

The Effect of Squeezing in the Attenuation Processes

Dedicated to Professor T. Toyoda on his 70th birthday

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Abstract — The rigorous description for the attenuation processes is discussed and the error probability for the optical communication processes is derived and it is computed in some concrete models. Moreover the effect of squeezing in the attenuation processes is considered from the quantum information theoretical points of view.

I. INTRODUCTION

The communication theory has been started by Shannon in discrete systems around 1948 [17] and it is followed by Kolmogorov in the measure theoretic framework [7]. This communication theory is often called “the commutative communication theory” because a system representing a signal has a commutative structure.

It is difficult to fully describe optical communication processes by the commutative communication theory because the optical signal should be a quantum object having a noncommutative structure. Therefore we need new communication theory “quantum communication theory” expressing quantum effects such as “quantum noise” associated to optical communication processes. Some rigorous studies related to quantum communication theory have been progressed in the fields of quantum entropy theory [10,12,21,22] and quantum control theory [3,5,6,11,26,27,28], rather independently.

In this paper we review a rigorous mathematical formulation of quantum communication processes and we derive error probability in each modulation and detection. Especially we show the rigorous formulation of error probability for a squeezed state taken as an input state and we discuss the effect of squeezing in the attenuation process. The whole content in this paper is one of the applications of “Information Dynamics” proposed by Ohya [13]. At first, in Section II, we review a mathematical

formulation of the quantum mechanical channel and a mathematical construction of the channel for optical communication processes [9,10,11,12]. In Section III we review the general expression for an attenuation process and discuss another simpler expression [13,15,16]. In this paper, we apply this expression to the derivation of each error probability. In Section IV we briefly review some basic facts of quantum coding and types of channel for the derivation of error probability given in Section V. In Sections V and VI, we give general expressions of the error probabilities in IM-DD (Intensity Modulation - Direct Detection) and COC (Coherent Optical Communication), respectively. In Section VII we present some numerical results of the error probabilities and we discuss the efficiency for each modulation and detection. Especially we emphasize the effect of squeezing in the attenuation process.

II. QUANTUM MECHANICAL CHANNEL AND ITS MATHEMATICAL CONSTRUCTION

In this section we review the general definition of a quantum mechanical channel and its mathematical expression for real optical communication processes [9,10,11,12].

A. Quantum Mechanical Channel

In order to construct the communication theory we have to set at least two dynamical systems : an input system and an output system. And each system can be characterized by each state. That is, once we fix a state in a dynamical system, we can get almost all properties of this system. Therefore we have only to know the relation between input states and output states. And a channel describes the effect of state change in the course of information transmission [10,13].

In the classical communication theory, each state of input and output systems is described by a probability distribution. So a channel causes the change of this probability distribution.

On the other hand, in the quantum communication theory, each state of input and output system should be described by quantum states such as density operators or general state on noncommutative systems. We mathematically describe quantum mechanical systems in the framework of Hilbert space. Let \mathcal{H}_1 and \mathcal{H}_2 be the separable complex Hilbert spaces describing an input space and an output space, respectively. Let $B(\mathcal{H}_k)$ ($k = 1, 2$) be the set of all bounded linear operators on \mathcal{H}_k , and $\mathfrak{S}(\mathcal{H}_k)$ be the set of all states (density operators) on the Hilbert spaces \mathcal{H}_k ; that is, $\mathfrak{S}(\mathcal{H}_k) = \{\rho \in B(\mathcal{H}_k) \mid \rho \geq 0, \rho^* = \rho, \text{tr}\rho = 1\}$

Then a mapping $\Lambda^* : \mathfrak{S}(\mathcal{H}_1) \rightarrow \mathfrak{S}(\mathcal{H}_2)$ is here called a quantum mechanical channel and it is a completely positive (CP) channel if the dual map $\Lambda : B(\mathcal{H}_2) \rightarrow B(\mathcal{H}_1)$ satisfies the completely positivity :

$$\sum_{i,j}^n B_i^* \Lambda(A_i^* A_j) B_j \geq 0 \quad \text{for } \forall B_i \in B(\mathcal{H}_1), \forall A_j \in B(\mathcal{H}_2) \quad \text{and } \forall n \in N. \quad (2.1)$$

Most of physical transformations satisfy the condition completely positivity, so that this definition is general enough to mathematically construct a concrete realistic channel for a quantum communication.

B. Mathematical Construction for Channel Λ^*

Let us find a mathematical expression for real optical communication processes by taking account of the effect of noise and loss in the course of information transmission.

For the purpose, in addition to the Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , we need two more Hilbert spaces \mathcal{K}_1 and \mathcal{K}_2 describing a noise system and a loss system, respectively. Then we have the following mathematical structure for optical communication processes.

$$\begin{array}{ccc}
 \text{(Noise)} \ \xi \in \mathfrak{S}(\mathcal{K}_1) & & \\
 \downarrow & & \\
 \mathfrak{S}(\mathcal{H}_1) \ni \rho & \xrightarrow{\quad\quad\quad} & \Lambda^* \rho \in \mathfrak{S}(\mathcal{H}_2) \\
 & & \downarrow \\
 & & \text{(Loss)} \ \mathfrak{S}(\mathcal{K}_2)
 \end{array}$$

Fig. 1 Quantum Mechanical Channel

Let $\rho \in \mathfrak{S}(\mathcal{H}_1)$ and $\xi \in \mathfrak{S}(\mathcal{K}_1)$ be quantum states representing an input state and a noise state, respectively. We need the following three mappings to construct a general form of a channel for optical communication processes :

- (1) the map a is an amplification from $B(\mathcal{H}_2)$ to $B(\mathcal{H}_2 \otimes \mathcal{K}_2)$ given by $a(A) = A \otimes I$ for any $A \in B(\mathcal{H}_2)$, where I is an identity operator on \mathcal{K}_2 ,
- (2) the map Π is a completely positive map from $B(\mathcal{H}_2 \otimes \mathcal{K}_2)$ to $B(\mathcal{H}_1 \otimes \mathcal{K}_1)$ with $\Pi(I) = I$ describing the physical mechanism of the transformation,
- (3) the map Γ is given by $\Gamma(Q) = \text{tr}_{\mathcal{K}_1} \xi Q$ for any $Q \in B(\mathcal{H}_1 \otimes \mathcal{K}_1)$. Here, $\text{tr}_{\mathcal{K}_1}$ is the partial trace : $\langle \Phi_1, \text{tr}_{\mathcal{K}_1} Q \Phi_2 \rangle \equiv \sum_n \langle \Phi_1 \otimes \Psi_n, Q \Phi_2 \otimes \Psi_n \rangle$ for any $Q \in B(\mathcal{H}_1 \otimes \mathcal{K}_1)$, any $\Phi_1, \Phi_2 \in \mathcal{H}_1$, and any CONS $\{\Psi_n\}$ of \mathcal{K}_1 .

Then we define a mapping Λ from $B(\mathcal{H}_2)$ to $B(\mathcal{H}_1)$ such that

$$\Lambda = \Gamma \circ \Pi \circ a. \quad (2.2)$$

$$\begin{array}{ccc}
 B(\mathcal{H}_2) & \xrightarrow{\Lambda} & B(\mathcal{H}_1) \\
 a \downarrow & & \uparrow \Gamma \\
 B(\mathcal{H}_2 \otimes \mathcal{K}_2) & \xrightarrow{\Pi} & B(\mathcal{H}_1 \otimes \mathcal{K}_1)
 \end{array}$$

Fig. 2 Λ

We next consider the dual maps of a , Π , Γ ;

- (1') the dual map a^* of a is a map from $\mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2)$ to $\mathfrak{S}(\mathcal{H}_2)$ such that $a^*(\theta) = \text{tr}_{\mathcal{K}_2} \theta$ for any $\theta \in \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2)$,

- (2') the dual map $\Pi^* : \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1) \longrightarrow \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2)$ is given by $\text{tr} \Pi^*(\theta)W = \text{tr} \theta \Pi(W)$ for any $\theta \in \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$ and any $W \in B(\mathcal{H}_2 \otimes \mathcal{K}_2)$,
- (3') the dual map $\Gamma^* : \mathfrak{S}(\mathcal{H}_1) \longrightarrow \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1)$ is given by $\Gamma^*(\rho) = \rho \otimes \xi$

$$\begin{array}{ccc} \mathfrak{S}(\mathcal{H}_1) & \xrightarrow{\Lambda^*} & \mathfrak{S}(\mathcal{H}_2) \\ \Gamma^* \downarrow & & \uparrow a^* \\ \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{K}_1) & \xrightarrow{\Pi^*} & \mathfrak{S}(\mathcal{H}_2 \otimes \mathcal{K}_2) \end{array}$$

Fig. 3 Channel Λ^*

Therefore, once we know the noise ξ and the mechanism of the transformation Π , we can write down a channel explicitly as

$$\Lambda^* = a^* \circ \Pi^* \circ \Gamma^*. \quad (2.3)$$

so that

$$\Lambda^*(\rho) = \text{tr}_{\mathcal{H}_2} \Pi^*(\rho \otimes \xi) \quad (2.4)$$

for any $\rho \in \mathfrak{S}(\mathcal{H}_1)$ [10].

