数理解析研究所講究録
第 859 巻 1994 年 113-126

Cat Paradox for C*-dynamical systems

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1. Introduction

In classical mechanics, the indeterminacy on the state of an object results only from the lack of our knowledge about the object, and hence it is resolved by our direct observation, which causes no disturbance of the object. In quantum mechanics, however, we have a different kind of indeterminacy; the indeterminacy of a microscopic object cannot be resolved by our observation, which inevitably disturbs the object and makes new uncertainty about it. This kind of indeterminacy is well delineated by the wave functions as blurring of variables of the object such as the cloud of negative electricity around the nucleus. Moreover, the statistical interpretation of the wave functions provides a clear relationship between the blurring in the microscopic objects and the macroscopic indeterminacy arising in the outcomes of measurements. However, Schrödinger posed a serious suspicion against the coexistence of the microscopic indeterminacy and the macroscopic one. The paradox of Serödinger's cat [3, §5] describes the discrepancy with extreme clarity:

> A cat is penned up in a steel chamber, along with the following diabolical device (which must be secured against direct interference by the cat): in a Geiger counter there is a tiny bit of radioactive substance, *so* small, that *perhaps* in the course of one hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour, one would say that the cat still lives *if* meanwhile no atoms has decayed. The first atomic decay would have poisoned it. The ψ-function of the entire system would express this by having in it the living and the dead (pardon the expression) mixed or smeared out in equal parts.

The simplest but rather schematized way to describe this process would be as follows. Let $\varphi_0 = |\text{nondecay}\rangle$ be the state of the radioactive substance in which no atoms will decay certainly in one hour, and $\varphi_1 = |\text{decay}\rangle$ the state in which at least one atoms will decay certainly in one hour—for simplicity, possibly huge degeneracy of eigenstates are neglected hereafter. Let U be the unitary operator for the time evolution of the isolated radioactive substance for one hour. Let $\varphi_0' = U\varphi_0$ and $\varphi_1' = U\varphi_1$. Let $\xi_0 = |\text{alive}\rangle$, the state of the cat alive, and $\xi_1 = |\text{dead}\rangle$, the state of the cat dead. Then the process in the steel chamber in one hour could be described as

$$\begin{aligned} \varphi_0 \otimes \xi_0 &\rightarrow \varphi_0' \otimes \xi_0, \\ \varphi_1 \otimes \xi_0 &\rightarrow \varphi_1' \otimes \xi_1. \end{aligned} \tag{1.1}$$

Since the decay will happen fifty percent of chance, the initial state φ of the substance is, for instance, $\varphi = (1/\sqrt{2})(\varphi_0 + \varphi_1)$—where φ_0 or φ_1 can be multiplied by any phase factor in general. Then by the linearity of the Schrödinger equation we have the following time evolution in the steel chamber

$$\frac{1}{\sqrt{2}}(\varphi_0 + \varphi_1) \otimes \xi_0 \rightarrow \frac{1}{\sqrt{2}}(\varphi_0' \otimes \xi_0 + \varphi_1' \otimes \xi_1). \tag{1.2}$$

The state of the whole system $\psi = (1/\sqrt{2})(\varphi_0' \otimes \xi_0 + \varphi_1' \otimes \xi_1)$ has indeed in it the living cat ξ_0 and the dead cat ξ_1 mixed or smeared out in equal parts.

Once the life of the cat has correlated with a microscopic system, we can no longer observe the cat without disturbing the whole system of the steel chamber. When we see the cat alive, we find the steel chamber in state $\varphi_0' \otimes \xi_0$, while when the cat dead, in $\varphi_1' \otimes \xi_1$. Since these cases happen fifty percent each, our interaction with the cat for the observation, e.g., shedding a light on the cat, changes the state of the steel chamber from the superposition $\psi = (1/\sqrt{2})(\varphi_0' \otimes \xi_0 + \varphi_1' \otimes \xi_1)$ to the mixture represented by the density operator

$$\rho = \frac{1}{2}(|\varphi_0' \otimes \xi_0\rangle\langle\varphi_0' \otimes \xi_0| + |\varphi_1' \otimes \xi_1\rangle\langle\varphi_1' \otimes \xi_1|).$$

This contradicts our fundamental belief that the macroscopic objects like cats can be observed without any disturbance. Since our macroscopic world is naturally interacting with microscopic objects, our every glance at any macroscopic object thus causes a change of the state of our whole physical world.

Although there have been many attempts to resolve this paradox, we have never had a generally acceptable solution. Based on the understanding of the difficulty in treating the problem within quantum mechanics, the recent interest appears to focus on the unification of classical mechanics and quantum mechanics (e.g., [1]). The unification of those two mechanics is expected to provide a new formalism of describing the process of measurement in which the measured object is described microscopically, but the measuring instrument is described macroscopically. A basic idea in this approach is to introduce the macroscopic observables, which are postulated to be observed without disturbing the system, and the goal is to describe the dynamics which explains how the information about the object is transmitted to values of a macroscopic observable in the measuring apparatus. If this is realized, we can obtain the information about the microscopic object by observing the macroscopic observable without making any further disturbance of the physical world. This solves the paradox. At the same time, this would realize Bohr's philosophy concerning the relationship between quantum mechanics and classical mechanics, usually called the Copenhagen interpretation; we will refer to this expecting approach as the *Copenhagen approach.*

In this paper, this program is examined mathematically and is shown to be unable to solve the problem as long as the dynamics of the isolated system is postulated to be reversible in time. Thus in the expecting unified framework of quantum mechanics and classical mechanics even an isolated system, if it consists of a quantum system and a classical system as interacting parts, will not obey the energy conservation law probably because of the huge difference of the scale parameters—and hence it is anticipated that for retaining the energy conservation law we need to extend the number system so as to describe such infinitely large scale differences.

