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Borel fields of Lie Groups and Algebras

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This paper is dedicated, with thanks and honour, to my teacher, colleague and friend, Professor Masamichi Takesaki, on the occasion of his Kanreki.

Introduction This paper provides an overview of recent joint work of the author and V. Ya. Golodets, Institute of Low Temperature Physics and Engineering, Kharkov; full details will appear elsewhere [GS 2].

The notion of a Borel field of Polish groups was introduced in [S], and arose naturally in the context of analysing actions of discrete amenable groups on von Neumann algebras. The concept, and the associated Cohomology Lemma, has proven vital for such investigations [J, O, JT, ST1, ST2, KST]. Subsequently, Golodets and the author have specialised and sharpened the technique to the context of locally compact Polish groups, establishing the existence of a suitably varying field of Haar measures, and the Borel nature of forming Pontryagin duals. The present work is a further refinement, establishing that the subclass of Lie groups themselves form a Borel family, and that the natural interplay between Lie groups and Lie Algebras (suitably parametrised) is Borel. One surprising fact emerges; while the classification of Lie algebras is smooth, this is not the case for Lie Groups. Thus the classification of Lie groups is intrinsically immensely more complicated than that of Lie algebras, and the passage from Lie algebras to Lie groups has a somewhat surprising complexity.

§1 Borel families of Lie groups We recall some basic facts from [S] concerning the parametrisation of Polish groups.

We let PG denote the space of pairs (μ, d) where $\mu : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ is a group law, and d is a metric on $\mathbf{N}$ invariant under left translations for this group structure. PG is a Borel subset of $\mathbf{N}^{\mathbf{N}\times\mathbf{N}} \times [0,1]^{\mathbf{N}\times\mathbf{N}}$, and hence a standard Borel space. For each $(\mu, d) \in PG$, we let $G(\mu, d)$ denote the completion of the group $(\mathbf{N}, \mu, d)$; evidently $G(\mu, d)$ is a Polish group, and every Polish group appears infinitely often up to isomorphism among $\{G(\mu, d) : (\mu, d) \in PG\}$.

By construction, the space $\mathcal{PG} = \cup\{G(\mu, d) : (\mu, d) \in PG\}$ carries a standard Borel structure for which

a) the projection π of $\mathcal{PG}$ on PG is Borel.

b) the relative and intrinsic Borel structures on $\pi^{-1}(\mu, d)$ coincide for each $(\mu, d) \in PG$.

c) the group operations $x \in \mathcal{PG} \to x^{-1}$ and $(x, y) \in \mathcal{PG} * \mathcal{PG} \to xy \in \mathcal{PG}$ are Borel .

d) there are metrics $\delta_{(\mu,d)}$ on $G(\mu, d)$, and a countable family $\{g_k\}$ of Borel maps from PG to$\mathcal{PG}$ such that

i) $\{g_k(\mu, d) : k \geq 1\}$ is dense in $G(\mu, d)$ for each $(\mu, d) \in PG$.

ii) $\delta_{(\mu,d)}$ is compatible with the topology on $G(\mu, d)$ for each $(\mu, d) \in PG$.

iii) the maps $x \in \mathcal{PG} \to \delta_{\pi(x)}(x, g_k(\pi(x)))$ are Borel for each $k \geq 1$.

If X is a standard Borel space, and $x \in X \to G_x$ is a field of Polish groups, we say $x \to G_x$ is Borel if there is a Borel injection $\varphi : X \to PG$ and isomorphisms of Polish groups $\theta_x : G_x \to G(\varphi(x))$ for each $x \in X$. By transferring the structure on $\mathcal{PG}$ to $\dot{\cup}_{x \in X} G_x$, via $\{\theta_x : x \in X\}$, every Borel field of Polish groups inherits a standard Borel structure, a countable family of Borel sections and metrics enjoying a) – d) above. The field $(\mu, d) \in \mathcal{PG} \to G(\mu, d)$ will be referred to as the universal field of Polish groups.

Theorem 1 $LG = \{(\mu, d) \in PG : G(\mu, d)$ is a connected Lie group$\}$ is Borel in $\mathcal{PG}$.

The proof is routine, based on the characterisation of Lie groups as those connected Polish groups without small subgroups.

By restriction, we now obtain a universal field of Lie groups $(\mu, d) \in LG \to G(\mu, d)$; an arbitrary field of Lie groups over a standard Borel space will be said to be Borel if it factors through this universal field as in the case of Polish groups. Note that although each $G(\mu, d)$ for $(\mu, d) \in LG$ carries (for example) a differentiable structure, one parameter subgroups, canonical coordinates etc, we are not yet able to assert that these vary in any controlled manner. This will be remedied subsequently (c.f. Corollary 8).

§2 **Borel families of Lie algebras** We now seek to emulate the parametrisation of Lie groups given above for Lie algebras. The first problem is to build a suitable parameter space; this is accomplished by considering all possible sets of structure constants for Lie algebras. (Here we will treat Lie algebras over $\mathbf{R}$; the case of complex Lie algebras is dealt with analogously).