Let us show that this mathematical expression Λ^* indeed becomes a CP quantum mechanical channel. We have only to show the completely positivity of the mapping Λ . We show the completely positivity of the mapping Γ by the following proof. Next we prove the completely positivity of the mapping Γ .

For any $A_i \in B(\mathcal{H}_1 \otimes \mathcal{K}_1)$, any $B_j \in B(\mathcal{H}_1)$, any CONS $\{\Phi_k^1\}$ of \mathcal{H}_1 , any CONS $\{\Psi_l^1\}$ of \mathcal{K}_1 , any $\xi \in \mathfrak{S}(\mathcal{K}_1)$, any $\Phi \in \mathcal{H}_1$ and any $n \in N$

$$\begin{aligned} & \langle \Phi, \sum_{i,j}^n B_i^* \Gamma(A_i^* A_j) B_j \Phi \rangle \\ &= \sum_{i,j}^n \langle B_i \Phi, \text{tr}_{\mathcal{K}_1} \xi A_i^* A_j B_j \Phi \rangle \\ &= \sum_{i,j}^n \sum_m \langle B_i \Phi \otimes \Psi_m^1, (I \otimes \xi) A_i^* A_j B_j \Phi \otimes \Psi_m^1 \rangle \\ &= \sum_{i,j}^n \sum_m \sum_{k,l} \langle B_i \Phi \otimes \Psi_m^1, (I \otimes \xi) A_i^* \Phi_k^1 \otimes \Psi_l^1 \rangle \langle \Phi_k^1 \otimes \Psi_l^1, A_j B_j \Phi \otimes \Psi_m^1 \rangle \\ &= \sum_{i,j}^n \sum_{k,l} \langle \Phi_k^1 \otimes \Psi_l^1, A_j (|B_j \Phi\rangle \langle B_j \Phi| \otimes I) (I \otimes \xi) A_i^* \Phi_k^1 \otimes \Psi_l^1 \rangle \\ &= \sum_{k,l}^n \sum_{i,j} \langle \Phi_k^1 \otimes \Psi_l^1, A_j (B_j \otimes I) (|\Phi\rangle \langle \Phi| \otimes I) \end{aligned}$$

$$\begin{aligned}
& \times (B_i^* \otimes I)(I \otimes \xi^{\frac{1}{2}})(I \otimes \xi^{\frac{1}{2}})A_i^* \Phi_k^1 \otimes \Psi_l^1 > \\
& = \sum_m \sum_{k,l} \sum_i^n \langle \Phi_k^1 \otimes \Psi_l^1, A_j(I \otimes \xi^{\frac{1}{2}})(B_j \otimes I)\Phi \otimes \Psi_m^1 \rangle \\
& \quad \times \sum_j^n \langle \Phi_k^1 \otimes \Psi_l^1, A_i(I \otimes \xi^{\frac{1}{2}})(B_i \otimes I)\Phi \otimes \Psi_m^1 \rangle \\
& = \sum_m \sum_{k,l} \left| \sum_i^n \langle \Phi_k^1 \otimes \Psi_l^1, A_j(I \otimes \xi^{\frac{1}{2}})(B_j \otimes I)\Phi \otimes \Psi_m^1 \rangle \right|^2 \geq 0
\end{aligned}$$

We can prove the completely positivity of the mapping a as similarly as above.

Therefore the mapping Λ given by Eq.(2.2) is completely positive, that is, the mapping Λ^* is a quantum mechanical CP channel.

III. ATTENUATION PROCESS

In real communication processes we suffer the loss of the information in the course of information transmission. Therefore we construct a more concrete model of the channel Λ^* by taking into account this attenuation of the information. We at first give the general expression for an attenuation process by using the Hamiltonian of each system [10,12]. Secondly, we discuss another simpler expression related to the concept "lifting"[1,13].

A. General Expression for an Attenuation Process [10,12]

Each quantum system composed of photons is described by the Hamiltonian $H = a^*a + 1/2$, where a^* and a are creation and annihilation operators of a photon, respectively. By solving the Schrödinger equation $Hx(q) = Ex(q)$, we can easily get the eigenvalue E_n ; $E_n = n + 1/2$ ($n = 0, 1, 2, \dots$) and the eigenvector $x_n(q)$; $x_n(q) = (1/(\pi^{1/2}n!)^{1/2})H_n(\sqrt{2}q)\exp(-q^2/2)$, where $H_n(q)$ is the n th Hermite function. The Hilbert space of each system is the closed linear span of the linear combinations $x_n(q)$ ($n = 0, 1, 2, \dots$).

Then a model for optical attenuation processes is considered as follows : When n_1 photons are transmitted from the input system, n_2 photons from the noise system add to the signal. Then m_1 photons are lost to the loss system through the channel, and m_2 photons are detected in the output system. The Hilbert spaces and their coordinates in this model are denoted in Table I below.

System	Hilbert Space	CONS	Coordinate
Input	\mathcal{H}_1	$x_{n_1}^{(1)}(q_1)$	q_1
Noise	\mathcal{K}_1	$y_{n_2}^{(1)}(t_1)$	t_1
Output	\mathcal{H}_2	$x_{m_1}^{(2)}(q_2)$	q_2
Loss	\mathcal{K}_2	$y_{m_2}^{(2)}(t_2)$	t_2

Table. I Quantum Systems

According to the conservation of energy ($n_1 + n_2 = m_1 + m_2$), we suppose the following linear transformation [20] among the coordinates q_1, t_1, q_2, t_2 of the input, noise, output, and loss systems, respectively :

$$\begin{cases} q_2 = \alpha q_1 + \beta t_1, \\ t_2 = -\beta q_1 + \alpha t_1, \end{cases} \quad (\alpha^2 + \beta^2 = 1)$$

By using this linear transformation, we define the mapping $\Pi = U(\cdot)U^*$ by

$$\begin{aligned} U(x_{n_1}^{(1)} \otimes y_{n_2}^{(1)})(q_2, t_2) &= x_{n_1}^{(2)} \otimes y_{n_2}^{(2)}(\alpha q_2 - \beta t_2, \beta q_2 + \alpha t_2) \\ &= \sum_{j=0}^{n_1+n_2} C_j^{n_1, n_2} x_j^{(2)} \otimes y_{n_1+n_2-j}^{(2)}(q_2, t_2) \end{aligned} \quad (3.1)$$

where $C_j^{n_1, n_2}$ is given by

$$\begin{aligned} C_j^{n_1, n_2} &= \int \int x_{n_1}^{(2)} \otimes y_{n_2}^{(2)}(\alpha q_2 - \beta t_2, \beta q_2 + \alpha t_2) \overline{x_j^{(2)} \otimes y_{n_1+n_2-j}^{(2)}(q_2, t_2)} dq_2 dt_2 \\ &= \sum_{r=L}^K (-1)^r \frac{\sqrt{n_1! n_2! j! (n_1 + n_2 - j)!}}{r! (n_1 - r)! (j - r)! (n_2 - j + r)!} \alpha^{n_2 - j + 2r} \beta^{n_1 + j - 2r} \end{aligned} \quad (3.2)$$

where $K = \min\{j, n_1\}$, $L = \max\{j - n_2, 0\}$.