2. Copenhagen approach to the Cat Paradox

In this section, we will describe the basic features of the Copenhagen approach. In this approach, we will still have concepts of observables, states, time evolutions, and interactions as in quantum mechanics or classical mechanics. We will set forth the postulates for the basic notions in a similar way with the Heisenberg picture in quantum mechanics.

2.1. Observables

Consider an isolated physical system $\mathbf{S}$ to be described. Let $\mathcal{Q}$ be the set of physical quantities of the system $\mathbf{S}$. The set $\mathcal{Q}$ should contain enough quantities to derive all detectable predictions, but does not need to be exhaustive; among others the set of bounded quantities is a mathematically tractable choice. We assume thus that $\mathcal{Q}$ can be identified with a set of bounded operators on a Hilbert space $\mathcal{H}$ so that the mathematical structure of $\mathcal{Q}$ is represented by that of the operators. It is postulated that $\mathcal{Q}$ has the identity operator 1 and is closed under the following operations: the addition, the scalar multiplication by the complex numbers, the product, and the making adjoint. Thus, if A, B are in $\mathcal{Q}$ then $\alpha A + \beta B$ (α, β are complex numbers), AB, and $A^\dagger$ (the adjoint of A) are also in $\mathcal{Q}$—in mathematical terminology, $\mathcal{Q}$ is a *-algebra of bounded operators on $\mathcal{H}$. A quantity A is called an *observable* if $A = A^\dagger$. Denote by $\mathcal{O}$ the set of all observables of the system $\mathbf{S}$. The observables are quantities to be measured by an *instantaneous* measurement. Every quantity A is written as $A = A_1 + iA_2$ with two observables A_1 and A_2 by putting $A_1 = (1/2)(A + A^\dagger)$ and $A_2 = (1/2i)(A - A^\dagger)$.

2.2. States

If we know that the system is in a specific state, we can predict the expectations of the outcomes of any instantaneous measurement in that state; hence the states determine the expectations. Denote the expectation of the outcome of a measurement of an observable A in a state ρ by $\mathrm{Ex}[A|\rho]$. Then the following relations should hold:

$$\mathrm{Ex}[\alpha A + \beta B|\rho] = \alpha \mathrm{Ex}[A|\rho] + \beta \mathrm{Ex}[B|\rho], \tag{2.3}$$

$$\mathrm{Ex}[A^2|\rho] \geq 0, \tag{2.4}$$

$$\mathrm{Ex}[1|\rho] = 1, \tag{2.5}$$

for any A, B in $\mathcal{O}$, and α, β in $\mathbf{R}$. Any function $A \mapsto \mathrm{Ex}[A|\rho]$ on $\mathcal{O}$ satisfying Eqs. (2.3)–(2.5) is called a normalized positive linear functional. In one approach, we can postulate that every normalized positive linear functional is realized as the expectation in a state. However, we will keep our option slightly open; the set of states is chosen depending on the choice of the Hilbert space $\mathcal{H}$, so that the states are determined by the vectors. Thus, we postulate that for any vector $\psi \in \mathcal{H}$ with unit length there is a state ρ such that

$$\mathrm{Ex}[A|\rho] = \langle \psi | A | \psi \rangle \tag{2.6}$$

for all $A \in \mathcal{O}$, and any state arises in this way. In this case, we will also write $\mathrm{Ex}[A|\psi] = \mathrm{Ex}[A|\rho]$. If the system is prepared in two different ways for which the physical laws lead to the same predictions of the expectations of the outcomes of measurements, these two preparations are attributed to the same state. Thus we say that two states ρ and ρ' are identical, if $\mathrm{Ex}[A|\rho] = \mathrm{Ex}[A|\rho']$ for all $A \in \mathcal{O}$. Note that even two orthogonal vectors may happen to determine the same state, when $\mathcal{Q}$ does not exhaust the all bounded operators.

For any quantity $A = A_1 + iA_2$ with $A_1,\ A_2 \in \mathcal{O}$, define $\mathrm{Ex}[A|\rho] = \mathrm{Ex}[A_1|\rho] + i\mathrm{Ex}[A_2|\rho]$; this cannot be interpreted as the expectation of the outcome of an instantaneous measurement, but can be interpreted as the ensemble average of the outcomes of the measurement of A which is carried out by the A_1-measurement and the A_2-measurement for the respective parts of the identically prepared ensemble of the identical systems.

2.3. Probability distributions

Since the states determine the expectations, they also determine the probability distributions of the outcomes of *instantaneous* measurements. Let ρ be a state of the system. Then for any observable A, and real polynomial $p(x)$, we have the expectation $\mathrm{Ex}[p(A)|\rho]$. By the Weierstrass approximation theorem and the Riez representation theorem, we have a unique right-continuous monotone increasing function $F(x)$ on $\mathbf{R}$ such that

$$\mathrm{Ex}[p(A)|\rho] = \int_{-\infty}^{\infty} p(\lambda)\, dF(\lambda). \tag{2.7}$$

By the basic postulate of probability theory, this shows that $F(\lambda)$ is the distribution function of the random variable A, and hence we put $\Pr[A \le \lambda|\rho] = F(\lambda)$ for the probability distribution of A. The probability distribution is determined independently of our choice of the Hilbert space $\mathcal{H}$.

