数理解析研究所講究録
第 856 巻 1994 年 107-111

Microlocal analysis of massless singularities

Takahiro KAWAI
Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606-01 JAPAN

The celebrated Coleman-Norton picture [CN] of the Landau-Nakanishi equations ([L], [N]) became more charming with the advent of microlocal analysis in that each (external) vertex is found to represent the cotangential component of the singularity spectrum of the relevant function. (Cf. [IS], [P], [I], [S], [K^3] Chap.III, §3 and references cited there.) The Coleman-Norton picture, however, becomes somewhat obscure when massless particles are relevant. It is not without reason, because the singularity spectrum of $1/(k^2+i0)$, i.e., the massless propagator [with the numerators ignored for simplicity], cannot be found as the limit ($m \downarrow 0$) of the singularity spectrum of $1/(k^2-m^2+i0)$ ($m>0$). Here and in what follows, k^2 denotes $k_0^2 - \sum_{j=0}^{3} k_j^2$ for a four-vector k. In fact, we find

$$\Box\left(\frac{1}{k^2+i0}\right) = -4\pi^2 i\delta^4(k),$$

where $\Box$ denotes the differential operator $\partial^2/\partial k_0^2 - \sum_{j=1}^{3} \partial^2/\partial k_j^2$ and $\delta^4(k)$ denotes $\prod_{j=0}^{3} \delta(k_j)$. (See [K^3], Chap.III, §5. See also [KS 1], §1.) Hence we can readily verify

$$\mathrm{SS}(1/(k^2+i0)) = \Lambda_0 \cup \Lambda_1,$$

where

$$\Lambda_0 = \{(k; iw\infty); k=0, w \in \mathbf{R}^4 - \{0\}\}$$

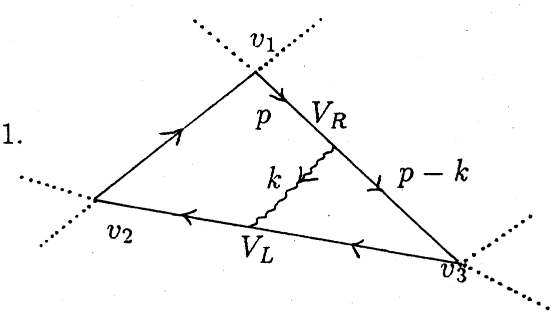
and

$$\Lambda_1 = \{(k; iw\infty); k^2=0, k \neq 0, w = ck\ (c>0)\}.$$

Thus the crucially important ingredient for the Coleman-Norton picture when some relevant particles are massless is the component Λ_0. Let us recall a mathematical interpretation of the Coleman-Norton picture is that it is a figure in the cotangent space of the energy-momentum space, i.e., the coordinate space. [This is consistent with the physical interpretation of micro-analyticity, i.e., macrocausality. See [IS], [I] and references cited there.] Sacrificing the dual interpretaion of a four vector in the original Coleman-Norton picture, i.e., abandonment of the understanding [based on the Minkowsky metric] to the effect that a four vector reprents both the energy-momentum vector [to describe the energy-momentum conservation at each vertex] and the cotangent vector [to describe the cotangential component of the singularity spectrum of the propagator], we obtain an analogue of the Coleman-Norton picture that describes the cotangential component of the singularity spectrum (originating from the singularity at $k = 0$ of the massless propagator $1/(k^2 + i0)$) of the relevant function, if we associate an arbitrary real non-zero four vector with each massless line in a Feynman graph, keeping the energy-momentum conservation relation at each endpoint of a massless line by setting the energy-momentum vector attached to the massless line to be 0.

A typical space-time diagram thus obtained is:

Figure 1.

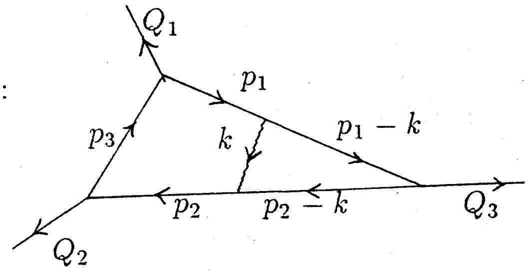


,

where a solid line corresponds to a massive propagator $1/(q^2 - m^2 + i0)$ and it reprensents a posive multiple αp of a four vector p in this diagram. The wiggly line corresponds to a massless propagator, and its slope is arbitrary. The energy-momentum conservation holds, by definition, at each of its endpoints V_L and V_R; this diagram is intended to describe the cotangential component of the singularity spectrum of the relevant Feynman function that originates from the point $k = 0$. Note that three relations $p^2 = m^2$, $(p-k)^2 = m^2$ and $k^2 = 0$ force $k = 0$ if both p and k are real and m is strictly positive.