Then the CP channel Λ^* is expressed as

$$\Lambda^* \rho = \text{tr}_{\mathcal{H}_2} U(\rho \otimes \xi) U^* \quad (3.3)$$

Here note that α^2 can be regarded as the transmission efficiency η for the channel Λ^* .

In this paper, we let a noise state ξ a vacuum state for simplicity. That is, $\xi = |y_0^{(1)}\rangle\langle y_0^{(1)}| = |0\rangle\langle 0| \in \mathfrak{S}(\mathcal{K}_1)$ is a noise state due to the ‘‘zero point fluctuation’’ of electromagnetic field ($y_0^{(1)}$ is a vacuum state vector in \mathcal{K}_1).

B. Lifting

The concept of ‘‘lifting’’ can be applied to the expression for an attenuation process [1,13,15].

Definition 3.1 [1,13]: Let \mathcal{H}, \mathcal{K} be Hilbert spaces and let $\mathcal{H} \otimes \mathcal{K}$ be a fixed tensor product of \mathcal{H} and \mathcal{K} . A lifting \mathcal{E}^* from \mathcal{H} to $\mathcal{H} \otimes \mathcal{K}$ is a continuous map

$$\mathcal{E}^* : \mathfrak{S}(\mathcal{H}) \longrightarrow \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}) \quad (3.4)$$

If \mathcal{E}^* is affine, we call it a linear lifting ; if it maps pure states into pure states, we call it pure.

When we may take $\mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$ and $\mathcal{K} = \mathcal{K}_1 = \mathcal{K}_2$,

$$\mathcal{E}^* : \rho \in \mathfrak{S}(\mathcal{H}) \rightarrow \Pi^*(\rho \otimes \xi) \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$$

is a lifting, and we can rewrite the channel :

$$\Lambda^* \rho = \text{tr}_K \mathcal{E}^* \rho. \quad (3.5)$$

By using the lifting, we can define a mapping V from \mathcal{H} to $\mathcal{H} \otimes \mathcal{K}$ as

$$V|\theta\rangle = |\alpha\theta\rangle \otimes |\beta\theta\rangle \quad (3.6)$$

where $|\theta\rangle$ represents a coherent vector [4,8].

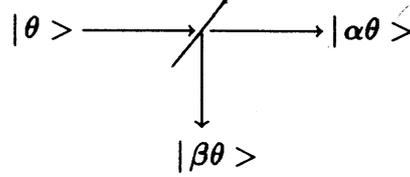


Fig. 4 Attenuation Process V

Now, let us show the equivalence of the above operator V and the operator U in the conventional expression.

$$\begin{aligned} V|\theta\rangle &= |\alpha\theta\rangle \otimes |\beta\theta\rangle \\ &= \exp\left(-\frac{|\alpha\theta|^2}{2}\right) \sum_n \frac{(\alpha\theta)^n}{\sqrt{n!}} |n\rangle \otimes \exp\left(-\frac{|\beta\theta|^2}{2}\right) \sum_m \frac{(\beta\theta)^m}{\sqrt{m!}} |m\rangle \\ &= \exp\left(-\frac{|\theta|^2}{2}\right) \sum_n \sum_m \frac{(\alpha\theta)^n (\beta\theta)^m}{\sqrt{n!m!}} |n\rangle \otimes |m\rangle \\ &= \exp\left(-\frac{|\theta|^2}{2}\right) \sum_{N=0}^{\infty} \frac{\theta^N}{\sqrt{N!}} \sum_{n=0}^{\infty} \alpha^n \beta^{N-n} \sqrt{\frac{N!}{n!(N-n)!}} |n\rangle \otimes |N-n\rangle, \end{aligned}$$

which implies, for any nonnegative integer N ,

$$V|N\rangle = \sum_{n=0}^N \alpha^n \beta^{N-n} \sqrt{\frac{N!}{n!(N-n)!}} |n\rangle \otimes |N-n\rangle$$

Thus U when $n_2 = 0$ in Eq.(3.1) equals to V by replacing β with $-\beta$.

Therefore the attenuation CP channel can be rewritten as

$$\Lambda^* \rho = \text{tr}_K V \rho V^*. \quad (3.7)$$

In this paper, we use this expression Eq.(3.7) to the derivation of error probabilities.

IV. QUANTUM CODING AND TYPES OF CHANNEL

In this section, before we derive concrete error probabilities, we review some basic facts for quantum coding and two types of channeling transformation.

Suppose that, by some procedure, we encode an information representing it by a sequence of letters $c^{(1)}, \dots, c^{(n)}, \dots$, where $c^{(k)}$ is an element in a set \mathcal{C} of symbols called the alphabet.

A quantum code is a map which associates to each symbol (or sequence of symbols) in \mathcal{C} a quantum state, representing an optical signal. This expression is called the quantum mechanical coding. Let ρ_i be the quantum code corresponding to a symbol $c_i \in \mathcal{C}$. We usually take

$$\mathcal{C} = \{0, 1\} \iff \Xi = \{\rho_0, \rho_1\}. \quad (4.1)$$

Then we assume that the noise state $\xi \in \mathfrak{S}(\mathcal{K}_1)$ is a vacuum state due to the "zero point fluctuation" of electromagnetic field. Therefore, when we derive error probabilities, we have to consider the following two types of channel: Z-type channel and X-type channel. Each type of channel corresponds to IM-DD and COC, respectively because the information associated to the input state is set by different manners in IM-DD and COC.

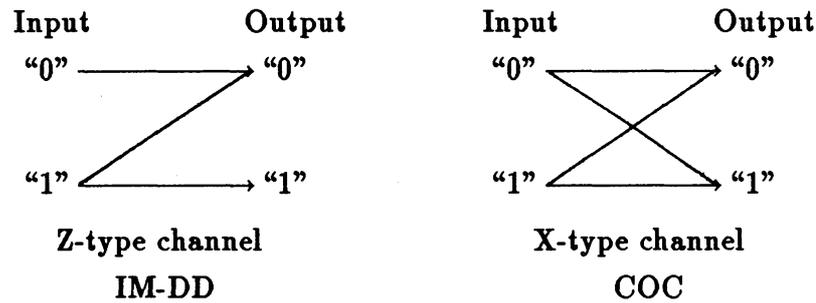


Fig. 5 Type of Channel

At first, in the case of IM-DD, we usually take ρ_0 for the vacuum state and ρ_1 for another state such as a coherent state or a squeezed state. Since the noise state ξ is a vacuum state, the input signal "0" represented by the state $\rho_0^{(1)}$, is error free in the sense that it always goes to the output signal "0" represented by $\rho_0^{(2)}$, while the input signal "1", represented by the state $\rho_1^{(1)}$, is not error free in the sense that its output may reach to both states $\rho_0^{(2)}$ and $\rho_1^{(2)}$ with different probabilities. We call this channel Z-type channel. Then the error probability P_e for IM-DD is given by

$$P_e = P_{e1} \quad (4.2)$$

where P_{e1} is the error probability that the signal "1" is read as the signal "0".

On the other hand, in the case of COC, the information is carried by amplitude, frequency or phase of the input state. Therefore, regardless of the noise state ξ , both of transmitted input signals "0" and "1" have a possibility to be suffered some mistake

in the output system. We call this channel X-type channel. Here we assume the input signals “0” and “1” are transmitted with equal probability 1/2, so that an error probability P_e for digital modulation is given by

$$P_e = \frac{P_{e0} + P_{e1}}{2} \quad (4.3)$$

where P_{e0} and P_{e1} are the error probabilities associated with the input signal “0” and the input signal “1”, respectively.

V. RIGOROUS DERIVATION OF ERROR PROBABILITY IN IM-DD

As discussed in [3], POVM (positive operator valued measure) is a useful tool to describe quantum measurement processes. Therefore we apply the attenuation channel and each POVM expression to the derivation of error probability for a coherent input state and a squeezed input state.

Direct detection is a measurement of photons in a transmitted state, so that the POVM for the direct detection is given by

$$E_{DD}(n) = | n \rangle \langle n | \quad (5.1)$$

where $| n \rangle$ is the n-th number photon vector in \mathcal{H}_2 .

In particular, in case of IM-DD, we consider only Z-type channel. That is, direct detection in IM-DD measures the number of photons in the transmitted state and decides whether the output state is vacuum or not.