Suppose that the state ρ corresponds to a vector $\psi \in \mathcal{H}$. Let $\{E_\lambda\}_{\lambda=-\infty}^{\infty}$ be the family of spectral projections on $\mathcal{H}$ of an observable $A \in \mathcal{O}$, then by the spectral theorem we have

$$\Pr[A \le \lambda|\rho] = \langle\psi|E_\lambda|\psi\rangle, \tag{2.8}$$

for any $\lambda \in \mathbf{R}$. Thus when $E_\lambda \in \mathcal{O}$, we have

$$\Pr[A \le \lambda|\rho] = \mathrm{Ex}[E_\lambda|\rho] \tag{2.9}$$

for any states ρ. Note that the spectral projections depend on the Hilbert space $\mathcal{H}$, and hence if we postulate only Eq. (2.3)–(2.5) but not Eq. (2.6), then Eq. (2.9) may fail to hold even if $E_\lambda \in \mathcal{O}$.

The procedure of determining the probability distributions in this subsection can be extended to the determination of the joint probability distributions of several mutually commutable observables $A_1, \ldots, A_k$, using the expectations $\mathrm{Ex}[p(A_1, \ldots, A_k)|\rho]$ for the polynomials p with several real variables; the resulting probability distributions are interpreted as the joint probability distributions of outcomes of simultaneous measurements of those observables.

2.4. Time evolutions

Once we know the state of the system at $t = 0$, we are supposed to be able to predict the expectation of the outcome of a measurement of any observable at any later time $t = \tau$ as long as the system is isolated from time $t = 0$ to $t = \tau$; both classical mechanics and quantum mechanics satisfy this requirement as a consequence of the energy conservation law. Thus the new theory will provide a law of time evolution of the system by which we can attribute at time t the element $A(t)$ in $\mathcal{O}$ to the observable A in such a way that the expectation of a measurement of the observable A at time t is given by $\mathrm{Ex}[A(t)|\rho]$. Obviously we should put $A(0) = A$. If we keep the principle of the reversibility of the time evolution of the isolated system, it is natural to assume that the correspondence $A(0) \mapsto A(t)$ is a one-to-one onto mapping of $\mathcal{O}$ and preserves the algebraic relations defined in $\mathcal{O}$; if $A(0) = \alpha B(0) + \beta C(0)$ then $A(t) = \alpha B(t) + \beta C(t)$, and if $A(0) = B(0)^2$ then $A(t) = B(t)^2$.

Remark. Define a map $h : \mathcal{Q} \to \mathcal{Q}$ by $h(A) = A_1(t) + iA_2(t)$ for $A = A_1 + iA_2$ with $A_1,\ A_2 \in \mathcal{O}$. Then h is a Jordan homomorphism, and from [2, Proposition 3.2.2] there is a projection $E \in \mathcal{Q}' \cap \mathcal{Q}''$, where the symbol $'$ means "the commutant of", such that $A \mapsto h(A)E$ is a homomorphism and $A \mapsto h(A)(1 - E)$ is an antihomomorphism. Thus, in particular, if $AB = BA$ then $A(t)B(t) = B(t)A(t)$.

2.5. Interactions

The interactions between two systems are described by the time evolution of the composite system of them. Consider a pair of systems $\mathbf{S}_{\mathrm{I}}$ and $\mathbf{S}_{\mathrm{II}}$. In order to describe the observables of their combined system $\mathbf{S}_{\mathrm{I+II}}$, consider the tensor-product Hilbert space $\mathcal{H}_{\mathrm{I+II}} = \mathcal{H}_{\mathrm{I}} \otimes \mathcal{H}_{\mathrm{II}}$. Then both systems $\mathbf{S}_{\mathrm{I}}$ and $\mathbf{S}_{\mathrm{II}}$ are represented on $\mathcal{H}_{\mathrm{I+II}}$, if any observable $A \in \mathcal{O}_{\mathrm{I}}$ on $\mathcal{H}_{\mathrm{I}}$ is replaced by $A \otimes 1$ on $\mathcal{H}_{\mathrm{I+II}}$, and if any observable $B \in \mathcal{O}_{\mathrm{II}}$ by $1 \otimes B$ on $\mathcal{H}_{\mathrm{I+II}}$. The set $\mathcal{O}_{\mathrm{I+II}}$ of observables of the composite system $\mathbf{S}_{\mathrm{I+II}}$ is thus contains all observables $A \otimes 1$ with $A \in \mathcal{O}_{\mathrm{I}}$ and all observables $1 \otimes B$ with $B \in \mathcal{O}_{\mathrm{II}}$. The *-algebra $\mathcal{Q}_{\mathrm{I}} \otimes \mathcal{Q}_{\mathrm{II}}$ generated by those operators is all bounded operators of the form $\sum_{i=1}^{n} A_i \otimes B_i$ with $A_i \in \mathcal{Q}_{\mathrm{I}}$ and $B_i \in \mathcal{Q}_{\mathrm{I}}$. However, a complicated interaction may not be described by an Jordan automorphism of this