For each $n \geq 1$, set

$$SC_n = \{\gamma = (\gamma_{ij}^k) \in \mathbf{R}^{n^3} : \gamma_{ij}^k \text{are structure constants}$$
$$\text{for a Lie algebra of dimension } n\}.$$

Thus $\gamma_{ij}^k \in SC_n$ if and only if

$$\gamma_{ij}^k = -\gamma_{ji}^k, \text{ and}$$

$$\sum_{\ell}(\gamma_{ij}^{\ell}\gamma_{\ell k}^{m} + \gamma_{jk}^{\ell}\gamma_{\ell i}^{m} + \gamma_{ki}^{\ell}\gamma_{\ell j}^{m}) = 0$$

for all i, j, k and m. Evidently, SC_n is an algebraic variety in $\mathbf{R}^{n^3}$, an observation which will be of some significance subsequently. We set

$$SC = \bigcup_{n \geq 0} SC_n \times [0, 1];$$

for each $(\gamma, t) \in SG$, we let $\mathcal{G}(\gamma, t)$ denote the space $\mathbf{R}^n$ (where $\gamma \in SC_n$) with the Lie algebra structure given by

$$[e_i, e_j] = \sum_k \gamma_{ij}^k e_k,$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbf{R}^n$.

As in the case of Polish or Lie groups, a field of Lie algebras $x \in X \to \mathcal{G}_x$, X standard Borel, is a Borel field if there is a Borel injection $\varphi : X \to SC$ and isomorphisms $\theta_x : \mathcal{G}_x \to \mathcal{G}(\varphi(x))$ for each $x \in X$. The following characterises Borel fields of Lie algebras of a fixed dimension.

Theorem 2. A field $x \in X \to \mathcal{G}_x$ of Lie algebras over a standard Borel space X with $dim(\mathcal{G}_x) = n$ for all x is Borel if and only if $\mathcal{G} = \dot{\cup}_{x \in X} \mathcal{G}_x$ carries a standard Borel structure for which

a) the projection $\pi : \mathcal{G} \to X$ is Borel.

b) the relative and intrinsic Borel structures on $\pi^{-1}(x)$ coincide for all $x \in X$.

c) there are Borel maps $x \in X \to A_j(x) \in \mathcal{G}$ with

   i) $\{A_j(x) : 1 \leq j \leq n\}$ a basis of $\mathcal{G}_x$ for each x,

   ii) $[A_i(x), A_j(x)] = \sum_{k=1}^{n} \gamma_{ij}^k(s) A_k(x)$ for some Borel functions γ_{ij}^k

We now turn to examine the natural equivalence relation of isomorphism induced on SC_n for each $n \geq 1$. If we identify $\mathbf{R}^{n^3}$ with $\mathbf{R}^n \otimes \mathbf{R}^n \otimes \mathbf{R}^n$ via $(\gamma_{ij}^k) \to \sum \gamma_{ij}^k e_i \otimes e_j \otimes e_k$, we may note that for $\gamma, \gamma' \in SC_n$, $\mathcal{G}(\gamma, t)$ is isomorphic with $\mathcal{G}(\gamma', t')$ if and only if $\gamma = (P \otimes P \otimes (P^{-1})^t)\gamma'$ for some invertible $n \times n$ matrix P; indeed P is just the change of basis matrix. We may thus view the space of isomorphism classes of Lie algebras as the disjoint union of the orbit spaces of algebraic actions of the algebraic groups $GL(n, \mathbf{R})$ on the algebraic varieties $SC_n \subseteq \mathbf{R}^n \otimes \mathbf{R}^n \otimes \mathbf{R}^n$. From [Z, p31] we thus obtain.

Theorem 3 The space of isomorphism classes of finite dimensional Lie algebras over $\mathbf{R}$ is countably separated.

The theorem indicates that there is a constructive way to specify a basis for each finite dimensional Lie algebra; while this can certainly be done for special classes of Lie algebras, the general case is far from clear.

§3. From Lie group to Lie algebra. (I) For each connected Lie group G, we let Lie (G) denote its Lie algebra.

Theorem 4 The field $(\mu, d) \in LG \to \text{Lie}(G(\mu, d))$ is a Borel filed.

The idea of the proof is to identify Lie (G) as a space with the set of one-parameter subgroups of G, $Hom(\mathbf{R}, G)$, and to use variants on the Campbell-Baker-Hausdorff formula (V, p114–121) to identify the Lie algebra structure in terms of the product operations in the groups. This generalises to fields of groups courtesy of one of the major results of [GS1]: if $x \to G_x$ and $x \to H_x$ are Borel fields of Polish groups with H_x locally compact, then $\dot{\cup}_{x \in X} Hom(G_x, H_x)$ admits a standard Borel structure compatible with the projection on X and for which there is a countable family of Borel sections whose values at each x are dense in the topology of convergence in measure.