Therefore, when the input state ρ_1 is transmitted to an output state $\Lambda^*(\rho_1)$, the general formula of the error probability q_e that the state $\Lambda^*(\rho_1)$ is recognized as a vacuum state by mistake is given by :

$$\begin{aligned} q_e &= \text{tr}_{\mathcal{H}_2} \Lambda^* \rho_1 E_{DD}(0) \\ &= \text{tr}_{\mathcal{H}_2} \text{tr}_{\mathcal{K}_2} V \rho_1 V^* E_{DD}(0) \end{aligned} \quad (5.2)$$

A. PPM

In the case of PPM, since each symbol pulse is used for each quantum code, the error probability P_e^{PPM} becomes

$$P_e^{PPM} = q_e. \quad (5.3)$$

1) Coherent state

From Eq.(5.2) and Eq.(5.3), the error probability $P_{e(CO)}^{PPM}$ for a coherent state $\rho_1 = | \theta \rangle \langle \theta |$ is given by

$$\begin{aligned} q_e &= \text{tr}_{\mathcal{H}_2} (\text{tr}_{\mathcal{K}_2} V | \theta \rangle \langle \theta | V^*) | 0 \rangle \langle 0 | \\ &= \text{tr}_{\mathcal{H}_2} (\text{tr}_{\mathcal{K}_2} | \alpha \theta \rangle \langle \alpha \theta | \otimes | \beta \theta \rangle \langle \beta \theta |) | 0 \rangle \langle 0 | \\ &= \text{tr}_{\mathcal{H}_2} | \alpha \theta \rangle \langle \alpha \theta | | 0 \rangle \langle 0 | \\ &= | \langle 0 | \alpha \theta \rangle |^2 = \exp(- | \alpha \theta |^2) = \exp(- \eta | \theta |^2) \end{aligned} \quad (5.4)$$

where $\eta = \alpha^2$ and η is constant describing the transmission efficiency for the channel.

2) Squeezed state

A squeezed state can be expressed by a unitary operator $U(z)$ ($z \in \mathbf{C}$) such that

$$\rho_1 = U(z) | \theta \rangle \langle \theta | U(z)^*$$

where $| \theta \rangle$ is a certain coherent vector. More concretely a squeezed vector $U(z) | \theta \rangle$ is expressed as [19,25].

$$\begin{aligned} U(z) | \theta \rangle &\equiv | \theta_{sq}; \mu, \nu \rangle \\ b &\equiv \mu a + \nu a^* \\ \theta &= \mu \theta_{sq} + \nu \overline{\theta_{sq}} \\ b | \theta_{sq}; \mu, \nu \rangle &= (\mu \theta_{sq} + \nu \overline{\theta_{sq}}) | \theta_{sq}; \mu, \nu \rangle \\ |\mu|^2 - |\nu|^2 &= 1 \\ \mu &= \cosh z \\ \nu &= \exp(i\phi) \sinh z \end{aligned}$$

Then, from Eq.(5.2) and Eq.(5.3), the error probability $P_{e(SQ)}^{PPM}$ for a squeezed state $\rho_1 = U(z) | \theta \rangle \langle \theta | U(z)^*$ is given by

$$\begin{aligned} q_e &= \text{tr}_{\mathcal{H}_2} (\text{tr}_{\mathcal{H}_1} V U(z) | \theta \rangle \langle \theta | U(z)^* V^* | 0 \rangle \langle 0 |) \\ &= \text{tr}_{\mathcal{H}_1} U(z) | \theta \rangle \langle \theta | U(z)^* (V^* (| 0 \rangle \langle 0 | \otimes I) V) \\ &= \langle U(z)\theta, V^* (| 0 \rangle \langle 0 | \otimes I) V U(z)\theta \rangle \\ &= \frac{1}{\pi^2} \int \int d^2 v d^2 w \langle U(z)\theta, w \rangle \langle \alpha w, 0 \rangle \langle \beta w, \beta v \rangle \\ &\quad \times \langle 0, \alpha v \rangle \langle v, U(z)\theta \rangle \end{aligned}$$

This can be computed by the following Gaussian type integration :

$$\frac{1}{\pi} \int d^2 w \exp\{-|w|^2 + aw + b\bar{w} + cw^2 + d\bar{w}^2\} = \frac{1}{\sqrt{1-4cd}} \exp\left\{\frac{a^2 d + ab + b^2 c}{1-4cd}\right\}. \quad (5.5)$$

The result is

$$q_e = \sqrt{\tau} \exp \left[\{(1-\eta)\tau - 1\} |\theta|^2 + \{1 - (1-\eta)^2 \tau\} \left\{ \frac{\bar{\nu}\theta^2}{2\mu} + \frac{\nu\bar{\theta}^2}{2\bar{\mu}} \right\} \right] \quad (5.6)$$

where $\tau = \{|\mu|^2 - (1-\eta)^2 |\nu|^2\}^{-1}$, μ and ν are complex numbers satisfying $|\mu|^2 - |\nu|^2 = 1$.

B. PCM

In the case that the code has the weight N (the number of symbol "1"), the j -multiple error probability in the output system is

$$P^{(j)} = {}_N C_j q_e^j (1 - q_e)^{N-j}, \quad (5.7)$$

where

$${}_N C_j = \frac{N!}{j!(N-j)!}.$$

Therefore, when the code with the weight N is transmitted, the error probability P_e^{PCM} for PCM modulation with t_0 -tuple error correcting code with the weight N is given by :

$$\begin{aligned} P_e^{PCM} &= \sum_{j=t_0+1}^N P^{(j)} \\ &= \sum_{j=t_0+1}^N {}_N C_j q_e^j (1 - q_e)^{N-j}. \end{aligned} \quad (5.8)$$

By substituting Eq.(5.4) and Eq.(5.6) in the above formula Eq.(5.8), we can easily compute the error probabilities $P_{e(CO)}^{PCM}$ and $P_{e(SQ)}^{PCM}$.

VI. RIGOROUS DERIVATION OF ERROR PROBABILITY IN COHERENT OPTICAL COMMUNICATION

A. P.D.F. for Each Detection

1) Homodyne Detection

Homodyne detection is a measurement of the real part of the complex amplitude of a transmitted state. Therefore the P.O.V.M. E_{HO} for homodyne detection is given by

$$E_{HO}(\Delta^{HO}) = \int_{\Delta^{HO}} |\theta_x\rangle \langle \theta_x| d\theta_x \quad (6.1)$$

where $|\theta_x\rangle$ is the eigenvector of the operator $a_x = (a + a^*)/2$ and a is the annihilation operator of photon, Δ^{HO} is the set of real variables θ_x .

The infinitesimal nonnegative definite Hermitian operator $dE_{HO}(\theta_x)$ is given by

$$dE_{HO}(\theta_x) = |\theta_x\rangle \langle \theta_x| d\theta_x \quad (6.2)$$

The probability density function $p^{HO}(\theta_x)$ of the outcomes is

$$\begin{aligned} p^{HO}(\theta_x) d\theta_x &= \text{tr} \mathcal{H}_2 \Lambda^* \rho dE_{HO}(\theta_x) \\ &= \text{tr} \mathcal{H}_2 \Lambda^* \rho |\theta_x\rangle \langle \theta_x| d\theta_x, \end{aligned}$$

so that the probability density function $p^{HO}(\theta_x)$ is

$$p^{HO}(\theta_x) = \text{tr} \mathcal{H}_2 \Lambda^* \rho | \theta_x \rangle \langle \theta_x | \quad (6.3)$$

We derive the probability density function $p_{CO}^{HO}(\theta_x)$ for a coherent input state.

$$\begin{aligned} p_{CO}^{HO}(\theta_x) &= \text{tr} \mathcal{H}_2 \Lambda^* (| \theta \rangle \langle \theta |) | \theta_x \rangle \langle \theta_x | \\ &= | \langle \theta_x | \alpha \theta \rangle |^2 \\ &= \sqrt{\frac{2}{\pi}} \exp(-2(\theta_x - \alpha \text{Re}(\theta))^2) \end{aligned} \quad (6.4)$$

This probability density function $p_{CO}^{HO}(\theta_x)$ is a Gaussian type. Then m_{CO} and σ_{CO}^2 , the average and the variance for this distribution $p_{CO}^{HO}(\theta_x)$, are calculated as

$$m_{CO}^{HO} = \alpha \text{Re}(\theta), \quad \sigma_{CO}^{HO^2} = \frac{1}{4}. \quad (6.5)$$