*-algebra. Thus our optional postulate is the following: The set $\mathcal{Q}_{\mathrm{I+II}}$ of physical quantities for the system $\mathbf{S}_{\mathrm{I+II}}$ consisting of two interacting systems $\mathbf{S}_{\mathrm{I}}$ and $\mathbf{S}_{\mathrm{II}}$ is a *-algebra of bounded operators on $\mathcal{H}_{\mathrm{I+II}}$ containing $\mathcal{Q}_{\mathrm{I}} \otimes \mathcal{Q}_{\mathrm{II}}$ and contained in the von Neumann algebra $(\mathcal{Q}_{\mathrm{I}} \otimes \mathcal{Q}_{\mathrm{II}})''$ on $\mathcal{H}_{\mathrm{I+II}}$ generated by $\mathcal{Q}_{\mathrm{I}} \otimes \mathcal{Q}_{\mathrm{II}}$.

2.6. Classical mechanics

Now, we will show that this formulation includes classical mechanics in this subsection and quantum mechanics in the next. In classical mechanics, for any system $\mathbf{S}$ we are given the phase space Ω of the system. The set $\mathcal{Q}$ of physical quantities can be given by the algebra $C_b(\Omega)$ of all bounded continuous functions on Ω. The states are usually determined by the probability measures on Ω. For any probability measure μ, let $L^2(\Omega, \mu)$ be the Hilbert space of μ-square-integrable functions on Ω. Then the quantity $f \in \mathcal{Q}$ is represented by the multiplication operator L_f on $L^2(\Omega, \mu)$ by the relation $(L_f g)(x) = f(x)g(x)$ for $g \in L^2(\Omega, \mu)$. In this case, the expectation with respect to probability measure μ is represented by

$$\mathrm{Ex}[f|\mu] = \langle 1|L_f|1\rangle = \int_{\Omega} f(x)\, d\mu(x),$$

where $1 \in L^2(\Omega, \mu)$ is the constant function with the value $1 \in \mathbf{R}$. In probability theory, the conditional expectation plays an important role. The conditional expectation of f given condition $g \le \lambda$ in the probability measure μ is given by

$$\mathrm{Ex}[f|g \le \lambda, \mu] = \frac{1}{\mu(\{x|g(x) \le \lambda\})} \int_{\{x|g(x)\le\lambda\}} f(x')d\mu(x').$$

In this case, there is a state which represents this conditional expectation. Let $\xi \in L^2(\Omega, \mu)$ be such that $\xi(x') = \mu(\{x|g(x) \le \lambda\})^{-1/2}$ if $g(x') \le \lambda$, and that $\xi(x') = 0$ otherwise. Then we have

$$\mathrm{Ex}[f|g \le \lambda, \mu] = \langle \xi|L_f|\xi\rangle.$$

Let S be a set of probability measures. A Hilbert space $\mathcal{H}$ with which every probability measure in S is realized as a state of the system $\mathbf{S}$ is the direct sum $\mathcal{H} = \sum_{\mu\in S}^{\oplus} L^2(\Omega, \mu)$, on which $f \in \mathbf{Q}$ is represented by the operator L_f defined by

$$(L_f \textstyle\sum^{\oplus} g_\mu) = \sum^{\oplus} f g_\mu,$$

where $g_\mu \in \mathrm{L}^2(\Omega, \mu)$ with $\sum_{\mu\in S} \|g_\mu\|^2 < \infty$. The time evolution is usually described by orbits $x(t; a)$ for any initial conditions $x(0; a) = a$. This induces the time evolution of observables $f \in \mathcal{Q}$,

$$f_t(a) = f(x(t; a)), \quad \text{and} \quad L_f(t) = L_{f_t}.$$

If the operators L_f are represented on the Hilbert space $\mathcal{H} = L^2(\Omega, m)$ of the square Lebesgue-integrable functions, the orbits $x(t; a)$ defines the one-parameter group of unitary operatos U_t on $\mathcal{H}$ by

$$U_t g(a) = g(x(t; a)),$$

for all $g \in L^2(\Omega, m)$. Then we have $L_f(t) = U_{-t} L_f U_t$. In $\mathcal{H} = L^2(\Omega, m)$, every density function ρ on Ω gives a state vector $\sqrt{\rho} \in \mathcal{H}$, i.e.,

$$\mathrm{Ex}[f|\rho] = \langle\sqrt{\rho}|L_f|\sqrt{\rho}\rangle.$$

2.7. Quantum mechanics

In quantum mechanics, $\mathcal{Q}$ is the set of all bounded operators on $L^2(\Gamma, m)$, where Γ is the configuration space and m is the Lebesgue measure. Then $\mathcal{O}$ is the set of bounded self-adjoint operators. The states are given by state vectors $\xi \in \mathcal{H}$, and the expectation is given by $\mathrm{Ex}[A|\xi] = \langle\xi|A|\xi\rangle$. The probability distribution is given by $\Pr[A \leq \lambda|\xi] = \langle\xi|E_\lambda|\xi\rangle$, where E_λ is the spectral family of A. In quantum statistical mechanics, the states are give by the density operators ρ on $L^2(\Gamma, m)$. In this case, we take the Hilbert space $\mathcal{H}$ as the Hilbert space HS of all Hilbert-Schmidt operators on $L^2(\Gamma, m)$. Then $\mathcal{Q}$ is represented as the set of all operators L_A on HS such that $L_A X = AX$ (multiplication by A from left) for all $X \in HS$. Then inner product on HS is defined by $\langle X|Y\rangle = \mathrm{Tr}[X^\dagger Y]$. For any density operator ρ on $\mathrm{L}^2(\Gamma, m)$, the square root $\sqrt{\rho}$ is in HS and we have

$$\mathrm{Ex}[A|\rho] = \mathrm{Tr}[\sqrt{\rho} A \sqrt{\rho}] = \langle\sqrt{\rho}|L_A|\sqrt{\rho}\rangle.$$