Corollary 5. If $x \in X \to G_x$ is a Borel field of Lie groups over a standard Borel space X, then $x \to \text{Lie}(G_x)$ is a Borel field.

Corollary 6. If G is a Lie group which acts ergodically on a standard measure space (X, μ), and $G_x = \{g \in G : gx = x\}$, the isomorphism class of $\text{Lie}(G_x)$ is constant μ-ae.

To prove Corollary 6, we need only note that $x \to G_x$ is Borel, so that so also is $x \to [\mathrm{Lie}(G_x)]$, the isomorphism class of $\mathrm{Lie}(G_x)$. Since the space of all such isomorphism classes is countably separated (Theorem 3) and the map is constant on G-orbits, it is constant a.e. by ergodicity.

§4 **From Lie algebra to Lie group** For each Lie algebra $\mathcal{G}$, we let $Gr(\mathcal{G})$ denote the simply connected Lie group with Lie algebra $\mathcal{G}$ (so that $Gr(\mathcal{G})$ is defined only up to isomorphism).

Theorem 7 Let map $(\gamma, t) \in LC \to Gr(\mathcal{G}(\gamma, t))$ is Borel.

The proof is, in essence, to adapt the ingredients of the classical proof of the existence of a Lie group with a specified Lie algebra to the context of Borel fields. (see [V, §3]). In particular, one must establish Borel versions of the Whitehead lemmas, and of the Levi-Malcev and Ado Theorems. There generalisations are conceptually simple even if somewhat complex in formation.

The arguments involved in the proofs of Theorems 4 and 7 lead to the following

Corollary 8 Let $x \in X \to G_x$ be a Borel field of Lie groups over a standard Borel space X. Then

a) $x \in X \to \pi_1(G_x)$ is a Borel field of discrete groups

b) the exponential map $\exp : \dot{\cup}\mathrm{Lie}(G_x) \to \dot{\cup} G_x$ is Borel, and is a Borel isomorphism on a Borel subset B of $\dot{\cup}\mathrm{Lie}(G_x)$ such that $B \cap \mathrm{Lie}(G_x)$ is an open neighbourhood of O in $\mathrm{Lie}(G_x)$ for each x.

Of course Corollary 8b) allows us to construct local coordinates for the field $x \to G_x$ in a Borel manner.

§5 From Lie Groups to Lie Algebras (II) The following example shows that the space of Lie groups with a fixed Lie algebra is not countably separated for the equivalence relation of isomorphism, and hence that the classification of Lie groups is qualitatively and inherently immensely more complex than that of Lie algebras.

Let $\mathcal{H}$ denote the 3-dimensional Heisenberg Lie algebra with generators X, Y, Z satisfying $[X, Y] = Z$, and let H be the corresponding simply connected Lie group; thus

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{R} \right\}$$

We set $\mathcal{G} = \mathbf{R} \oplus \mathcal{H}$, and let $G = \mathbf{R} \times H$ be the corresponding simply connected Lie group.

Any connected Lie group with Lie algebra $\mathcal{G}$ is of the form G/D where D is a discrete central subgroup; we also have G/D_1 isomorphic with G/D_2 if and only $D_1 = \alpha(D_2)$ for some automorphism α of G.

Identify the centre $Z(G)$ of G with $\{(w, z) \in \mathbf{R}^2\}$; any automorphism of G restricts on the centre of G to a map of the form $(w, z) \rightarrow (aw, bz + cw)$ where $ac \neq 0$, and all such maps occur.

For each pair of linearly independent vectors α and β in $Z(G)$, let $D_{\alpha,\beta} = \mathbf{Z}\alpha + \mathbf{Z}\beta$, and $G_{\alpha,\beta} = G/D_{\alpha,\beta}$. Identifying α, β with the matrix (α, β) in $GL(2, \mathbf{R})$ whose columns are α and β, we see that $D_{\alpha,\beta} = D_{\alpha',\beta'}$ if and only if $(\alpha, \beta)GL(2, \mathbf{Z}) = (\alpha', \beta')GL(2, \mathbf{Z})$, and hence that $G_{\alpha,\beta}$ is isomorphic with $G_{\alpha',\beta'}$ if and only if $L(\alpha, \beta)GL(Z, \mathcal{Z}) = L(\alpha', \beta')GL(2, \mathbf{Z})$, where L is the group of lower triangular matrices

$$L = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c \in \mathbf{R}, ac \neq 0 \right\}.$$

The space of isomorphism classes of the groups $G_{\alpha,\beta}, (\alpha, \beta) \in GL(2, \mathbf{R})$ may thus be identified with the double coset space $L\backslash GL(2, \mathbf{R})/GL(2, \mathbf{Z})$; that this space is not smoth follows routinely from Moore's ergodicity theorem $(Z, p19)$.

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