On the other hand, in the case of a squeezed input state, we derive the probability density function p_{SQ}^{HO} from Eq.(6.3).

$$\begin{aligned} p_{SQ}^{HO}(\theta_x) &= \text{tr} \mathcal{H}_2 (\text{tr} \mathcal{K}_2 (VU(z) | \theta \rangle \langle \theta | U(z)^* V^*) | \theta_x \rangle \langle \theta_x | \\ &= \text{tr} \mathcal{H}_1 U(z) | \theta \rangle \langle \theta | U(z)^* (V^* (| \theta_x \rangle \langle \theta_x | \otimes I) V) \\ &= \frac{1}{\pi^2} \int \int d^2 v d^2 w \langle U(z)\theta, w \rangle \langle \alpha w, \theta_x \rangle \langle \beta w, \beta v \rangle \\ &\quad \times \langle \theta_x, \alpha v \rangle \langle v, U(z)\theta \rangle \\ &= \frac{1}{\sqrt{2\pi \{ \frac{1}{4}\eta |\mu - \nu|^2 + \frac{1}{4}(1 - \eta) \}}} \exp \left(-\frac{(\theta_x - \alpha \text{Re}((\bar{\mu} - \bar{\nu})\theta))^2}{2 \{ \frac{1}{4}\eta |\mu - \nu|^2 + \frac{1}{4}(1 - \eta) \}} \right) \end{aligned} \quad (6.6)$$

This probability density function $p_{SQ}^{HO}(\theta_x)$ is again a Gaussian type. Then m_{SQ} and σ_{SQ}^2 , the average and the variance of this distribution Eq.(6.6), are calculated as

$$m_{SQ}^{HO} = \alpha \text{Re}((\bar{\mu} - \bar{\nu})\theta), \quad \sigma_{SQ}^{HO^2} = \frac{1}{4}\eta |\mu - \nu|^2 + \frac{1}{4}(1 - \eta). \quad (6.7)$$

2) Heterodyne Detection

Heterodyne detection is a simultaneous measurement of the real and the imaginary parts of the complex amplitude in a transmitted state. Therefore the heterodyne detection may not depend on the effect of squeezing, so that we derive the error probabilities for a coherent input state only.

Let E_{HE} be the P.O.V.M. for heterodyne detection.

$$E_{HE}(\Delta^{HE}) = \int_{\Delta^{HE}} | \theta \rangle \langle \theta | \frac{d^2 \theta}{\pi} \quad (6.8)$$

where $|\theta\rangle$ is a coherent vector, and Δ^{HE} is the set of complex variables θ .

Therefore the infinitesimal nonnegative definite Hermitian operator $dE_{HE}(\theta)$ is given by

$$dE_{HE}(\theta) = |\theta\rangle\langle\theta| \frac{d^2\theta}{\pi} \quad (6.9)$$

The joint probability density function $p^{HE}(\theta_x, \theta_y)$ of the outcomes becomes

$$\begin{aligned} p^{HE}(\theta_x, \theta_y) d^2\theta &= \text{tr} \mathcal{H}_2 \Lambda^* \rho dE_{HE}(\theta) \\ &= \text{tr} \mathcal{H}_2 \Lambda^* \rho |\theta\rangle\langle\theta| \frac{d^2\theta}{\pi} \end{aligned}$$

so that the joint probability density function $p^{HE}(\theta_x, \theta_y)$ is :

$$p^{HE}(\theta_x, \theta_y) = \frac{1}{\pi} \text{tr} \mathcal{H}_2 \Lambda^* \rho |\theta\rangle\langle\theta| \quad (6.10)$$

For a coherent state $\rho = |\theta\rangle\langle\theta|$, $p^{HE}(\theta_x, \theta_y)$ is concretely derived as :

$$\begin{aligned} p^{HE}(\theta_x, \theta_y) &= \frac{1}{\pi} \text{tr} \mathcal{H}_2 \Lambda^* \rho |\theta\rangle\langle\theta| \\ &= \frac{1}{\pi} |\langle\theta| \alpha\theta_s \rangle|^2 \\ &= \frac{1}{\pi} \exp(-|\theta - \alpha\theta_s|^2) \end{aligned} \quad (6.11)$$

where the index "s" represents the signal "0" or "1".

Then the coherent detection demodulate the part "cos wt" from the transmitted signal. We let $p_{co}^{HE}(\theta_x)$ the marginal probability density function of $p^{HE}(\theta_x, \theta_y)$, and from Eq.(6.11) $p_{co}^{HE}(\theta_x)$ is given by

$$\begin{aligned} p_{co}^{HE}(\theta_x) &= \int p^{HE}(\theta_x, \theta_y) d\theta_y \\ &= \frac{1}{\sqrt{\pi}} \exp(-(\theta_x - \alpha \text{Re}(\theta_s))^2). \end{aligned} \quad (6.12)$$

This probability density function $p_{co}^{HE}(\theta_x)$ is also a Gaussian type. Then m_{co}^{HE} and $\sigma_{co}^{HE^2}$, the average and the variance of this distribution Eq.(6.12), are calculated as

$$m_{co}^{HE} = \alpha \text{Re}(\theta_s), \quad \sigma_{co}^{HE^2} = \frac{1}{2} \quad (6.13)$$

On the other hand, the envelope detection [18] demodulates the envelope of the transmitted signal. Let $g(r)$ be the probability density function for the envelope detection. It is well known that we can get the following probability density function $g(r)$ from Eq.(6.11):

$$g(r) = 2r \exp(-r^2 - |\alpha\theta_s|^2) I_0(2r |\alpha\theta_s|) \quad (6.14)$$

where I_0 is the zeroth-order modified Bessel function of the first kind. The distribution $g(r)$ in Eq.(6.14) is called a Rice distribution [18].

B. OOK Homodyne Detection

In OOK, ρ_0 is a vacuum state and ρ_1 is another state such as a coherent state and a squeezed state in the input system \mathcal{H}_1 .

1) Coherent state

The probability density functions $p_{0(CO)}^{HO}(\theta_x)$ and $p_{1(CO)}^{HO}(\theta_x)$ for the signal "0" and "1" are respectively obtained by Eq.(6.4) as

$$p_{0(CO)}^{HO}(\theta_x) = \sqrt{\frac{2}{\pi}} \exp(-2\theta_x^2) \quad (6.15)$$

$$p_{1(CO)}^{HO}(\theta_x) = \sqrt{\frac{2}{\pi}} \exp(-2(\theta_x - \alpha \operatorname{Re}(\theta_1))^2) \quad (6.16)$$

Every error probability of OOK for each signal turns to be identical. That is,

$$P_{e0(CO)}^{OOK-HO} = P_{e1(CO)}^{OOK-HO} = \int_{\alpha \operatorname{Re}(\theta_1)/2}^{\infty} p_{0(CO)}^{HO}(\theta_x) d\theta_x \quad (6.17)$$

Hence the error probability $P_{e(CO)}^{OOK-HO}$ is given by

$$P_{e(CO)}^{OOK-HO} = \frac{1}{2} \operatorname{erfc}\left(\frac{\sqrt{\eta} \operatorname{Re}(\theta_1)}{\sqrt{2}}\right), \quad (6.18)$$

where $\operatorname{erfc}(x)$ is the complementary error function given by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-t^2) dt \quad (6.19)$$

2) Squeezed state

Under the similar discussion as the case of a coherent state, the error probability is given by

$$P_{e(SQ)}^{OOK-HO} = \frac{1}{2} \operatorname{erfc}\left(\frac{\sqrt{\eta} \operatorname{Re}((\bar{\mu} - \bar{\nu})\theta_1)}{\sqrt{2\eta} |\mu - \nu|^2 + 2(1 - \eta)}\right) \quad (6.20)$$

C. BPSK Homodyne Detection

In BPSK, ρ_0 is a state with the phase 0 and ρ_1 is a state with the phase π .