3. Macroscopic observables

Let us examine how the macroscopic nature of an object is characterized in this formalism. Our basic requirement for this matter is that *we can measure an observable of a macroscopic object without disturbing the system.* Let A be an observable of a system **S**, and suppose that A can be measured without disturbing the system. Consider an ensemble $\mathcal{E}$ of the systems identical with **S** and prepared uniformly in a state σ. Suppose that we measure A for each system in $\mathcal{E}$ once, and divide the ensemble $\mathcal{E}$ into two parts $\mathcal{E}_1$ and $\mathcal{E}_2$: $\mathcal{E}_1$ is the ensemble of the systems such that the outcome of the measurement is $\leq \lambda$, and $\mathcal{E}_2$ is such that the outcome is $> \lambda$, where λ is an arbitrarily fixed number such that $0 < \Pr[A \leq \lambda|\sigma] <$. Then each system chosen randomly from $\mathcal{E}_1$ just after the measurement is uniformly prepared in a state σ_1 satisfying

$$\Pr[A \leq \lambda|\sigma_1] = 1, \tag{3.10}$$

and each system from $\mathcal{E}_2$ in a state σ_2 satisfying

$$\Pr[A > \lambda|\sigma_2] = 1. \tag{3.11}$$

Now suppose that the system $\mathcal{S}$ is in a state σ' just after the measurement, and that we measure another observable X in this state. Since, the system $\mathcal{S}$ is chosen from $\mathcal{E}_1$ with probability $\Pr[A \leq \lambda|\sigma]$ (say, $= p_1$), and from $\mathcal{E}_2$ with probability $\Pr[A > \lambda|\sigma]$ (say, $= p_2$), the expectation of the outcome of this measurement of A satisfies

$$\mathrm{Ex}[X|\sigma'] = p_1\mathrm{Ex}[X|\sigma_1] + p_2\mathrm{Ex}[X|\sigma_2].$$

Since the measurement of A does not disturb the system $\mathbf{S}$, the state σ' is identical with the state σ, and hence $\mathrm{Ex}[X|\sigma'] = \mathrm{Ex}[X|\sigma]$. Thus we have reached the relation

$$\mathrm{Ex}[X|\sigma] = \Pr[A \leq \lambda|\sigma]\mathrm{Ex}[X|\sigma_1] + \Pr[A > \lambda|\sigma]\mathrm{Ex}[X|\sigma_2]. \tag{3.12}$$

for all $X \in \mathcal{O}$.

Motivated by the above consideration, we say that an observable $A \in \mathcal{O}$ is *macroscopic*, if for any state σ of the system $\mathbf{S}$ there are states σ_1 and σ_2 satisfying relations (3.10)–(3.12) for all $X \in \mathcal{O}$ and $\lambda \in \mathbf{R}$ with $0 < \Pr[A \leq \lambda|\sigma] < 1$. Thus any observables which we can measure without disturbing the system are macroscopic. Then we can prove the following characterization of macroscopic observables.

Theorem 3.1. *An observable A in $\mathcal{O}$ is an macroscopic observable if and only if A commutes with all observables in $\mathcal{O}$.*

Proof. Let $A \in \mathcal{O}$ be an observable with the spectral family $\{E_\lambda\}_{-\infty}^{\infty}$ on $\mathcal{H}$. First, suppose that $[A, X] = 0$ for all $X \in \mathcal{O}$. Let σ be a state with vector $\xi \in \mathcal{H}$, and $\lambda \in \mathbf{R}$ with $0 < \Pr[A \leq \lambda|\sigma] < 1$. Let $\xi_1 = E_\lambda\xi/\|E_\lambda\xi\|$, and $\xi_2 = (1-E_\lambda)\xi/\|(1-E_\lambda)\xi\|$. Define states σ_1 and σ_2 as the sates determined by ξ_1 and ξ_2, respectively. Then it is easy to check that relations (3.10)–(3.12) hold for σ, σ_1, and σ_2. Conversely, suppose that A is macroscopic. Let $\xi \in \mathcal{H}$ be an arbitrary vector with unit length, and σ the state of the system $\mathbf{S}$ corresponding to ξ. Let σ_1 and σ_2 be states satisfying relations (3.10)–(3.12) for σ. Let ξ_i $(i = 1, 2)$ be a vector in $\mathcal{H}$ corresponding to σ_i. Let $\lambda \in \mathbf{R}$ with $0 < \|E_\lambda\xi\| < 0$. Then from relation (3.12),

