T, and Ts is a computable sequence of Y. Then

T preserves the computability iff T is bounded.

This theorem has many applications in analysis and physics.

Examples. 1. C[u,v] with the uniform norm. Put S=
{f/f is a sequence of functions on [u,v] uniformly computable in the sense of 5}
S is a computability structure. The sequence of monomials (1,x,xx,xxx,...) is an effective generating set. The solution of a wave propagation does not necessarily preserve computability, while in the same function space with the energy norm the computability is preserved.

2. A sequence f of L-p[u,v] is L-p-computable if there is a double sequence g, which is computable in the sense of 5 above and such that ||g(n,k)-f(n)|| converges to zero as k tends to the infinity, effectively in n and k. The monomials form an effective generating set.

3. There is a Hilbert space H with a computability structure S such that (1) the space H is separable, (2) the computable elements of H are dense in H and (3) <H,S> does not have an effective generating set.

7. Computability structure in metric spaces

Let M = (X,d) be a metric space. We propose a family of squences from X,say S, be a computability structure for M if the following hold.

(1) Let x and y be sequences in S. Then {d(x(m),y(n))} is a computable, double sequence of reals.

(2) Let z be a double sequence in S, and let x be a sequence from X. If d(z(m,n),x(m)) tends to 0 as n tends to the infinity effectively in m and n,

then x belongs to S.

(3) Let a be a one-to-one recursive function. If x belongs to S, then $\{x(a(n))\}$ also belongs to S.

If a sequence w in S is d-dense in X, then w is said to be an effective separating set of X.

A double sequence x in S is said to be effectively Cauchy if there is a recursive function a such that

$$\text{for every } m \text{ and } p, \text{ for every } n,k>a(m,p), d(x(m,n),x(m,k))<1/\exp(2,p).$$

Fact. If w is an effective separating set of X, then it is "effectively dense" in X, and w determines a unique computability strucrue containing it.

8. Completion of computability structure

Define a partial function from SxS to reals, $d^\wedge$, as follows. Let x and y be double sequences in S. For any fixed m,

$$d^\wedge(\{x(m,q)\},\{y(m,q)\}) = \lim d(x(m,q),y(m,q)) \quad \text{as } q \text{ tends to the infinity,}$$

where $\{x(m,q)\}$ represents a sequence in q with m fixed. Define $eq(d^\wedge; ,)$ as

$$eq(d^\wedge;\{x(m,q)\},\{y(m,q)\}) \text{ iff } d^\wedge(\{x(m,q)\},\{y(m,q)\}) = 0.$$

$$S^\wedge = \{x \text{ in } S \mid x \text{ is effectively d-Cauchy}\}$$

$$\text{For } x \text{ in } S,\ [x] = \{y \text{ in } S \mid eq(d^\wedge;x,y)\}$$

$$[x] = [y] \text{ if } eq(d^\wedge;x,y)$$

Proposition. (1) $S^\wedge$ is a subset of $X^\wedge$, the completion of X.

(2) $\langle S^\wedge,d^\wedge,eq(d^\wedge; , )\rangle$ is a metric space.

(3) $S^\wedge$ is "complete" with respect to $d^\wedge$ (in the classical sense).

Proposition. S is complete in the following sense. Let x be a triple sequence (of l,n,q) in S, such that it is also a sequence of $d^\wedge$-Cauchy sequences from $S^\wedge$. That is, $[x(l,n,q)]$ as a sequence of q belongs to $S^\wedge$ and $\{[x(l,n,q)]\}$ as a sequence of q and n is a $d^\wedge$-Cauchy sequence. Then there is a sequence from $S^\wedge$, say $\{[y(l,q)]\}$, such that $\lim d^\wedge([x(l,n,q)],[y(l,q)]) = 0$ as n tends to the infinity uniformly in l.

Examples. 1. $X = Q$, $d(x,y) = |x-y|$

$S = \{f |\ f$ is a recursive function (from natural numbers to Q)$\}$

X can be identified with the subset of S, $\{(x,x,x,...) | x$ in $Q\}$.

f belongs to $S^\wedge$ iff f is an effectively Cauchy sequence of rationals.

An effective enumeration of Q, say w, is an effective separability set.

$S^\wedge$ = the set of computable reals

2. $X = R$ or $= [0,1]$, $d(x,y) = |x-y|$

S = the set of computable sequences of reals

w as above is an effective separability set for X.

$S^\wedge$ is isomorphic to S.

3. Let $B=(X,||\ ||)$ be a Banach space, and let S be its computability structure.

Define a metric for X by $d(x,y) = ||x-y||$.

S is a computability structure for (X,d).

Let e be an effective generating set of X as a Banach space. Then, an effective enumeration of all the linear forms of e with respect to rational coefficients is an effective separability set for (X,d).

There are many open problems with regards to the computability properties in metric spaces. For example, which open sets and closed sets are computable? Do the effective versions of Urysohn's lemma, Baire catogory theorem etc. hold? Do metrizations of a Frechet space have computability structures?

We have just started our project.

## References

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