1) Coherent state

The probability density functions $p_{0(CO)}^{HO}(\theta_x)$ and $p_{1(CO)}^{HO}(\theta_x)$ for the signal "0" and "1" are respectively obtained from Eq.(6.4).

$$p_{0(CO)}^{HO}(\theta_x) = \sqrt{\frac{2}{\pi}} \exp(-2(\theta_x - \alpha|\theta|)^2) \quad (6.21)$$

$$p_{1(CO)}^{HO}(\theta_x) = \sqrt{\frac{2}{\pi}} \exp(-2(\theta_x + \alpha|\theta|)^2) \quad (6.22)$$

where $|\theta|$ is the amplitude in an input state ρ_0 or ρ_1 .

We obtain the error probability for BPSK

$$P_{e0(CO)}^{BPSK-HO} = P_{e1(CO)}^{BPSK-HO} = \int_0^\infty p_{1(CO)}^{HO}(\theta_x) d\theta_x \quad (6.23)$$

Then the error probability $P_{e(CO)}^{BPSK-HO}$ is given by

$$P_{e(CO)}^{BPSK-HO} = \frac{1}{2} \operatorname{erfc}(\sqrt{2\eta}|\theta|) \quad (6.24)$$

2) Squeezed state

Under the similar discussion as the case of a coherent state, the error probability is given by

$$P_{e(SQ)}^{BPSK-HO} = \frac{1}{2} \operatorname{erfc}\left(\frac{\sqrt{2\eta}|\theta| \operatorname{Re}(\bar{\mu} - \bar{\nu})}{\sqrt{\eta|\mu - \nu|^2 + (1 - \eta)}}\right) \quad (6.25)$$

D. OOK Heterodyne Coherent Detection

From Eq.(6.12) the probability density functions $p_{0(CO)}^{HE}(\theta_x)$ and $p_{1(CO)}^{HE}(\theta_x)$ for the signal "0" and "1" are respectively given by

$$p_{0(CO)}^{HE}(\theta_x) = \frac{1}{\sqrt{\pi}} \exp(-\theta_x^2) \quad (6.26)$$

$$p_{1(CO)}^{HE}(\theta_x) = \frac{1}{\sqrt{\pi}} \exp(-(\theta_x - \alpha \operatorname{Re}(\theta_1))^2) \quad (6.27)$$

As is analogized from the case OOK - homodyne, the error probability $P_{e(CO)}^{OOK-HE}$ is given by

$$P_{e(CO)}^{OOK-HE} = \frac{1}{2} \operatorname{erfc}\left(\frac{\sqrt{\eta} \operatorname{Re}(\theta_1)}{2}\right) \quad (6.28)$$

E. OOK Heterodyne Envelope Detection

From Eq.(6.14) the probability density functions $g_0(r)$ and $g_1(r)$ for the signal "0" and "1" are respectively given by

$$g_0(r) = 2r \exp(-r^2) \quad (6.29)$$

$$g_1(r) = 2r \exp(-r^2 - |\alpha\theta_1|) I_0(2r|\alpha\theta_1|) \quad (6.30)$$

Therefore, by a proper approximation given in [18], the error probability $P_{e(EN)}^{OOK-HE}$ becomes :

$$P_{e(EN)}^{OOK-HE} = \frac{1}{2} \exp\left(-\frac{\eta |\theta_1|^2}{4}\right) \quad (6.31)$$

F. FSK Heterodyne Coherent Detection

In FSK, ρ_0 is a state with the frequency w_0 and ρ_1 is a state with w_1 . The transmitted state $\Lambda^*\rho_0$ or $\Lambda^*\rho_1$ is separated by IF(intermediate frequency) dual filter and demodulated by coherent detectors [18]. Here we can consider only the case that the signal "0" is transmitted without loss of generality. Let A and B be the above two coherent detectors, and let θ_A and θ_B be each outcome of A and B, respectively. From Eq.(6.12), the probability density functions $p_{A(CO)}^{HE}(\theta_A)$ and $p_{B(CO)}^{HE}(\theta_B)$ for the outcomes of A and B are respectively given by

$$p_{A(CO)}^{HE}(\theta_A) = \frac{1}{\sqrt{\pi}} \exp(-(\theta_A - \alpha Re(\theta_0))^2) \quad (6.32)$$

$$p_{B(CO)}^{HE}(\theta_B) = \frac{1}{\sqrt{\pi}} \exp(-\theta_B^2) \quad (6.33)$$

The error probability $P_{e(CO)}^{FSK-HE}$ is given by

$$\begin{aligned} P_{e(CO)}^{FSK-HE} &= \text{Prob}(\theta_A - \theta_B < 0) \\ &= \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\theta_{A-B} - \alpha Re(\theta_0))^2}{2}\right) d\theta_{A-B} \\ &= \frac{1}{2} \text{erfc}\left(\frac{\sqrt{\eta} Re(\theta_0)}{\sqrt{2}}\right) \end{aligned} \quad (6.34)$$

where $\theta_{A-B} = \theta_A - \theta_B$.

G. FSK Heterodyne Envelope Detection

Here, we consider the case without loss of generality that the signal "1" is transmitted. From Eq.(6.14) the probability density functions $g_A(r)$ and $g_B(r)$ for the outcomes of the band pass filter "A" and "B" are respectively given by

$$g_A(r_A) = 2r_A \exp(-r_A^2) \quad (6.35)$$

$$g_B(r_B) = 2r_B \exp(-r_B^2 - |\alpha\theta_1|) I_0(2r_B |\alpha\theta_1|) \quad (6.36)$$

Whenever $r_A > r_B$, an error occurs. Thus we can get the following formula

$$\begin{aligned} P_{e(EN)}^{FSK-HE} &= \int_{r_B=0}^{\infty} g_B(r_B) \left(\int_{r_A=r_B}^{\infty} g_A(r_A) dr_A \right) dr_B \\ &= \frac{1}{2} \exp\left(-\frac{\eta |\theta_1|^2}{4}\right), \end{aligned} \quad (6.37)$$

where we applied the approximation given in [18] to this derivation.

H. BPSK Heterodyne Coherent Detection

From Eq.(6.12) the probability density functions $p_{0(CO)}^{HE}(\theta_x)$ and $p_{1(CO)}^{HE}(\theta_x)$ for the signal "0" and "1" are respectively given by

$$p_{0(CO)}^{HE}(\theta_x) = \sqrt{\frac{1}{\pi}} \exp(-(\theta_x - \alpha | \theta |)^2) \quad (6.38)$$

$$p_{1(CO)}^{HE}(\theta_x) = \sqrt{\frac{1}{\pi}} \exp(-(\theta_x + \alpha | \theta |)^2) \quad (6.39)$$

where $| \theta |$ is the amplitude in an input state ρ_0 or ρ_1 .

By an analogy of the case in BPSK - Homodyne, the error probability $P_{e(CO)}^{BPSK-HE}$ is given by

$$P_{e(CO)}^{BPSK-HE} = \frac{1}{2} \operatorname{erfc}(\sqrt{\eta} | \theta |) \quad (6.40)$$

I. BPSK Heterodyne Differential Detection

This system is often called "DPSK". The information is represented by the change of phase between two successive signals. Therefore the signal is demodulated by the product of two successive outcomes, that is, the signal is recognized as "0" when the product is positive and the signal is recognized as "1" when the product is negative. Then the error probability $P_e^{DPSK-HE}$ is given by

$$P_e^{DPSK-HE} = \frac{1}{2} \exp(-\eta | \theta_s |^2) \quad (6.41)$$

using some results obtained in [18].

VII. NUMERICAL RESULTS

Fig.7 shows that each error probability for PPM is smaller than that for PCM at any transmission efficiency η . In this simulation, PCM does not have any error-corrections, however, if PCM has some error-correction, then the relation between them may be opposite to this result. On the other hand, this result tells us that the stronger the effect of squeezing becomes, the better the efficiency becomes. However, in the case of IM-DD, the information is represented by the number of photons contained in each pulse. Therefore it is generally difficult to examine the effect of squeezing for the parameter θ_x . The comparison between PCM and PPM has been already well discussed in our previous paper [14].

Next, let us discuss the efficiency about COC for a coherent input state and a squeezed input state.