$$\langle\xi|Y|\xi\rangle = \langle\xi|E_\lambda|\xi\rangle\langle\xi_1|Y|\xi_1\rangle + \langle\xi|1 - E_\lambda|\xi\rangle\langle\xi_2|Y|\xi_2\rangle, \tag{3.13}$$

for all $Y = X \in \mathcal{O}$, and hence Eq. (3.13) still holds for all $Y \in \mathcal{O}''$. From Eqs. (3.10) and (3.11),

$$\langle \xi_1 | E_\lambda | \xi_1 \rangle = 1 \quad \text{and} \quad \langle \xi_2 | E_\lambda | \xi_2 \rangle = 0,$$

whence $E_\lambda \xi_1 = \xi_1$ and $E_\lambda \xi_2 = 0$. Let $X \in \mathcal{O}$. Then, from Eq. (3.13) with $Y = XE_\lambda \in \mathcal{O}''$,

$$\begin{aligned} \langle \xi | XE_\lambda | \xi \rangle &= \langle \xi | E_\lambda | \xi \rangle \langle \xi_1 | XE_\lambda | \xi_1 \rangle + \langle \xi | 1 - E_\lambda | \xi \rangle \langle \xi_2 | XE_\lambda | \xi_2 \rangle \\ &= \langle \xi | E_\lambda | \xi \rangle \langle \xi_1 | X | \xi_1 \rangle. \end{aligned}$$

Similarly,

$$\langle \xi | E_\lambda X | \xi \rangle = \langle \xi | E_\lambda | \xi \rangle \langle \xi_1 | X | \xi_1 \rangle.$$

Thus $\langle \xi | XE_\lambda | \xi \rangle = \langle \xi | E_\lambda X | \xi \rangle$, so that, by the arbitrariness of ξ and λ, we have $[X, A] = 0$. □

4. Measuring processes

Now we consider the interaction between an object system $\mathbf{S}_\mathrm{I}$ and a measuring apparatus $\mathbf{S}_\mathrm{II}$. Suppose that we measure an observable A in $\mathcal{O}_\mathrm{I}$, and that the measuring interaction transmits the information about A to an observable B in $\mathcal{S}_\mathrm{II}$. The interaction is turned on from time $t = 0$ to $t = \tau$, and the combined system $\mathbf{S}_\mathrm{I+II}$ is isolated during this time interval. The condition for this measurement to give the correct outcome is that if $A(0) \le \lambda$ with probability p then the interaction realizes $B(\tau) \le \lambda$ with probability p. Now, we say that an observable A of a system $\mathbf{S}_\mathrm{I}$ is *transduced* to an observable B of another system $\mathbf{S}_\mathrm{II}$ in state σ_0 by the interaction from $t = 0$ to $t = \tau$, if we have

$$\Pr[A(0) \le \lambda | \sigma \otimes \sigma_0] = \Pr[B(\tau) \le \lambda | \sigma \otimes \sigma_0], \tag{4.1}$$

for any state σ of the system $\mathbf{S}_\mathrm{I}$. An observable A is called *transduceable* to an observable B of $\mathbf{S}_\mathrm{II}$, if there is an interaction between $\mathbf{S}_\mathrm{I}$ and $\mathbf{S}_\mathrm{II}$, and there is a state σ_0 of $\mathbf{S}_\mathrm{II}$ such that A is transduced to B by them.

In the conventional approach, the Schrödinger picture is usually adopted for description of measuring processes. The argument runs as follows [5, Ch. VI]. Suppose that an observable $A = \sum_n a_n |\varphi_n\rangle\langle\varphi_n|$ of a quantum system $\mathbf{S}_\mathrm{I}$ is measured by an apparatus $\mathbf{S}_\mathrm{II}$ which is also described as a quantum system. The measurement is carried out by the interaction between these two system from time $t = 0$ to $t = \tau$, and just after the interaction the pointer observable $B = \sum_n b_n|\xi_n\rangle\langle\xi_n|$ of the system $\mathbf{S}_{\mathrm{II}}$ is observed by the observer. The system $\mathbf{S}_{\mathrm{II}}$ is supposed to be prepared in a vector state $\xi \in \mathcal{H}_{\mathrm{II}}$ at the time of measurement $(t = 0)$. Then the interaction is usually supposed to satisfy the condition

$$\varphi_n \otimes \xi \rightarrow \varphi_n \otimes \xi_n. \tag{4.2}$$

Thus if $A(0) = a_n$ with probability 1 then $B(\tau) = b_n$ with probability 1. Let the initial state of the system $\mathbf{S}_{\mathrm{I}}$ be the superposition $\varphi = \sum_n c_n\varphi_n$. Then $A(0) = a_n$ with probability $|c_n|^2$. By the linearity of the time evolution we have

$$\big(\sum_n c_n\varphi_n\big) \otimes \xi \rightarrow \sum_n c_n\varphi_n \otimes \xi_n,$$

and hence we have $B(\tau) = b_n$ with probability $|c_n|^2$. Without any loss of generality we can assume $a_n = b_n$. Then we have

$$\Pr[A(0) \leq \lambda|\varphi \otimes \xi] = \Pr[B(\tau) \leq \lambda|\varphi \otimes \xi],$$

for all $\lambda \in \mathbf{R}$. Thus the interaction (4.2) transduces A to B of $\mathbf{S}_{\mathrm{II}}$ in ξ, and hence our condition is satisfied in the standard description of a measuring process. Note that more general interactions satisfying, instead of Eq. (4.2), the condition

$$\varphi_n \otimes \xi \rightarrow \varphi'_n \otimes \xi_n, \tag{4.3}$$

for an arbitrary sequence $\{\varphi'_n\}$, also satisfies Eq. (4.1), i.e., transduces A to B of $\mathbf{S}_{\mathrm{II}}$ in ξ. This type of measurements do not necessarily satisfy the so-called von Neumann's repeatability hypothesis, but satisfy the Born statistical formula for outcomes of measurements.