Basic facts in microlocal analysis (e.g. [K³], Chap.III, §1 and §2) entails the fact that the external vertices of the above diagram represent the cotangential component of the singularity spectrum of the Feynman function associated with the graph G below:

G :



Here Q_j $(j = 1, 2, 3)$ denotes the sum of energy-momentum vectors incident upon the vertex in question.

Leaving the detailed discussion to the reader, we show what we call the Landau table (cf. [KS3]), where w shall be specified later.

Table 1.

	dQ_1	dQ_2	dQ_3	dp_1	dp_2	dp_3	dk
v_1	1			1		-1	
v_2		1			-1	1	
v_3			1	-1	1		
α_1				p_1			
α_2				$p_1 - k$			$k - p_1$
α_3					p_2		
α_4					$p_2 - k$		$k - p_2$
α_5						p_3	
α_6							w

According to the rules for the manipulation of hyperfunctions, the singularity of the Feynman function associated with the graph G exists (if we neglect the ultraviolet divergence, which is not our main concern here), only when the column vectors labeled as dp_j $(j = 1, 2, 3)$ or dk in Table 1 sum up to 0 after suitably multiplied by v_j $(j = 1, 2, 3)$ or non-negative constants α_j $(j = 1, \cdots, 6)$ with some $\alpha_l \neq 0$, under the obvious constraints due to the energy momentum conservation at each vertex and the refined mass-shell conditions such as $\alpha_1(p_1^2 - m^2) = 0$ etc. [If we do not use microlocal analysis, it it is better to consider the Feynman function after deleting the over-all energy-momentum conservation δ-function $\delta^4(Q_1+Q_2+Q_3)$ and considering the function on the submanifold $\{Q \in \mathbf{R}^{12}; Q_1+Q_2+Q_3=0\}$; we have then to regard (v_1, v_2, v_3) as a representative of $\mathbf{R}^{12}/T$, where $T = \{(a,a,a); a \in \mathbf{R}^4\}$.] Furthermore the cotangential component of its singularity spectrum is given by (v_1, v_2, v_3) unless $v_1 = v_2 = v_3$. If $k \neq 0$, then $w = k$ in Table 1; however one can readily verify that the above condition of "summing up to 0" cannot be met then. Hence $k = 0$ in Table 1, and correspondingly w is an arbitrary (non-zero) real four-vector. Hence Figure 1 is a visualization of Table 1. Note that, although the vector k is set to be 0 at the end, it plays an important role in forming Table 1 through $\mathrm{grad}_k((p_j-k)^2-m^2)$ $(j=1,2)$. We have thus obtained a diagramatic interpretation of the singularity (spectrum) of a Feynman function. An important fact is, although the non-negativity of α_j's gives some restrictions on the location of V_L and V_R, still they can freely move on the segments v_2v_3 and v_1v_3, respectively. Since the simple holonomicity of an integral cannot be expected in such circumstances [even the holonomicity is far from being trivial when some vertices are flexible], the functional character of the Feynman function in question is an interesting question. The answer is given in [KS 2]; the simplicity is lost, but it is an amenable holonomic function. The loss of simplicity is closely tied up with the infrared catastrophe in that the resulting singularity is incompatible with the known fall-off properties of charged particle propagators at large distances. A remedy for such a catastrophe is given in [S], and some mathematical claims needed in [S] are confirmed in [KS 3].

References

[CN] Coleman, S. and R.E. Norton: Singularities in the physical region. *Nuovo Cimento* **38** (1965), 438-442.

[I] Iagolnitzer, D.: The S-matrix. North-Holland, 1978.

[IS] Iagolnitzer, D. and H.P. Stapp: Macroscopic causality and physical region analyticity in S-matrix theory. *Commun. math. Phys.,* **14** (1969), 15-55.

[K^3] Kashiwara, M., T. Kawai and T. Kimura: Foundations of Algebraic Analysis, Princeton Univ. Press, 1986. (Translation from the Japanese edition published by Kinokuniya)

[KS 1] Kawai, T. and H.P. Stapp: Microlocal analysis of infrared singularities. Algebraic Analysis, I, Academic Press, 1988, pp.309–330.

[KS 2] ———: Infrared catastrophe in a massless Feynman function. To appear.

[KS 3] ———: Quantum electrodynamics at large distances. LBL Technical Report LBL-25819 (1993).

[L] Landau, L.D.: On analytic properties of vertex parts in quantum field theory. *Nuclear Phys.*, **13** (1959, October), 183–192. (Submitted on April 27, 1959)

[N] Nakanishi, N.: Ordinary and anomalous thresholds in perturbation theory. *Prog. Theor. Phys.*, **22** (1959, July), 128–144. (Submitted on March 24, 1959)

[P] Pham, F: Microanalyticité de la matrice S. Lecture Notes in Math., **449**, Springer, 1975, pp. 85–101.

[S] Stapp, H.P.: Discontinuity formula for multiparticle amplitudes. Structural Analysis of Collision Amplitudes, ed. by R. Balian and D. Iagolnitzer North Holland, (1976), 191–294.