In the case that an input state is a coherent state, the results of Fig.8 has the same relation with the numerical results in [24]. But our derivation is entirely different from that of [26,27,28]. The derivation in this paper is so general that we can find the

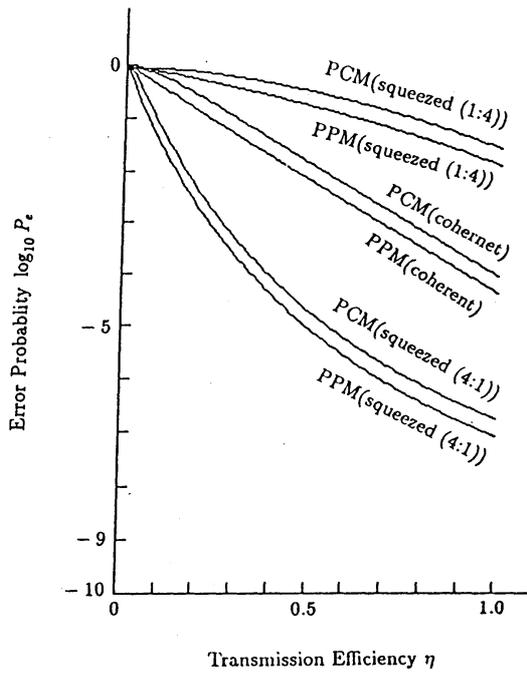


Fig.7 Error probability for IM-DD

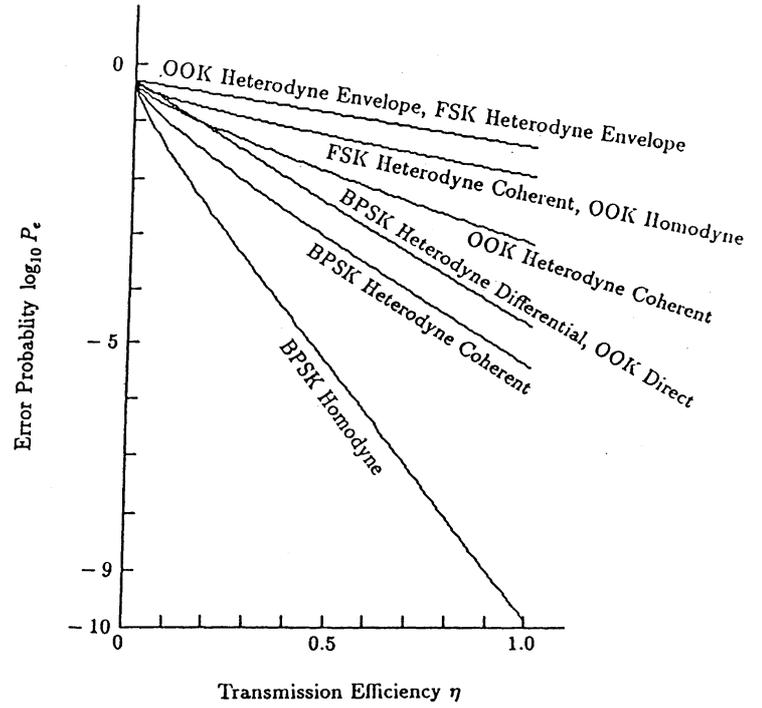


Fig.8 Error probability for each coherent state

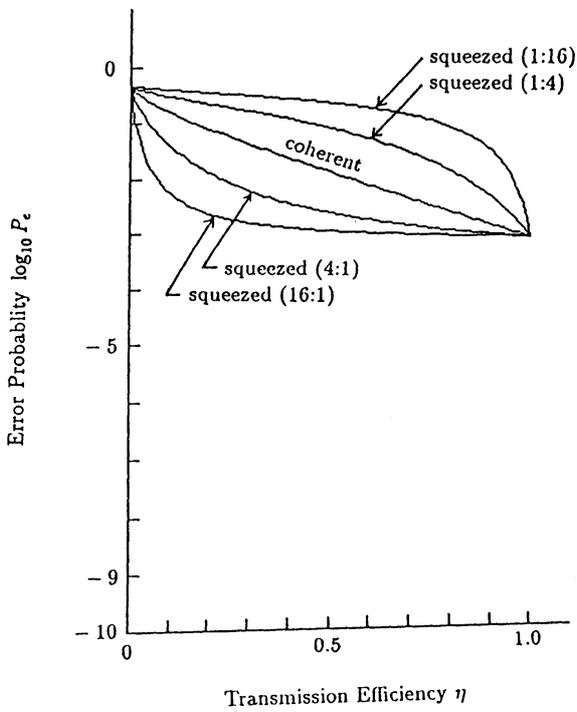


Fig.9 Error probability for OOK homodyne detection

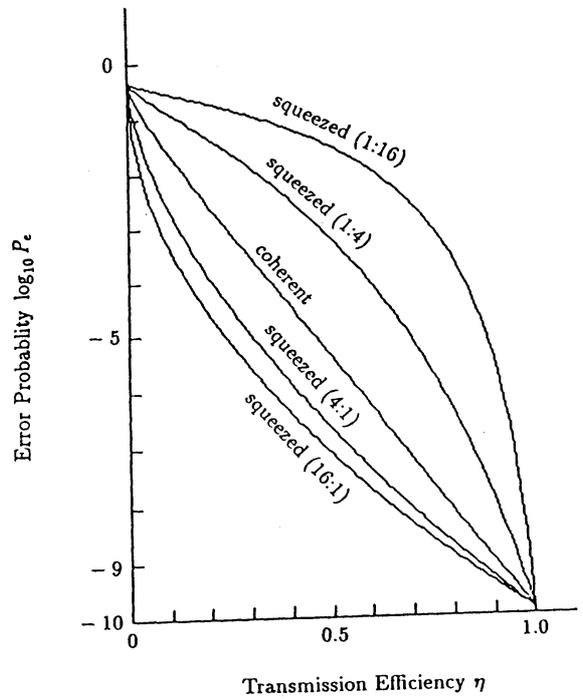


Fig.10 Error probability for BPSK homodyne detection

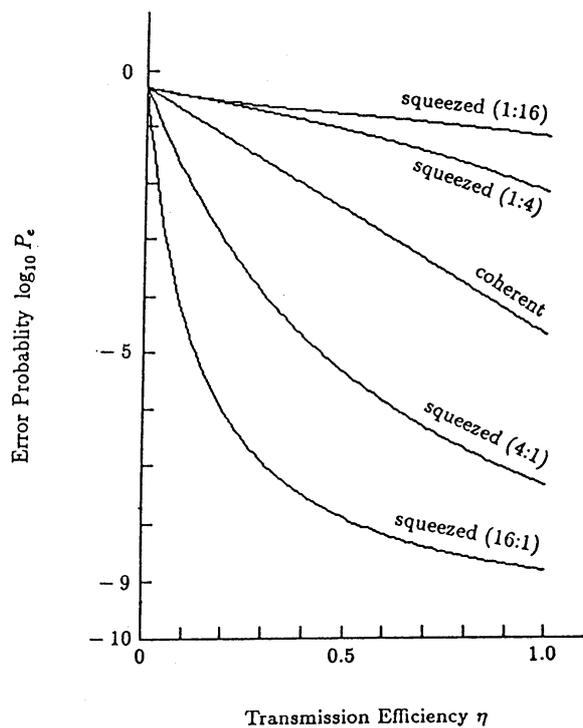


Fig.11 Error probability for OOK direct detection

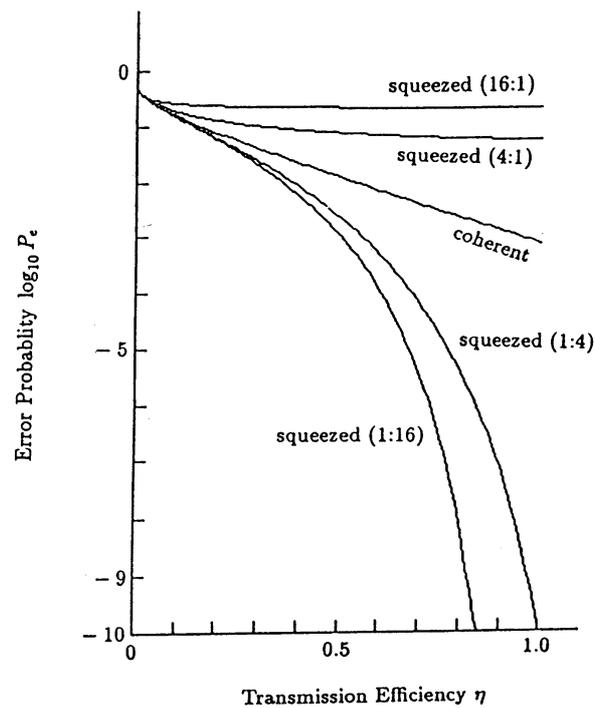


Fig.12 Error probability for OOK homodyne detection

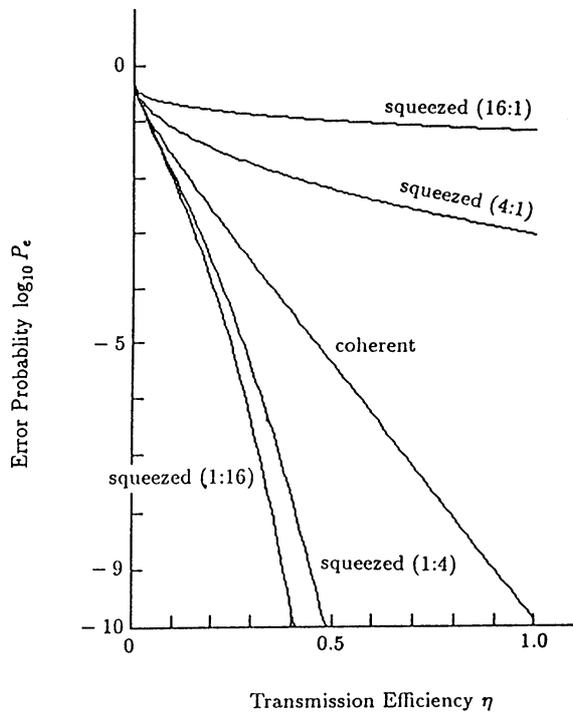


Fig.13 Error probability for BPSK homodyne detection

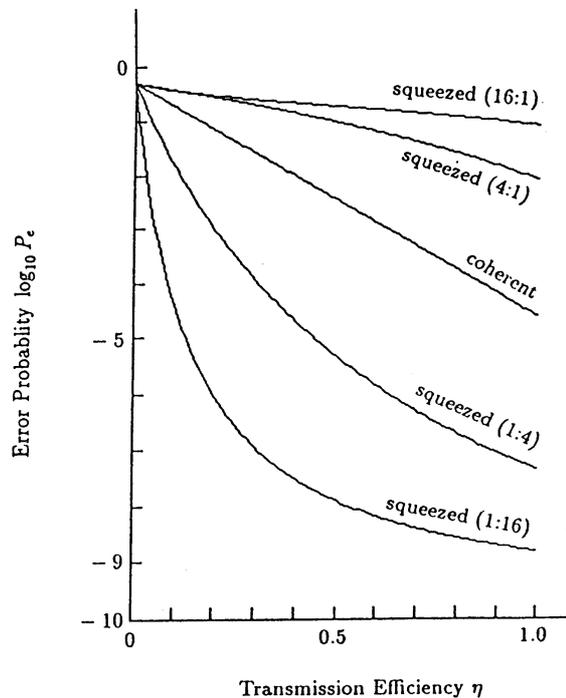


Fig.14 Error probability for OOK direct detection

error probability for a squeezed input state [15,16]. The relation among modulation, detection and demodulation are given by the following (Fig.8):

$$\begin{aligned} \text{modulation} &: P_e^{BPSK} \leq P_e^{FSK} \leq P_e^{OOK} \\ \text{detection} &: P_e^{\text{Homodyne}} \leq P_e^{\text{Heterodyne}}, P_e^{\text{Direct}} \\ \text{demodulation} &: P_e^{\text{Coherent}} \leq P_e^{\text{Differential}} \leq P_e^{\text{Envelope}} \end{aligned}$$

In particular, concerning the detection, it is obvious that the efficiency for homodyne detection is better than that for heterodyne detection because the quantum limits on the homodyne detection is smaller. However we can not compare the efficiency for heterodyne detection and direct detection quantitatively, because the observable for these detections are different from each other.

As we see in Fig.8, the efficiency for BPSK with homodyne detection is the best of all. Therefore, in this paper, we consider this ultimate efficiency for BPSK with homodyne detection, that is, that for a squeezed input state.

In order to study the efficiency for a squeezed input state, we consider two cases for the first setting. One case is that the average number of photons in a coherent state before squeezing is fixed (Fig.9-11). The other case is that the average number of photons in a squeezed state is fixed (Fig.12-14).

From Fig.9-11, we have

$$P_e(16:1) \leq P_e(4:1) \leq P_e(1:1) \leq P_e(1:4) \leq P_e(1:16) \quad (7.1)$$

On the contrary, from Fig.12-14, we have

$$P_e(1:16) \leq P_e(1:4) \leq P_e(1:1) \leq P_e(4:1) \leq P_e(16:1) \quad (7.2)$$