The presumable goal of the Copenhagen approach is to find a measuring interaction which transduces the object observable A to a *macroscopic* observable of B of a measuring apparatus. If this is achieved, Schrödinger's cat paradox is resolved, since we can observe the macroscopic observable B just after the measuring interaction *without* any disturbance of the combined system. Unfortunately, this program cannot be accomplished at all.

Theorem 4.1. *If an observable A of a system $\mathbf{S}_{\mathrm{I}}$ is transduceable to a macroscopic observable of another system $\mathbf{S}_{\mathrm{II}}$, then A is also a macroscopic observable.*

Proof. Suppose that an observable A of a system $\mathbf{S}_{\mathrm{I}}$ is transduced to a macroscopic observable B of a system $\mathbf{S}_{\mathrm{II}}$ in state σ_0 by an interaction from $t = 0$ to $t = \tau$.

Let $\xi \in \mathcal{H}_{\mathrm{II}}$ be a vector corresponding to σ_0. Let $V : \mathcal{H}_{\mathrm{I}} \to \mathcal{H}_{\mathrm{I}} \otimes \mathcal{H}_{\mathrm{II}}$ be the isometry defined by $V\varphi = \varphi \otimes \xi$ for any $\varphi \in \mathcal{H}_{\mathrm{I}}$. Then the map $\mathcal{E} : \mathcal{L}(\mathcal{H}_{\mathrm{I}} \otimes \mathcal{H}_{\mathrm{II}}) \to \mathcal{L}(\mathcal{H}_{\mathrm{I}})$ defined by $\mathcal{E}[X] = V^* X V$ for all $X \in \mathcal{L}(\mathcal{H}_{\mathrm{I}} \otimes \mathcal{H}_{\mathrm{II}})$ satisfies the relation

$$X\mathcal{E}[W(Y \otimes 1)] = \mathcal{E}[(X \otimes 1)W(Y \otimes 1)] = \mathcal{E}[(X \otimes 1)W]Y, \tag{4.4}$$

for all $X,\ Y \in \mathcal{L}(\mathcal{H}_{\mathrm{I}})$, and $W \in \mathcal{L}(\mathcal{H}_{\mathrm{I}} \otimes \mathcal{H}_{\mathrm{II}})$. By Eq. (4.1) we have

$$\mathrm{Ex}[A(0)|\sigma \otimes \sigma_0] = \mathrm{Ex}[B(\tau)|\sigma \otimes \sigma_0]. \tag{4.5}$$

It follows that

$$\langle \varphi | A(0) | \varphi \rangle = \langle \varphi | \mathcal{E}[B(\tau)] | \varphi \rangle,$$

for all $\varphi \in \mathcal{H}_{\mathrm{I}}$, and hence $A = \mathcal{E}[B(\tau)]$. Let $X \in \mathcal{O}_{\mathrm{I}}$. By Theorem 3.1, the macroscopic observable B commutes with all observables in $\mathcal{O}_{\mathrm{II}}$, and hence $B(0) = 1 \otimes B$ commutes with all observables in $\mathcal{O}_{\mathrm{I+II}}$. It follows from Remark in Subsection 2.4 that $B(\tau)$ commute with all observables in $\mathcal{O}_{\mathrm{I+II}}$. Thus, by Eq. (4.4), for any $X \in \mathcal{O}_{\mathrm{I}}$

$$\begin{aligned} XA &= X\mathcal{E}[B(\tau)] \\ &= \mathcal{E}[(X \otimes 1)B(\tau)] \\ &= \mathcal{E}[B(\tau)(X \otimes 1)] \\ &= \mathcal{E}[B(\tau)]X \\ &= AX. \end{aligned}$$

This concludes that A is a macroscopic observable. □

Remark. From the above proof, the assertion of the theorem can be strengthened as follows. *Let B be a macroscopic observable, and σ_0 a state of* $\mathbf{S}_{\mathrm{II}}$. *If the time evolution from $t = 0$ to $t = \tau$ of the composite system* $\mathbf{S}_{\mathrm{I+II}}$ *determined by the interaction satisfies Eq. (4.5) for any state σ of* $\mathbf{S}_{\mathrm{I}}$, *then A is also a macroscopic observable.*

5. C*- and W*-dynamical systems

In what follows we will discuss the related approaches based on C*-dynamical systems and W*-dynamical systems.