Let us consider the reason why we got the above inequalities. In the former case (Fig.9-11), the squeezing is not effective for the attenuation communication processes. Moreover this result is just opposite to the result expected. This is because the coherent state loses the energy for squeezing if the number of photons in a coherent state before squeezing is fixed. In the case of $\sigma_x : \sigma_y = 1 : 16$ for squeezing the parameter θ_x , we need the highest energy of all cases above. Here we examine the result by changing a squeezed input state in the attenuation processes. We derive the probability density function p_{SQ}^{HO} for an imaginary part θ_y of a complex amplitude as same as Eq.(6.6)

$$\begin{aligned} p_{SQ}^{HO}(\theta_y) &= \text{tr} \mathcal{H}_2 (\text{tr} \mathcal{K}_2 (VU(z) | \theta \rangle \langle \theta | U(z)^* V^*) | \theta_y \rangle \langle \theta_y | \\ &= \text{tr} \mathcal{H}_1 U(z) | \theta \rangle \langle \theta | U(z)^* (V^* (| \theta_y \rangle \langle \theta_y | \otimes I) V) \\ &= \frac{1}{\pi^2} \int \int d^2 v d^2 w \langle U(z) \theta, w \rangle \langle \alpha w, \theta_y \rangle \langle \beta w, \beta v \rangle \\ &\quad \times \langle \theta_y, \alpha v \rangle \langle v, U(z) \theta \rangle \\ &= \frac{1}{\sqrt{2\pi \{ \frac{1}{4} \eta | \mu + \nu |^2 + \frac{1}{4} (1 - \eta) \}}} \exp \left(- \frac{(\theta_y - \alpha \text{Re}((\bar{\mu} + \bar{\nu}) \theta))^2}{2 \{ \frac{1}{4} \eta | \mu + \nu |^2 + \frac{1}{4} (1 - \eta) \}} \right) \quad (7.3) \end{aligned}$$

This probability density function $p_{SQ}^{HO}(\theta_y)$ is again a Gaussian type. Then m'_{SQ} and σ'^2_{SQ} , the average and the variance of this distribution Eq.(7.3), are calculated as

$$m'_{SQ} = \alpha Re((\bar{\mu} + \bar{\nu})\theta), \quad \sigma'^2_{SQ} = \frac{1}{4}\eta |\mu + \nu|^2 + \frac{1}{4}(1 - \eta). \quad (7.4)$$

Therefore, from Eq.(6.7) and Eq.(7.4), the variance of each part of a complex amplitude for a squeezed state in the attenuation processes is given by :

$$\begin{cases} \sigma_x^2 = \frac{1}{4}\eta |\mu - \nu|^2 + \frac{1}{4}(1 - \eta) \\ \sigma_y^2 = \frac{1}{4}\eta |\mu + \nu|^2 + \frac{1}{4}(1 - \eta) \end{cases}$$

This implies that a squeezed state in the attenuation processes is not a minimum uncertainty state. That is, the effect of squeezing is losing in the attenuation processes.

On the other hand, if the number of photons in a squeezed state is fixed, the squeezing is effective for the optical communication. In this case we have only to consider the loss of energy in the attenuation processes, which is shown in Fig.9-14. One of further discussions is to find the optimal method for use of a squeezed state in the optical communications [16].

This paper totally studied rigorous mathematical expressions for quantum communication processes and applied them to derive various error probabilities in a general standing point. This rigorous and general theory contains most of the results shown by many authors such as Yuen and Shapiro [26,27,28].

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