5.1. C*-dynamical systems

A *C*-dynamical system* is a triple $\{\mathcal{A}, G, \alpha\}$, where $\mathcal{A}$ is a (unital) C*-algebra, G is a locally compact group, and α is a strongly continuous representation of G in the automorphism group of $\mathcal{A}$. In the C*-algebraic approach, a system **S** is associated with a C*-dynamical system $\{\mathcal{A}, \mathbf{R}, \alpha\}$; it is postulated that the observables are the self-adjoint elements of $\mathcal{A}$, the states are the normalized positive linear functionals on $\mathcal{A}$, and the time evolution is given by $A(t) = \alpha_t[A]$ for observable A. An interaction of the two systems $\mathbf{S}_{\mathrm{I}}$ and $\mathbf{S}_{\mathrm{II}}$ associated with C*-algebras $\mathcal{A}_{\mathrm{I}}$ and $\mathcal{A}_{\mathrm{II}}$ is represented by a C*-dynamical system $\{\mathcal{A}_{\mathrm{I+II}}, \mathbf{R}, \alpha\}$ such that $\mathcal{A}$ is the injective tensor product of $\mathcal{A}_{\mathrm{I}}$ and $\mathcal{A}_{\mathrm{II}}$, i.e., $\mathcal{A}_{\mathrm{I+II}} = \mathcal{A}_{\mathrm{I}} \otimes_{\mathrm{min}} \mathcal{A}_{\mathrm{II}}$ [4, p. 207]. These postulates coincide with our formulation, if the Hilbert space $\mathcal{H}$ is taken as the universal representation of $\mathcal{A}$, and $\mathcal{Q}$ is taken as $\mathcal{A}$ acting on $\mathcal{H}$. In this case, the Hilbert space $\mathcal{H}_{\mathrm{I+II}}$ corresponding to the composite system $\mathbf{S}_{\mathrm{I+II}}$ is the tensor product of the Hilbert spaces $\mathbf{H}_{\mathrm{I}}$ and $\mathcal{H}_{\mathrm{II}}$ of the universal representations of $\mathcal{A}_{\mathrm{I}}$ and $\mathcal{A}_{\mathrm{II}}$. Then the set $\mathcal{Q}_{\mathrm{I+II}}$ of the quantities of $\mathbf{S}_{\mathrm{I+II}}$ is identical with $\mathcal{A}_{\mathrm{I+II}}$ when $\mathcal{Q}_{\mathrm{I+II}}$ is taken as the uniform closure of the *-algebra $\mathcal{Q}_{\mathrm{I}} \otimes \mathcal{Q}_{\mathrm{II}}$. Thus our conclusion applies to this approach, i.e., C*-dynamical systems with $\mathcal{A}_{\mathrm{I}} \otimes_{\mathrm{min}} \mathcal{A}_{\mathrm{II}}$ describe no interactions that transduces an observable $A \in \mathcal{A}_{\mathrm{I}}$ to an macroscopic observable $B \in \mathcal{A}_{\mathrm{II}}$ unless A itself is macroscopic.

5.2. W*-dynamical systems

A *W*-dynamical system* is a triple $\{\mathcal{M}, G, \alpha\}$, where $\mathcal{M}$ is a W*-algebra, G is a locally compact group, and α is a weakly continuous representation of G in the automorphism group of $\mathcal{M}$. In the W*-algebraic approach, a system **S** is associated with a W*-dynamical system $\{\mathcal{M}, \mathbf{R}, \alpha\}$; it is postulated that the observables are the self-adjoint elements of $\mathcal{M}$, the states are the normal normalized positive linear functionals on $\mathcal{M}$, and the time evolution is given by $A(t) = \alpha_t[A]$ for observable A. An interaction of the two systems $\mathbf{S}_{\mathrm{I}}$ and $\mathbf{S}_{\mathrm{II}}$ associated with W*-algebras $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$ is represented by a W*-dynamical system $\{\mathcal{M}_{\mathrm{I+II}}, \mathbf{R}, \alpha\}$ such that $\mathcal{M}_{\mathrm{I+II}}$ is the W*-tensor product of $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$, i.e., $\mathcal{M}_{\mathrm{I+II}} = \mathcal{M}_{\mathrm{I}} \bar{\otimes} \mathcal{M}_{\mathrm{II}}$ [4, p.221]. These postulates coincide with our formulation again, if the Hilbert space $\mathcal{H}$ is taken as the universal normal representation of $\mathcal{M}$, and $\mathcal{Q}$ is taken as $\mathcal{M}$ acting on $\mathcal{H}$. In this case, the Hilbert space $\mathcal{H}_{\mathrm{I+II}}$ corresponding to the composite system $\mathbf{S}_{\mathrm{I+IL}}$ is the tensor product of the Hilbert spaces $\mathcal{H}_{\mathrm{I}}$ and $\mathcal{H}_{\mathrm{II}}$ of the universal normal representations of $\mathcal{M}_{\mathrm{I}}$ and $\mathcal{M}_{\mathrm{II}}$. Then the set $\mathcal{Q}_{\mathrm{I+II}}$ of the quantities is identical with $\mathcal{M}_{\mathrm{I+II}}$ when it is taken as the weak closure of the algebra $\mathcal{Q}_{\mathrm{I}} \otimes \mathcal{Q}_{\mathrm{II}}$. Thus our conclusion applies to this approach, i.e., W*-dynamical systems with $\mathcal{M}_{\mathrm{I}} \otimes \mathcal{M}_{\mathrm{II}}$ describe no interactions that transduces an observable $A \in \mathcal{M}_{\mathrm{I}}$ to an macroscopic observable $B \in \mathcal{M}_{\mathrm{II}}$ unless A itself is macroscopic.

Acknowledgments
The author thanks Izumi Ojima for stimulating discussions.

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